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THEORETICAL ANALYSIS OF DISCRETE CONTACT PROBLEMS WITH COULOMB FRICTION

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Abstract. A discrete model of the two-dimensional Signorini problem with Coulomb friction and a coefficient of friction $\mathcal{F}$ depending on the spatial variable is analysed. It is shown that a solution exists for any $\mathcal{F}$ and is globally unique if $\mathcal{F}$ is sufficiently small. The Lipschitz continuity of this unique solution as a function of $\mathcal{F}$ as well as a function of the load vector $\boldsymbol{f}$ is obtained. Furthermore, local uniqueness of solutions for arbitrary $\mathcal{F} > 0$ is studied. The question of existence of locally Lipschitz-continuous branches of solutions with respect to the coefficient $\mathcal{F}$ is converted to the question of existence of locally Lipschitz-continuous branches of solutions with respect to the load vector $\boldsymbol{f}$. A condition guaranteeing the existence of locally Lipschitz-continuous branches of solutions in the latter case and results for determining their directional derivatives are given. Finally, the general approach is illustrated on an elementary example, whose solutions are calculated exactly.

Keywords: unilateral contact, Coulomb friction, local uniqueness, qualitative behaviour

MSC 2010: 74M10, 74G20, 74G55

1. Introduction

Contact problems describe the behaviour of loaded deformable bodies in mutual contact. On the contacting parts one often has to take into account non-penetration as well as frictional conditions. The Coulomb law of friction leads to a complicated mathematical problem, in which a lot of issues are still open. In the static case of linear elasticity, existence results have been obtained for a small coefficient of friction $\mathcal{F}$ (see e.g. [13], [4]). More recently, it has been proved in [14] that if a solution possesses a certain property, it is unique provided that $\mathcal{F}$ is small enough.

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On the other hand, some examples of non-uniqueness are known for large $\mathcal{F}$ ([9], [10]).

In the finite element setting it is known that the discretized problem admits always a solution. There are even results guaranteeing uniqueness of the solution (see e.g. [7]). However, most of them are of global nature and need the assumption on the magnitude of the coefficient of friction $\mathcal{F}$ again. To the author's knowledge the only result concerning local uniqueness of solutions, which admits even large $\mathcal{F}$, has been presented in [11]. Therein, the discrete problem is formulated as a system of non-smooth equations and a suitable version of the implicit function theorem is employed to establish the result.

Having been inspired by this approach, the present paper deals with the local behaviour of discrete solutions. It analyses dependence of solutions not only on the coefficient $\mathcal{F}$ as in [11] but also on the loading. In fact, the role of loading seems to be important, as well (see e.g. the discrete model with non-unique solutions in [12]). Besides, qualitative properties of solutions are established.

The paper is organized as follows: Section 2 is devoted to discrete contact problems with given friction, which form basis of our study of problems with Coulomb friction. In Section 3 we prove that the discrete contact problem with Coulomb friction admits always a solution and that the solution is globally unique provided that the coefficient $\mathcal{F}$ is small enough. Moreover, we show that the unique solution is a Lipschitz-continuous function of $\mathcal{F}$. To get local uniqueness results we first reformulate the problem with Coulomb friction as a system of generalized equations. Using a generalization of the implicit function theorem to this case we show that there exist locally Lipschitz-continuous branches of solutions as functions of the coefficient $\mathcal{F}$ if there are locally Lipschitz-continuous branches of the solutions as functions of the load vector $\boldsymbol{f}$. Consequently, we focus on the dependence of the solutions on $\boldsymbol{f}$ and show that it is Lipschitz-continuous provided that the magnitude of $\mathcal{F}$ does not exceed the bound guaranteeing the global uniqueness of the solutions derived before. Next, we present the formulation of the problem consisting of piecewise-differentiable equations. Making use of the implicit function theorem corresponding to this case we arrive at a condition on the determinant sign of particular Jacobians that ensures the existence of locally Lipschitz-continuous branches of the solutions with respect to $\boldsymbol{f}$. Results for determining directional derivatives to these branches are also obtained. Finally, Section 4 deals with an elementary example with one contact node. After calculating all its solutions, we discuss the question of their uniqueness or local uniqueness with regard to the previous general results.

We consider a linearly elastic body whose reference configuration is given by a bounded domain $\Omega \subset \mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega$. Let Γ_u, Γ_p, and Γ_c be three disjoint, (relatively) open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_p \cup \overline{\Gamma}_c$. The body is fixed on Γ_u, surface tractions of density p act on Γ_p while a rigid foundation S supports the body unilaterally along Γ_c. In addition, the body is subject to volume forces of density f. We seek the equilibrium state of the body. In the sequel, we shall suppose that there is no gap between S and Γ_c.

The so-called Signorini problem consists in finding a displacement vector $u\colon \Omega \to \mathbb{R}^2$ satisfying the following equations and boundary conditions:

$$\begin{aligned} -\operatorname{div}\sigma(u) &= f && \text{in } \Omega,\\ \sigma(u) &= \mathcal{C}\varepsilon(u) && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_u,\\ \sigma(u)\nu &= p && \text{on } \Gamma_p,\\ u_\nu \leqslant 0, \quad \sigma_\nu(u) \leqslant 0, &\quad u_\nu\sigma_\nu(u) = 0 && \text{on } \Gamma_c. \end{aligned}$$

Here $\sigma(u)$ is the stress tensor, $\varepsilon(u) = 1/2(\nabla u + \nabla^\top u)$ is the linearized strain tensor and $\mathcal{C}$ is the 4th order elasticity tensor. By ν we denote the unit outward normal vector to $\partial\Omega$ and $u_\nu := u\cdot\nu$, $\sigma_\nu(u) := (\sigma(u)\nu)\cdot\nu$ stand for the normal components of the displacement vector u and of the stress vector $\sigma(u)\nu$ on Γ_c, respectively.

To take into account effects of friction, let t be a unit tangent vector orthogonal to ν. Then $u_t := u\cdot t$ and $\sigma_t(u) := (\sigma(u)\nu)\cdot t$ denote the tangential displacement and the tangential contact stress on Γ_c, respectively. The *Coulomb law* of friction reads as follows:

$$\left.\begin{aligned} |\sigma_t(u)(x)| &\leqslant -\mathcal{F}(x)\sigma_\nu(u)(x),\\ u_t(x) \neq 0 \implies \sigma_t(u)(x) &= \mathcal{F}(x)\sigma_\nu(u)(x)\frac{u_t(x)}{|u_t(x)|} \end{aligned}\right\}\quad x \in \Gamma_c.$$

Throughout the paper we shall use the following notation: $(\cdot,\cdot)_n$ stands for the scalar product in $\mathbb{R}^n$, $\|\cdot\|_n$ for the corresponding norm, whereas $\|\cdot\|_{n,\infty}$ denotes the max-norm in $\mathbb{R}^n$:

$$\|\boldsymbol{v}\|_{n,\infty} = \max_{i=1,\dots,n} |v_i|, \quad \boldsymbol{v} = (v_1,\dots,v_n) \in \mathbb{R}^n.$$

The symbol $\|\cdot\|_n$ is also used for the matrix norm in $\mathbb{R}^{n\times n}$ generated by the vector norm $\|\cdot\|_n$.

2. Discrete contact problems with given friction

This section deals with the two-dimensional Signorini problem with *given* friction in which the threshold for the magnitude of σ_t is set to be the product of $\mathcal{F}$ and a given slip bound g. It is assumed that $\mathcal{F}$ depends on the spatial variable, i.e. $\mathcal{F} = \mathcal{F}(x)$. A finite element approximation of this model leads to the variational inequality (for more details see e.g. [8]):

$$(\boldsymbol{P}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g})) \qquad \begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{K} \text{ such that} \\ (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_n + \displaystyle\sum_{i=1}^{p} \mathcal{F}_i g_i (|(\boldsymbol{B}_t \boldsymbol{v})_i| - |(\boldsymbol{B}_t \boldsymbol{u})_i|) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_n \\ \hfill \forall\, \boldsymbol{v} \in \boldsymbol{K}, \end{cases}$$

where $\boldsymbol{u}$ represents the displacement vector and $\boldsymbol{K}$ is the convex set of all kinematically admissible displacements:

$$\boldsymbol{K} = \{\boldsymbol{v} \in \mathbb{R}^n \colon\ \boldsymbol{B}_\nu \boldsymbol{v} \leqslant \boldsymbol{0}\}$$

with n being the number of degrees of freedom. By $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ we denote the stiffness matrix satisfying:

$$(2.1) \qquad \begin{cases} \text{(i) } \boldsymbol{A} = \boldsymbol{A}^\top; \\ \text{(ii) } \exists\, \gamma > 0\colon\ (\boldsymbol{A}\boldsymbol{v}, \boldsymbol{v})_n \geqslant \gamma \|\boldsymbol{v}\|_n^2 \quad \forall\, \boldsymbol{v} \in \mathbb{R}^n. \end{cases}$$

The matrices $\boldsymbol{B}_\nu, \boldsymbol{B}_t \in \mathbb{R}^{p\times n}$, where p is the number of the contact nodes, represent the linear mappings associating with a displacement vector its normal and tangential component on the contact zone, respectively. Hence we may suppose that

$$(2.2) \quad \begin{cases} \text{(j) the Euclidean norm of each row vector of } \boldsymbol{B}_\nu,\ \boldsymbol{B}_t \text{ is equal to one;} \\ \text{(jj) each column of } \boldsymbol{B}_\nu,\ \boldsymbol{B}_t \text{ contains at most one nonzero element;} \\ \text{(jjj) } \boldsymbol{B}_\nu^\top \boldsymbol{\mu}_\nu + \boldsymbol{B}_t^\top \boldsymbol{\mu}_t = \boldsymbol{0} \iff (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) = (\boldsymbol{0}, \boldsymbol{0}) \in \mathbb{R}^{2p}. \end{cases}$$

Note that (jjj) holds if and only if there exists $\beta > 0$ such that

$$(2.3) \qquad \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^n} \frac{(\boldsymbol{\mu}_\nu, \boldsymbol{B}_\nu \boldsymbol{v})_p + (\boldsymbol{\mu}_t, \boldsymbol{B}_t \boldsymbol{v})_p}{\|\boldsymbol{v}\|_n} \geqslant \beta \|(\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t)\|_{2p} \quad \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \in \mathbb{R}^{2p}.$$

Further, $\boldsymbol{\mathcal{F}} = (\mathcal{F}_1, \ldots, \mathcal{F}_p)$, $\boldsymbol{g} = (g_1, \ldots, g_p) \in \mathbb{R}^p_+$ characterize the distribution of the coefficient of friction $\mathcal{F}$ and of the given slip bound g in the contact nodes, respectively, and $\boldsymbol{f} \in \mathbb{R}^n$ denotes the load vector.

Using the Lagrange-multiplier sets $\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}, \boldsymbol{g})$ defined by

$$\begin{aligned} \boldsymbol{\Lambda}_\nu &= \mathbb{R}^p_-, \\ \boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}, \boldsymbol{g}) &= \{\boldsymbol{\mu}_t = (\mu_{t,1}, \ldots, \mu_{t,p}) \in \mathbb{R}^p \colon\ |\mu_{t,i}| \leqslant \mathcal{F}_i g_i\ \forall\, i = 1, \ldots, p\}, \end{aligned}$$

one can reformulate problem $(\boldsymbol{P}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$ as follows:

$$(\boldsymbol{M}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g})) \qquad \begin{cases} \text{Find } (\boldsymbol{u},\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_t) \in \mathbb{R}^n \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}},\boldsymbol{g}) \text{ such that} \\ (\boldsymbol{Au},\boldsymbol{v})_n = (\boldsymbol{f},\boldsymbol{v})_n + (\boldsymbol{\lambda}_\nu,\boldsymbol{B}_\nu\boldsymbol{v})_p + (\boldsymbol{\lambda}_t,\boldsymbol{B}_t\boldsymbol{v})_p \quad \forall\,\boldsymbol{v}\in\mathbb{R}^n, \\ (\boldsymbol{\mu}_\nu-\boldsymbol{\lambda}_\nu,\boldsymbol{B}_\nu\boldsymbol{u})_p + (\boldsymbol{\mu}_t-\boldsymbol{\lambda}_t,\boldsymbol{B}_t\boldsymbol{u})_p \geqslant 0 \\ \qquad\qquad\qquad\qquad \forall\,(\boldsymbol{\mu}_\nu,\boldsymbol{\mu}_t)\in\boldsymbol{\Lambda}_\nu\times\boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}},\boldsymbol{g}). \end{cases}$$

Properties of both problems are summarized in the following proposition.

Proposition 2.1. *Let* (2.1) *and* (2.2) *be satisfied. Then for any* $\boldsymbol{\mathcal{F}},\boldsymbol{g}\in\mathbb{R}^p_+$ *and any* $\boldsymbol{f}\in\mathbb{R}^n$ *there exists a unique solution to* $(\boldsymbol{P}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$ *as well as to* $(\boldsymbol{M}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$*. Moreover, if* $(\boldsymbol{u},\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_t)$ *is the solution of* $(\boldsymbol{M}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$ *then* $\boldsymbol{u}$ *solves* $(\boldsymbol{P}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$ *and we have*

$$\|\boldsymbol{u}\|_n \leqslant \frac{\|\boldsymbol{f}\|_n}{\gamma}, \tag{2.4}$$

$$\|(\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_t)\|_{2p} \leqslant \frac{\|\boldsymbol{f}\|_n}{\beta}\Big(\frac{\|\boldsymbol{A}\|_n}{\gamma}+1\Big), \tag{2.5}$$

where β *is the constant from* (2.3).

Proof. Since $(\boldsymbol{P}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$ is a variational inequality of the second kind, its solvability and uniqueness are established in [6]. To prove that $(\boldsymbol{M}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$ has a unique solution, one can introduce an equivalent saddle-point formulation and apply results from [5] showing also the mutual relation between the solutions to $(\boldsymbol{P}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$ and $(\boldsymbol{M}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$. Inserting $\boldsymbol{v}:=\boldsymbol{0}\in\boldsymbol{K}$ into $(\boldsymbol{P}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))$, we obtain

$$-(\boldsymbol{Au},\boldsymbol{u})_n - \sum_{i=1}^p \mathcal{F}_i g_i |(\boldsymbol{B}_t\boldsymbol{u})_i| \geqslant -(\boldsymbol{f},\boldsymbol{u})_n.$$

Using (2.1), we get:

$$\gamma\|\boldsymbol{u}\|_n^2 \leqslant (\boldsymbol{Au},\boldsymbol{u})_n + \sum_{i=1}^p \mathcal{F}_i g_i |(\boldsymbol{B}_t\boldsymbol{u})_i| \leqslant (\boldsymbol{f},\boldsymbol{u})_n \leqslant \|\boldsymbol{f}\|_n\|\boldsymbol{u}\|_n,$$

which yields (2.4). To prove (2.5) we employ $(\boldsymbol{M}(\boldsymbol{f},\boldsymbol{\mathcal{F}},\boldsymbol{g}))_2$:

$$(\boldsymbol{\lambda}_\nu,\boldsymbol{B}_\nu\boldsymbol{v})_p + (\boldsymbol{\lambda}_t,\boldsymbol{B}_t\boldsymbol{v})_p = (\boldsymbol{Au},\boldsymbol{v})_n - (\boldsymbol{f},\boldsymbol{v})_n \leqslant \|\boldsymbol{A}\|_n\|\boldsymbol{u}\|_n\|\boldsymbol{v}\|_n + \|\boldsymbol{f}\|_n\|\boldsymbol{v}\|_n \qquad \forall\,\boldsymbol{v}\in\mathbb{R}^n.$$

From this, (2.3), and (2.4) we have:

$$\begin{aligned}\beta\|(\boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)\|_{2p} &\leqslant \sup_{\mathbf{0}\neq \boldsymbol{v}\in\mathbb{R}^n} \frac{(\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v})_p + (\boldsymbol{\lambda}_t, \boldsymbol{B}_t \boldsymbol{v})_p}{\|\boldsymbol{v}\|_n} \\ &\leqslant \|\boldsymbol{A}\|_n\|\boldsymbol{u}\|_n + \|\boldsymbol{f}\|_n \leqslant \|\boldsymbol{A}\|_n \frac{\|\boldsymbol{f}\|_n}{\gamma} + \|\boldsymbol{f}\|_n.\end{aligned}$$

□

It is worth mentioning that the bounds (2.4) and (2.5) are independent of $\boldsymbol{\mathcal{F}}, \boldsymbol{g} \in \mathbb{R}^p_+$.

3. Discrete contact problems with Coulomb friction

Having studied the problems with given friction, we are able to present the fixed-point formulation of the problems with Coulomb friction. To this end let $\boldsymbol{f} \in \mathbb{R}^n$ be fixed and let us introduce the mapping $\boldsymbol{\mathcal{G}}\colon\ \mathbb{R}^p_+ \times \mathbb{R}^p_+ \to \mathbb{R}^p_+$ by

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, \boldsymbol{g}) = -\boldsymbol{\lambda}_\nu, \quad \boldsymbol{\mathcal{F}}, \boldsymbol{g} \in \mathbb{R}^p_+,$$

where $\boldsymbol{\lambda}_\nu := \boldsymbol{\lambda}_\nu(\boldsymbol{\mathcal{F}}, \boldsymbol{g})$ is the second component of the solution to $(\boldsymbol{M}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$.

Definition 3.1. Let $\boldsymbol{f} \in \mathbb{R}^n$ and $\boldsymbol{\mathcal{F}} \in \mathbb{R}^p_+$ be given. Any triplet $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ is called a solution of the discrete contact problem with Coulomb friction if it solves $(\boldsymbol{M}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu))$, i.e. $-\boldsymbol{\lambda}_\nu$ is a fixed point of the mapping $\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, \cdot)$:

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu) = -\boldsymbol{\lambda}_\nu.$$

Lemma 3.1. *Assume that* (2.1), (2.2) *hold and* $\boldsymbol{f} \in \mathbb{R}^n$, $\boldsymbol{\mathcal{F}}, \overline{\boldsymbol{\mathcal{F}}}, \boldsymbol{g}, \overline{\boldsymbol{g}} \in \mathbb{R}^p_+$ *are arbitrary. Let* $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$, $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_t)$ *be the solutions to* $(\boldsymbol{M}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$ *and* $(\boldsymbol{M}(\boldsymbol{f}, \overline{\boldsymbol{\mathcal{F}}}, \overline{\boldsymbol{g}}))$, *respectively. Then*

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_n \leqslant \frac{\|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\gamma}\|\boldsymbol{g} - \overline{\boldsymbol{g}}\|_p + \frac{\|\overline{\boldsymbol{g}}\|_p}{\gamma}\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}, \tag{3.1}$$

$$\|(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{\lambda}_t - \bar{\boldsymbol{\lambda}}_t)\|_{2p} \leqslant \frac{\|\boldsymbol{A}\|_n\|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\beta\gamma}\|\boldsymbol{g} - \overline{\boldsymbol{g}}\|_p + \frac{\|\boldsymbol{A}\|_n\|\overline{\boldsymbol{g}}\|_p}{\beta\gamma}\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}. \tag{3.2}$$

In particular, if $\boldsymbol{\mathcal{F}} = \overline{\boldsymbol{\mathcal{F}}}$ *then*

$$\|\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu\|_p \leqslant \frac{\|\boldsymbol{A}\|_n\|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\beta\gamma}\|\boldsymbol{g} - \overline{\boldsymbol{g}}\|_p, \tag{3.3}$$

i.e. $\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, \cdot)\colon\ \mathbb{R}^p_+ \to \mathbb{R}^p_+$ *is Lipschitz-continuous in* $\mathbb{R}^p_+$.

P r o o f. Inserting $\boldsymbol{v} := \bar{\boldsymbol{u}} \in \boldsymbol{K}$ in $(\boldsymbol{P}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$ and $\boldsymbol{v} := \boldsymbol{u} \in \boldsymbol{K}$ in $(\boldsymbol{P}(\boldsymbol{f}, \overline{\boldsymbol{\mathcal{F}}}, \overline{\boldsymbol{g}}))$, we have

$$(\boldsymbol{A}\boldsymbol{u}, \bar{\boldsymbol{u}} - \boldsymbol{u})_n + \sum_{i=1}^{p} \mathcal{F}_i g_i(|(\boldsymbol{B}_t\bar{\boldsymbol{u}})_i| - |(\boldsymbol{B}_t\boldsymbol{u})_i|) \geqslant (\boldsymbol{f}, \bar{\boldsymbol{u}} - \boldsymbol{u})_n,$$
$$(\boldsymbol{A}\bar{\boldsymbol{u}}, \boldsymbol{u} - \bar{\boldsymbol{u}})_n + \sum_{i=1}^{p} \overline{\mathcal{F}}_i \overline{g}_i(|(\boldsymbol{B}_t\boldsymbol{u})_i| - |(\boldsymbol{B}_t\bar{\boldsymbol{u}})_i|) \geqslant (\boldsymbol{f}, \boldsymbol{u} - \bar{\boldsymbol{u}})_n.$$

Summing the two inequalities and using (2.1) and (2.2), we arrive at

$$\begin{aligned}\gamma\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_n^2 &\leqslant \big(\boldsymbol{A}(\bar{\boldsymbol{u}} - \boldsymbol{u}), \bar{\boldsymbol{u}} - \boldsymbol{u}\big)_n \leqslant \sum_{i=1}^{p} (\mathcal{F}_i g_i - \overline{\mathcal{F}}_i \overline{g}_i)(|(\boldsymbol{B}_t\bar{\boldsymbol{u}})_i| - |(\boldsymbol{B}_t\boldsymbol{u})_i|) \\ &\leqslant \sum_{i=1}^{p} |\mathcal{F}_i(g_i - \overline{g}_i)||(\boldsymbol{B}_t\bar{\boldsymbol{u}} - \boldsymbol{B}_t\boldsymbol{u})_i| + \sum_{i=1}^{p} |(\mathcal{F}_i - \overline{\mathcal{F}}_i)\overline{g}_i||(\boldsymbol{B}_t\bar{\boldsymbol{u}} - \boldsymbol{B}_t\boldsymbol{u})_i| \\ &\leqslant \|\boldsymbol{\mathcal{F}}\|_{p,\infty}\|\boldsymbol{g} - \overline{\boldsymbol{g}}\|_p\|\bar{\boldsymbol{u}} - \boldsymbol{u}\|_n + \|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}\|\overline{\boldsymbol{g}}\|_p\|\bar{\boldsymbol{u}} - \boldsymbol{u}\|_n,\end{aligned}$$

which leads to (3.1). Furthermore, the difference $(\boldsymbol{M}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))_2 - (\boldsymbol{M}(\boldsymbol{f}, \overline{\boldsymbol{\mathcal{F}}}, \overline{\boldsymbol{g}}))_2$ results in

$$(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{B}_\nu\boldsymbol{v})_p + (\boldsymbol{\lambda}_t - \bar{\boldsymbol{\lambda}}_t, \boldsymbol{B}_t\boldsymbol{v})_p = \big(\boldsymbol{A}(\boldsymbol{u} - \bar{\boldsymbol{u}}), \boldsymbol{v}\big)_n \leqslant \|\boldsymbol{A}\|_n\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_n\|\boldsymbol{v}\|_n \quad \forall\, \boldsymbol{v} \in \mathbb{R}^n.$$

From this and (2.3) we obtain

$$\begin{aligned}\beta\|(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{\lambda}_t - \bar{\boldsymbol{\lambda}}_t)\|_{2p} &\leqslant \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^n} \frac{(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{B}_\nu\boldsymbol{v})_p + (\boldsymbol{\lambda}_t - \bar{\boldsymbol{\lambda}}_t, \boldsymbol{B}_t\boldsymbol{v})_p}{\|\boldsymbol{v}\|_n} \\ &\leqslant \|\boldsymbol{A}\|_n\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_n,\end{aligned}$$

which together with (3.1) completes the proof. □

Let

$$\boldsymbol{B}_R(\boldsymbol{0}) = \{\boldsymbol{\mu} \in \mathbb{R}^p \colon \|\boldsymbol{\mu}\|_p \leqslant R\}, \quad R > 0.$$

The next theorem guarantees the existence and under an additional assumption also the uniqueness of the fixed points we seek.

Theorem 3.1. *Suppose that* (2.1) *and* (2.2) *are satisfied. For any* $\boldsymbol{f} \in \mathbb{R}^n$ *and any* $\boldsymbol{\mathcal{F}} \in \mathbb{R}^p_+$ *there exists at least one fixed point of the mapping* $\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, \cdot)$. *All the fixed points are contained in* $\mathbb{R}^p_+ \cap \boldsymbol{B}_R(\boldsymbol{0})$ *with* $R = \|\boldsymbol{f}\|_n/\beta \cdot (\|\boldsymbol{A}\|_n/\gamma + 1)$. *In addition, the fixed point is unique provided that* $\|\boldsymbol{\mathcal{F}}\|_{p,\infty} < \beta\gamma/\|\boldsymbol{A}\|_n$.

P r o o f. It follows from the Brouwer and the Banach fixed-point theorems by making use of Proposition 2.1 and Lemma 3.1. □

Corollary 3.1. *Let* (2.1) *and* (2.2) *be satisfied. For any* $\boldsymbol{\mathcal{F}} \in \mathbb{R}^p_+$, $\|\boldsymbol{\mathcal{F}}\|_{p,\infty} < \beta\gamma/\|\boldsymbol{A}\|_n$, *and any* $\boldsymbol{f} \in \mathbb{R}^n$ *the discrete contact problem with Coulomb friction has a unique solution. In addition, the method of successive approximations converges for any choice of the initial approximation.*

Confining ourselves to $\boldsymbol{\mathcal{F}}$ such that $\|\boldsymbol{\mathcal{F}}\|_{p,\infty} \leqslant \mathcal{F}_{\max}$ for an arbitrary $\mathcal{F}_{\max} \in [0, \beta\gamma/\|\boldsymbol{A}\|_n)$, we shall show that the solution of the contact problem with Coulomb friction is a Lipschitz-continuous function of $\boldsymbol{\mathcal{F}}$. For this purpose we define a mapping $\boldsymbol{\mathcal{S}_f}\colon \mathbb{R}^p_+ \to \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p$ for a *fixed* $\boldsymbol{f} \in \mathbb{R}^n$ by

$$\boldsymbol{\mathcal{S}_f}(\boldsymbol{\mathcal{F}}) = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t), \quad \boldsymbol{\mathcal{F}} \in \mathbb{R}^p_+,\ \|\boldsymbol{\mathcal{F}}\|_{p,\infty} < \frac{\beta\gamma}{\|\boldsymbol{A}\|_n},$$

where $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ is the unique solution to the contact problem with Coulomb friction with the coefficient $\boldsymbol{\mathcal{F}}$ and the load vector $\boldsymbol{f}$.

Theorem 3.2. *Let* (2.1) *and* (2.2) *be satisfied and let* $\boldsymbol{f} \in \mathbb{R}^n$ *be arbitrary. Then for any* $\mathcal{F}_{\max} \in [0, \beta\gamma/\|\boldsymbol{A}\|_n)$ *there exists* $\delta > 0$ *such that:*

$$\|\boldsymbol{\mathcal{S}_f}(\boldsymbol{\mathcal{F}}) - \boldsymbol{\mathcal{S}_f}(\overline{\boldsymbol{\mathcal{F}}})\|_{n+2p} \leqslant \delta\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \quad \forall\, \boldsymbol{\mathcal{F}}, \overline{\boldsymbol{\mathcal{F}}} \in \mathbb{R}^p_+,\ \|\boldsymbol{\mathcal{F}}\|_{p,\infty}, \|\overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \leqslant \mathcal{F}_{\max}.$$

P r o o f. For given $\mathcal{F}_{\max} \in [0, \beta\gamma/\|\boldsymbol{A}\|_n)$ and $\boldsymbol{\mathcal{F}}, \overline{\boldsymbol{\mathcal{F}}} \in \mathbb{R}^p_+$ with $\|\boldsymbol{\mathcal{F}}\|_{p,\infty}, \|\overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \leqslant \mathcal{F}_{\max}$ let $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) := \boldsymbol{\mathcal{S}_f}(\boldsymbol{\mathcal{F}})$, $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_t) := \boldsymbol{\mathcal{S}_f}(\overline{\boldsymbol{\mathcal{F}}})$. Further, let $\{\boldsymbol{g}^k\}$, $\{\overline{\boldsymbol{g}}^k\}$ be sequences defined by

$$\begin{aligned} &(3.4) \qquad \boldsymbol{g}^0 = \overline{\boldsymbol{g}}^0 \in \mathbb{R}^p_+, \quad \|\boldsymbol{g}^0\|_p \leqslant \frac{\|\boldsymbol{f}\|_n}{\beta}\Big(\frac{\|\boldsymbol{A}\|_n}{\gamma} + 1\Big), \\ &\boldsymbol{g}^{k+1} = \boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, \boldsymbol{g}^k), \quad \overline{\boldsymbol{g}}^{k+1} = \boldsymbol{\mathcal{G}}(\overline{\boldsymbol{\mathcal{F}}}, \overline{\boldsymbol{g}}^k), \quad k = 1, 2, \ldots \end{aligned}$$

From Corollary 3.1 we know that

$$\lim_{k\to\infty} \boldsymbol{g}^k = -\boldsymbol{\lambda}_\nu, \quad \lim_{k\to\infty} \overline{\boldsymbol{g}}^k = -\bar{\boldsymbol{\lambda}}_\nu.$$

First, (3.2) and (3.4) give

$$\begin{aligned} (3.5) \quad \|\boldsymbol{g}^1 - \overline{\boldsymbol{g}}^1\|_p = \|\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, \boldsymbol{g}^0) - \boldsymbol{\mathcal{G}}(\overline{\boldsymbol{\mathcal{F}}}, \boldsymbol{g}^0)\|_p &\leqslant \frac{\|\boldsymbol{A}\|_n\|\boldsymbol{g}^0\|_p}{\beta\gamma}\,\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \\ &\leqslant \frac{\|\boldsymbol{A}\|_n\|\boldsymbol{f}\|_n}{\beta^2\gamma}\left(\frac{\|\boldsymbol{A}\|_n}{\gamma} + 1\right)\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} = c\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}, \end{aligned}$$

where $c := \|\boldsymbol{A}\|_n \|\boldsymbol{f}\|_n/(\beta^2\gamma) \cdot (\|\boldsymbol{A}\|_n/\gamma + 1)$. From (2.5), (3.2), and (3.5) we obtain

$$\begin{aligned}
\|\boldsymbol{g}^2 - \overline{\boldsymbol{g}}^2\|_p &= \|\boldsymbol{\mathcal{G}}(\boldsymbol{\mathcal{F}}, \boldsymbol{g}^1) - \boldsymbol{\mathcal{G}}(\overline{\boldsymbol{\mathcal{F}}}, \overline{\boldsymbol{g}}^1)\|_p \\
&\leqslant \frac{\|\boldsymbol{A}\|_n \|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\beta\gamma} \|\boldsymbol{g}^1 - \overline{\boldsymbol{g}}^1\|_p + \frac{\|\boldsymbol{A}\|_n \|\overline{\boldsymbol{g}}^1\|_p}{\beta\gamma} \|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \\
&\leqslant q\|\boldsymbol{g}^1 - \overline{\boldsymbol{g}}^1\|_p + c\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \leqslant (cq + c)\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}
\end{aligned}$$

with $q := \mathcal{F}_{\max}\|\boldsymbol{A}\|_n/(\beta\gamma) < 1$. Thus by induction,

$$\begin{aligned}
\|\boldsymbol{g}^{k+1} - \overline{\boldsymbol{g}}^{k+1}\|_p &\leqslant c\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} + q\|\boldsymbol{g}^k - \overline{\boldsymbol{g}}^k\|_p \\
&\leqslant c\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} + q(c + cq + \ldots + cq^{k-1})\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \\
&\leqslant \frac{c}{1-q} \|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}.
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\|\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu\|_p \leqslant \frac{c}{1-q} \|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}. \tag{3.6}$$

Inserting $\boldsymbol{g} := -\boldsymbol{\lambda}_\nu$ and $\overline{\boldsymbol{g}} := -\bar{\boldsymbol{\lambda}}_\nu$ into (3.1), using (3.6) and Theorem 3.1 we see that

$$\begin{aligned}
\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_n &\leqslant \frac{\|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\gamma} \|\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu\|_p + \frac{\|\bar{\boldsymbol{\lambda}}_\nu\|_p}{\gamma} \|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \\
&\leqslant \left(\frac{c\mathcal{F}_{\max}}{\gamma(1-q)} + \frac{\|\boldsymbol{f}\|_n}{\beta\gamma} \left(\frac{\|\boldsymbol{A}\|_n}{\gamma} + 1 \right) \right) \|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}.
\end{aligned}$$

Finally, (3.2) with $\boldsymbol{g} := -\boldsymbol{\lambda}_\nu$ and $\overline{\boldsymbol{g}} := -\bar{\boldsymbol{\lambda}}_\nu$ together with Theorem 3.1 and (3.6) ensures that

$$\|\boldsymbol{\lambda}_t - \bar{\boldsymbol{\lambda}}_t\|_p \leqslant q\|\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu\|_p + c\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \leqslant \frac{c}{1-q} \|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty}.$$

□

In the sequel, we shall restrict ourselves to the coefficients of friction $\boldsymbol{\mathcal{F}}$ with positive components solely. On the other hand, *no* upper bounds will be imposed, i.e. $\boldsymbol{\mathcal{F}}$ will belong to the set $\boldsymbol{\mathcal{A}}$ defined by

$$\boldsymbol{\mathcal{A}} = \{\boldsymbol{\mathcal{F}} \in \mathbb{R}^p \colon \mathcal{F}_i > 0 \; \forall\, i = 1, \ldots, p\}.$$

As we know from Theorem 3.1, there exists *at least* one solution to the contact problem with Coulomb friction for any $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$. Next, we shall study the behaviour of such solutions as functions of $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$ and of the load vector $\boldsymbol{f} \in \mathbb{R}^n$, respectively. For this purpose we introduce an alternative definition of the problem in which the Lagrange-multiplier set $\boldsymbol{\Lambda}_t(\cdot)$ does not depend on $\boldsymbol{\mathcal{F}}$.

Let $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$, $\boldsymbol{g} \in \mathbb{R}^p_+$ be given and set

$$\boldsymbol{\Lambda}_t(\boldsymbol{g}) = \{\boldsymbol{\mu}_t \in \mathbb{R}^p \colon\ |\mu_{t,i}| \leqslant g_i\ \forall\, i = 1, \ldots, p\}.$$

As an alternative to $(\boldsymbol{M}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$, a mixed formulation of the problem with given friction reads as follows:

$$(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g})) \qquad \begin{cases} \text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \mathbb{R}^n \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\boldsymbol{g}) \text{ such that} \\ (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v})_n = (\boldsymbol{f}, \boldsymbol{v})_n + (\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v})_p + (\boldsymbol{F}\boldsymbol{\lambda}_t, \boldsymbol{B}_t \boldsymbol{v})_p \quad \forall\, \boldsymbol{v} \in \mathbb{R}^n, \\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{u})_p + (\boldsymbol{F}(\boldsymbol{\mu}_t - \boldsymbol{\lambda}_t), \boldsymbol{B}_t \boldsymbol{u})_p \geqslant 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \in \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\boldsymbol{g}), \end{cases}$$

where $\boldsymbol{F} := \boldsymbol{F}(\boldsymbol{\mathcal{F}}) = \operatorname{diag}\{\mathcal{F}_1, \ldots, \mathcal{F}_p\} \in \mathbb{R}^{p \times p}$.

Clearly, the triplet $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ solves problem $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$ if and only if $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{F}\boldsymbol{\lambda}_t)$ is a solution of $(\boldsymbol{M}(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$. Hence, the existence and the uniqueness of the solution to $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$ result from Proposition 2.1.

Now we are ready to rewrite Definition 3.1.

Definition 3.2. Let $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$ be given. Any triplet $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ is said to be a solution of the discrete contact problem with Coulomb friction if it solves $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu))$.

Since there is a one-to-one correspondence between the solutions established by this and the former definition, the existence and uniqueness results remain valid.

Next, we derive an equivalent formulation of the contact problem using Definition 3.2. Let $\boldsymbol{f} \in \mathbb{R}^n$ be *fixed* and let $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ be the corresponding solution of $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu))$. The inequality $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu))_3$ can be written as

$$-\boldsymbol{B}_\nu \boldsymbol{u} \in \boldsymbol{N}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu), \quad -\boldsymbol{F}\boldsymbol{B}_t \boldsymbol{u} \in \boldsymbol{N}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t),$$

where $\boldsymbol{N}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\mu})$, $\boldsymbol{N}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\mu})$ denote the normal cones of $\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)$, respectively, at a point $\boldsymbol{\mu} \in \mathbb{R}^p$. Consequently, the solution of the discrete contact problem with Coulomb friction can be characterized as a solution to the system of generalized equations

$$\text{Find } \boldsymbol{y} \in \mathbb{R}^{n+2p} \text{ such that } \boldsymbol{0} \in \boldsymbol{C}_{\boldsymbol{f}}(\boldsymbol{\mathcal{F}}, \boldsymbol{y}) + \boldsymbol{Q}(\boldsymbol{y}), \tag{3.7}$$

where $\boldsymbol{C_f}\colon \boldsymbol{\mathcal{A}}\times\mathbb{R}^{n+2p}\to\mathbb{R}^{n+2p}$ and $\boldsymbol{Q}\colon \mathbb{R}^{n+2p}\rightrightarrows\mathbb{R}^{n+2p}$ are the single-valued, continuously differentiable function and the set-valued mapping, respectively, defined by

$$\boldsymbol{C_f}(\boldsymbol{\mathcal{F}},\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}_\nu^\top & -\boldsymbol{B}_t^\top\boldsymbol{F} \\ \boldsymbol{B}_\nu & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}\boldsymbol{B}_t & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda}_\nu \\ \boldsymbol{\lambda}_t \end{pmatrix} - \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix},$$
$$\boldsymbol{Q}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{N}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu) \\ \boldsymbol{N}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t) \end{pmatrix},$$
$$\boldsymbol{\mathcal{F}} = (\mathcal{F}_1,\dots,\mathcal{F}_p)\in\boldsymbol{\mathcal{A}},\ \ \boldsymbol{y}\equiv(\boldsymbol{u},\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_t)\in\mathbb{R}^{n+2p},$$

with $\boldsymbol{F} := \boldsymbol{F}(\boldsymbol{\mathcal{F}}) = \operatorname{diag}\{\mathcal{F}_1,\dots,\mathcal{F}_p\}$.

Interpreting $\boldsymbol{\mathcal{F}}$ as a perturbation parameter and following the technique used in [1], we shall analyse this system according to [15] (see also [3]):

Let $\boldsymbol{\mathcal{F}}^0\in\boldsymbol{\mathcal{A}}$ be a reference point. Assume that $\boldsymbol{y}^0\in\mathbb{R}^{n+2p}$ is such that

$$\boldsymbol{0}\in\boldsymbol{C_f}(\boldsymbol{\mathcal{F}}^0,\boldsymbol{y}^0)+\boldsymbol{Q}(\boldsymbol{y}^0).$$

Define multi-valued functions $\boldsymbol{\mathcal{S}}_{\boldsymbol{f}}^*\colon \boldsymbol{\mathcal{A}}\rightrightarrows\mathbb{R}^{n+2p}$, $\boldsymbol{\Sigma_f}\colon \mathbb{R}^{n+2p}\rightrightarrows\mathbb{R}^{n+2p}$ by

$$\text{(3.8)}\quad \boldsymbol{\mathcal{S}}_{\boldsymbol{f}}^*(\boldsymbol{\mathcal{F}}) = \{\boldsymbol{y}\in\mathbb{R}^{n+2p}\colon\ \boldsymbol{0}\in\boldsymbol{C_f}(\boldsymbol{\mathcal{F}},\boldsymbol{y})+\boldsymbol{Q}(\boldsymbol{y})\},\quad \boldsymbol{\mathcal{F}}\in\boldsymbol{\mathcal{A}},$$
$$\boldsymbol{\Sigma_f}(\boldsymbol{\xi}) = \{\boldsymbol{y}\in\mathbb{R}^{n+2p}\colon\ \boldsymbol{\xi}\in\boldsymbol{C_f}(\boldsymbol{\mathcal{F}}^0,\boldsymbol{y}^0)+\boldsymbol{\nabla_y C_f}(\boldsymbol{\mathcal{F}}^0,\boldsymbol{y}^0)(\boldsymbol{y}-\boldsymbol{y}^0)+\boldsymbol{Q}(\boldsymbol{y})\},\quad \boldsymbol{\xi}\in\mathbb{R}^{n+2p},$$

where $\boldsymbol{\nabla_y C_f}(\boldsymbol{\mathcal{F}}^0,\boldsymbol{y}^0)$ stands for the gradient of $\boldsymbol{C_f}$ with respect to $\boldsymbol{y}$ at $(\boldsymbol{\mathcal{F}}^0,\boldsymbol{y}^0)$. In other words, $\boldsymbol{\mathcal{S}}_{\boldsymbol{f}}^*(\boldsymbol{\mathcal{F}})$ is the solution set of (3.7) for a given coefficient $\boldsymbol{\mathcal{F}}\in\boldsymbol{\mathcal{A}}$ and the load vector $\boldsymbol{f}\in\mathbb{R}^n$. Furthermore, $\boldsymbol{\Sigma_f}(\boldsymbol{\xi})$ is the solution set to the generalized equation obtained by the partial linearization of $\boldsymbol{C_f}(\boldsymbol{\mathcal{F}},\boldsymbol{y})$ in (3.7) with respect to the second variable around the reference point $(\boldsymbol{\mathcal{F}}^0,\boldsymbol{y}^0)$.

The following generalization of the implicit function theorem holds (see [3, Theorem 5.1]).

Theorem 3.3. *Assume that there exist a single-valued Lipschitz function $\boldsymbol{\phi_f}$ from a neighbourhood $\boldsymbol{W}$ of $\boldsymbol{0}\in\mathbb{R}^{n+2p}$ into $\mathbb{R}^{n+2p}$ and a neighbourhood $\tilde{\boldsymbol{V}}$ of $\boldsymbol{y}^0$ such that*

$$\boldsymbol{\phi_f}(\boldsymbol{0}) = \boldsymbol{y}^0 \qquad\text{and}\qquad \boldsymbol{\phi_f}(\boldsymbol{\xi}) = \boldsymbol{\Sigma_f}(\boldsymbol{\xi})\cap\tilde{\boldsymbol{V}}\quad\forall\,\boldsymbol{\xi}\in\boldsymbol{W}.$$

Then there exist neighbourhoods $\boldsymbol{U}$ and $\boldsymbol{V}$ of $\boldsymbol{\mathcal{F}}^0$ and $\boldsymbol{y}^0$, respectively, and a single-valued Lipschitz map $\boldsymbol{\sigma_f}\colon \boldsymbol{U}\to\boldsymbol{V}$ with

$$\boldsymbol{\sigma_f}(\boldsymbol{\mathcal{F}}^0) = \boldsymbol{y}^0 \qquad\text{and}\qquad \boldsymbol{\sigma_f}(\boldsymbol{\mathcal{F}}) = \boldsymbol{\mathcal{S}}_{\boldsymbol{f}}^*(\boldsymbol{\mathcal{F}})\cap\boldsymbol{V}\quad\forall\,\boldsymbol{\mathcal{F}}\in\boldsymbol{U}.$$

Let us mention that if $\boldsymbol{Q} \equiv \mathbf{0}$, the single-valuedness of $\boldsymbol{\Sigma_f}$ in a neighbourhood of $\mathbf{0}$ in the assumption of the previous theorem corresponds to the nonsingularity of $\boldsymbol{\nabla_y C_f}(\boldsymbol{\mathcal{F}}^0, \boldsymbol{y}^0)$. Hence Theorem 3.3 is a generalization of the classical implicit function theorem.

Next, we analyse the assumptions of the theorem. Obviously, $\boldsymbol{\Sigma_f}(\boldsymbol{\xi})$ with $\boldsymbol{\xi} := (\boldsymbol{\xi}_u, \boldsymbol{\xi}_\nu, \boldsymbol{\xi}_t) \in \mathbb{R}^{n+2p}$ is the set of all $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ satisfying

$$\begin{cases} \mathbf{0} = \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^\top \boldsymbol{\lambda}_\nu - \boldsymbol{B}_t^\top \boldsymbol{F}^0 \boldsymbol{\lambda}_t - \boldsymbol{f} - \boldsymbol{\xi}_u, \\ \mathbf{0} \in \boldsymbol{B}_\nu \boldsymbol{u} - \boldsymbol{\xi}_\nu + \boldsymbol{N}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu), \\ \mathbf{0} \in \boldsymbol{F}^0 \boldsymbol{B}_t \boldsymbol{u} - \boldsymbol{\xi}_t + \boldsymbol{N}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t), \end{cases} \tag{3.9}$$

where $\boldsymbol{F}^0 := \boldsymbol{F}^0(\boldsymbol{\mathcal{F}}^0) = \operatorname{diag}\{\mathcal{F}_1^0, \ldots, \mathcal{F}_p^0\}$. Substitution

$$\boldsymbol{w} := \boldsymbol{u} - \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_t \end{pmatrix},$$

where $\begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix}^+$ denotes the Moore-Penrose pseudo-inverse of $\begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix}$, leads to the following transformation of (3.9):

$$\begin{cases} \mathbf{0} = \boldsymbol{A}\boldsymbol{w} - \boldsymbol{B}_\nu^\top \boldsymbol{\lambda}_\nu - \boldsymbol{B}_t^\top \boldsymbol{F}^0 \boldsymbol{\lambda}_t - \boldsymbol{f} + \boldsymbol{A} \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_t \end{pmatrix} - \boldsymbol{\xi}_u, \\ \mathbf{0} \in \boldsymbol{B}_\nu \boldsymbol{w} + \boldsymbol{N}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu), \\ \mathbf{0} \in \boldsymbol{F}^0 \boldsymbol{B}_t \boldsymbol{w} + \boldsymbol{N}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t). \end{cases} \tag{3.10}$$

Indeed,

$$\begin{aligned} \begin{pmatrix} \boldsymbol{B}_\nu \boldsymbol{w} \\ \boldsymbol{F}^0 \boldsymbol{B}_t \boldsymbol{w} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix} \boldsymbol{w} = \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix} \boldsymbol{u} - \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_t \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{B}_\nu \boldsymbol{u} - \boldsymbol{\xi}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \boldsymbol{u} - \boldsymbol{\xi}_t \end{pmatrix}. \end{aligned}$$

Comparing this with (3.7), it is readily seen that the triplet $(\boldsymbol{w}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ satisfies (3.10) if and only if it is a solution to the contact problem with Coulomb friction with the coefficient $\boldsymbol{\mathcal{F}}^0$ and the new load vector $\boldsymbol{\xi}_f$,

$$\boldsymbol{\xi}_f := \boldsymbol{f} - \boldsymbol{A} \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0 \boldsymbol{B}_t \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_t \end{pmatrix} + \boldsymbol{\xi}_u,$$

being a perturbation of $\boldsymbol{f}$. That is,

$$(\boldsymbol{w}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \boldsymbol{\mathcal{S}}^*_{\boldsymbol{\xi}_f}(\boldsymbol{\mathcal{F}}^0),$$

where $\boldsymbol{\mathcal{S}}^*_{\boldsymbol{\xi}_f}(\boldsymbol{\mathcal{F}}^0)$ is defined by (3.8) with $\boldsymbol{f} := \boldsymbol{\xi}_f$ and $\boldsymbol{\mathcal{F}} := \boldsymbol{\mathcal{F}}^0$.

To summarize the results we now introduce for a *fixed* $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$ the set-valued mapping $\boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p$ by

$$\boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}) = \{(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)\}, \quad \boldsymbol{f} \in \mathbb{R}^n,$$

where $\{(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)\}$, $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) := (\boldsymbol{u}(\boldsymbol{f}), \boldsymbol{\lambda}_\nu(\boldsymbol{f}), \boldsymbol{\lambda}_t(\boldsymbol{f}))$, denotes the set of all solutions to the contact problem with Coulomb friction with the coefficient $\boldsymbol{\mathcal{F}}$ and the load vector $\boldsymbol{f}$.

Theorem 3.4. *Let us suppose that $\boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}^0}$ has a locally Lipschitz-continuous branch containing $\boldsymbol{y}^0$ in a vicinity of $\boldsymbol{f} \in \mathbb{R}^n$, i.e. there exist a single-valued Lipschitz-continuous function $\varphi_{\boldsymbol{\mathcal{F}}^0}$ from a neighbourhood $\boldsymbol{O}$ of $\boldsymbol{f}$ into $\mathbb{R}^{n+2p}$ and a neighbourhood $\hat{\boldsymbol{V}}$ of $\boldsymbol{y}^0$ such that*

$$\varphi_{\boldsymbol{\mathcal{F}}^0}(\boldsymbol{f}) = \boldsymbol{y}^0 \qquad \text{and} \qquad \varphi_{\boldsymbol{\mathcal{F}}^0}(\boldsymbol{\xi}_f) = \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}^0}(\boldsymbol{\xi}_f) \cap \hat{\boldsymbol{V}} \quad \forall\, \boldsymbol{\xi}_f \in \boldsymbol{O}.$$

Then there are neighbourhoods $\boldsymbol{U}$, $\boldsymbol{V}$ of $\boldsymbol{\mathcal{F}}^0$, $\boldsymbol{y}^0$, respectively, and a single-valued Lipschitz-continuous function $\boldsymbol{\sigma_f} \colon \boldsymbol{U} \to \boldsymbol{V}$ satisfying

$$\boldsymbol{\sigma_f}(\boldsymbol{\mathcal{F}}^0) = \boldsymbol{y}^0 \qquad \text{and} \qquad \boldsymbol{\sigma_f}(\boldsymbol{\mathcal{F}}) = \boldsymbol{S}^*_{\boldsymbol{f}}(\boldsymbol{\mathcal{F}}) \cap \boldsymbol{V} \qquad \forall\, \boldsymbol{\mathcal{F}} \in \boldsymbol{U}.$$

P r o o f. One can easily verify the assumptions of Theorem 3.3 for

$$\phi_{\boldsymbol{f}}(\boldsymbol{\xi}) := \varphi_{\boldsymbol{\mathcal{F}}^0}\Bigl(\boldsymbol{f} - \boldsymbol{A}\begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0\boldsymbol{B}_t \end{pmatrix}^{+}\begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_t \end{pmatrix} + \boldsymbol{\xi}_u\Bigr) + \begin{pmatrix} \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{F}^0\boldsymbol{B}_t \end{pmatrix}^{+}\begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_t \end{pmatrix} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix},$$
$$\boldsymbol{\xi} = (\boldsymbol{\xi}_u, \boldsymbol{\xi}_\nu, \boldsymbol{\xi}_t) \in \boldsymbol{W},$$

with a sufficiently small neighbourhood $\boldsymbol{W}$ of $\boldsymbol{0} \in \mathbb{R}^{n+2p}$. □

The previous theorem says that the analysis of local dependence of a solution to the contact problem with Coulomb friction on the coefficient $\boldsymbol{\mathcal{F}}$ can be converted to the analysis of local dependence of the solution on the load vector $\boldsymbol{f}$. For this reason, we shall focus on the study of the set-valued mapping $\boldsymbol{f} \mapsto \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f})$, $\boldsymbol{f} \in \mathbb{R}^n$, for $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$ *fixed*.

To start with, using the same technique as in the proof of Lemma 3.1 one can get the following auxiliary result.

Lemma 3.2. *Let (2.1) and (2.2) be satisfied and let $\boldsymbol{\mathcal{F}} = (\mathcal{F}_1, \ldots, \mathcal{F}_p) \in \boldsymbol{\mathcal{A}}$, $\boldsymbol{f}, \overline{\boldsymbol{f}} \in \mathbb{R}^n$ and $\boldsymbol{g}, \overline{\boldsymbol{g}} \in \mathbb{R}^p_+$ be arbitrary. Denote the unique solutions of $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, \boldsymbol{g}))$,*

$(\boldsymbol{M}^*(\overline{\boldsymbol{f}}, \boldsymbol{\mathcal{F}}, \overline{\boldsymbol{g}}))$ *by* $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ *and* $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_t)$, *respectively. Then*

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_n \leqslant \frac{1}{\gamma}\|\boldsymbol{f} - \overline{\boldsymbol{f}}\|_n + \frac{\|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\gamma}\|\boldsymbol{g} - \overline{\boldsymbol{g}}\|_p, \tag{3.11}$$

$$\|\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu\|_p \leqslant \frac{1}{\beta}\left(\frac{\|\boldsymbol{A}\|_n}{\gamma} + 1\right)\|\boldsymbol{f} - \overline{\boldsymbol{f}}\|_n + \frac{\|\boldsymbol{A}\|_n\|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\beta\gamma}\|\boldsymbol{g} - \overline{\boldsymbol{g}}\|_p, \tag{3.12}$$

$$\|\boldsymbol{\lambda}_t - \bar{\boldsymbol{\lambda}}_t\|_p \leqslant \frac{1}{\beta\mathcal{F}_{\min}}\left(\frac{\|\boldsymbol{A}\|_n}{\gamma} + 1\right)\|\boldsymbol{f} - \overline{\boldsymbol{f}}\|_n + \frac{\|\boldsymbol{A}\|_n\|\boldsymbol{\mathcal{F}}\|_{p,\infty}}{\beta\gamma\mathcal{F}_{\min}}\|\boldsymbol{g} - \overline{\boldsymbol{g}}\|_p, \tag{3.13}$$

where $\mathcal{F}_{\min} = \min\limits_{i=1,\ldots,p} \mathcal{F}_i$.

Now we shall suppose for a moment that all components of the fixed coefficient $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$ are strictly bounded by $\beta\gamma/\|\boldsymbol{A}\|_n$ from above, i.e. $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{B}}$ with

$$\boldsymbol{\mathcal{B}} := \left\{\boldsymbol{\mathcal{F}} \in \mathbb{R}^p \colon\ 0 < \mathcal{F}_i < \frac{\beta\gamma}{\|\boldsymbol{A}\|_n}\ \forall\, i = 1, \ldots, p\right\}.$$

Then $\boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}$ is single-valued on $\mathbb{R}^n$ for any such $\boldsymbol{\mathcal{F}}$ according to Corollary 3.1. Owing to the previous lemma it can be proved in a way similar to that in Theorem 3.2 that $\boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}$ is even Lipschitz-continuous on $\mathbb{R}^n$.

Theorem 3.5. *Assume that* (2.1) *and* (2.2) *are satisfied and* $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{B}}$ *is arbitrary but fixed. Then there exists* $\delta_{\boldsymbol{\mathcal{F}}} > 0$ *such that*

$$\|\boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}) - \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\overline{\boldsymbol{f}})\|_{n+2p} \leqslant \delta_{\boldsymbol{\mathcal{F}}}\|\boldsymbol{f} - \overline{\boldsymbol{f}}\|_n \quad \forall\, \boldsymbol{f}, \overline{\boldsymbol{f}} \in \mathbb{R}^n.$$

As a consequence of this and Theorem 3.4 we arrive at a result, which is weaker than that of Theorem 3.2.

Corollary 3.2. *Let* (2.1) *and* (2.2) *hold and let* $\boldsymbol{f} \in \mathbb{R}^n$ *be arbitrary but fixed. Then* $\boldsymbol{\mathcal{S}}^*_{\boldsymbol{f}}$ *is locally Lipschitz-continuous in* $\boldsymbol{\mathcal{B}}$, *i.e. for any* $\boldsymbol{\mathcal{F}}^0 \in \boldsymbol{\mathcal{B}}$ *there exist a neighbourhood* $\boldsymbol{U} \subseteq \boldsymbol{\mathcal{B}}$ *of* $\boldsymbol{\mathcal{F}}^0$ *and* $\delta_{\boldsymbol{f}} > 0$ *such that:*

$$\|\boldsymbol{\mathcal{S}}^*_{\boldsymbol{f}}(\boldsymbol{\mathcal{F}}) - \boldsymbol{\mathcal{S}}^*_{\boldsymbol{f}}(\overline{\boldsymbol{\mathcal{F}}})\|_{n+2p} \leqslant \delta_{\boldsymbol{f}}\|\boldsymbol{\mathcal{F}} - \overline{\boldsymbol{\mathcal{F}}}\|_{p,\infty} \quad \forall\, \boldsymbol{\mathcal{F}}, \overline{\boldsymbol{\mathcal{F}}} \in \boldsymbol{U}.$$

In the rest of this section we shall suppose again that $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$, i.e. no upper bounds on $\boldsymbol{\mathcal{F}}$ are imposed. Our aim is to analyse the mapping $\boldsymbol{f} \mapsto \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f})$, $\boldsymbol{f} \in \mathbb{R}^n$, for such $\boldsymbol{\mathcal{F}}$ fixed with the aid of the implicit function theorem for piecewise-differentiable functions presented in [16] (see also Appendix).

First, we shall formulate the discrete contact problem with Coulomb friction as a system of non-smooth equations. Let $r > 0$ be an arbitrary parameter and let

$\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$ be fixed. If $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f})$, i.e. $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)$ solves $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu))$, the inequality $(\boldsymbol{M}^*(\boldsymbol{f}, \boldsymbol{\mathcal{F}}, -\boldsymbol{\lambda}_\nu))_3$ multiplied by $(-r)$ gives

$$\tag{3.14} \begin{cases} (\mu_{\nu,i} - \lambda_{\nu,i})((\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i - \lambda_{\nu,i}) \leqslant 0, & i = 1, \ldots, p,\ \forall\, \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu, \\ (\mu_{t,i} - \lambda_{t,i})((\boldsymbol{\lambda}_t - r\boldsymbol{B}_t \boldsymbol{u})_i - \lambda_{t,i}) \leqslant 0, & i = 1, \ldots, p,\ \forall\, \boldsymbol{\mu}_t \in \boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu). \end{cases}$$

Since $\boldsymbol{\lambda}_\nu \in \boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\lambda}_t \in \boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)$, the equivalent expression of (3.14) is

$$\boldsymbol{\lambda}_\nu = \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u}), \quad \boldsymbol{\lambda}_t = \boldsymbol{P}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t \boldsymbol{u}).$$

Here $\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}\colon\ \mathbb{R}^p \to \boldsymbol{\Lambda}_\nu$ and $\boldsymbol{P}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}\colon\ \mathbb{R}^p \to \mathbb{R}^p$ are vector functions with the components

$$(\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu})_i(\boldsymbol{\mu}) = P_{(-\infty,0]}(\mu_i), \quad i = 1, \ldots, p,\ \boldsymbol{\mu} \in \mathbb{R}^p,$$
$$(\boldsymbol{P}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)})_i(\boldsymbol{\mu}) = \begin{cases} P_{[\lambda_{\nu,i}, -\lambda_{\nu,i}]}(\mu_i) & \text{if } \lambda_{\nu,i} \leqslant 0, \\ -P_{[-\lambda_{\nu,i}, \lambda_{\nu,i}]}(\mu_i) & \text{if } \lambda_{\nu,i} > 0, \end{cases} \qquad i = 1, \ldots, p,\ \boldsymbol{\mu} \in \mathbb{R}^p,$$

where $P_{(-\infty,0]}$, $P_{[a,b]}$ stand for the projections of $\mathbb{R}^1$ onto $(-\infty, 0]$ and $[a, b]$, $-\infty < a \leqslant b < \infty$, respectively. It is readily seen that $\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}$ is the projection of $\mathbb{R}^p$ onto $\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{P}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}$ is the projection of $\mathbb{R}^p$ onto $\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)$ whenever $\boldsymbol{\lambda}_\nu \in \boldsymbol{\Lambda}_\nu$.

Let $\boldsymbol{\mathcal{H}}^*\colon\ \mathbb{R}^n \times \mathbb{R}^{n+2p} \to \mathbb{R}^{n+2p}$ be defined by

$$\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}, \boldsymbol{y}) = \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^\top \boldsymbol{\lambda}_\nu - \boldsymbol{B}_t^\top \boldsymbol{F} \boldsymbol{\lambda}_t - \boldsymbol{f} \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u}) \\ \boldsymbol{\lambda}_t - \boldsymbol{P}_{\boldsymbol{\Lambda}_t(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t \boldsymbol{u}) \end{pmatrix}, \quad \boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \mathbb{R}^{n+2p}.$$

Then $\boldsymbol{y} \in \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f})$, $\boldsymbol{f} \in \mathbb{R}^n$, if and only if $\boldsymbol{y}$ solves the following problem:

$$\tag{3.15} \text{Find } \boldsymbol{y} \in \mathbb{R}^{n+2p} \text{ such that } \boldsymbol{\mathcal{H}}^*(\boldsymbol{f}, \boldsymbol{y}) = \boldsymbol{0}.$$

We shall view this problem as an equation parametrized by $\boldsymbol{f}$.

Below we shall show that $\boldsymbol{\mathcal{H}}^*$ is a piecewise-differentiable function. Obviously, it is continuous. Moreover, let $(\boldsymbol{f}^0, \boldsymbol{y}^0) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}$, $\boldsymbol{y}^0 \equiv (\boldsymbol{u}^0, \boldsymbol{\lambda}^0_\nu, \boldsymbol{\lambda}^0_t)$, be an arbitrarily chosen vector. To construct a set of selection functions for $\boldsymbol{\mathcal{H}}^*$ at $(\boldsymbol{f}^0, \boldsymbol{y}^0)$

we introduce in a way similar to [2] the following index sets (see Fig. 1):

$$\begin{aligned}
I_\nu^{\rm s}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ (\boldsymbol{\lambda}_\nu^0 - r\boldsymbol{B}_\nu\boldsymbol{u}^0)_i < 0\},\\
I_\nu^{0}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ (\boldsymbol{\lambda}_\nu^0 - r\boldsymbol{B}_\nu\boldsymbol{u}^0)_i > 0\},\\
I_\nu^{w}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ (\boldsymbol{\lambda}_\nu^0 - r\boldsymbol{B}_\nu\boldsymbol{u}^0)_i = 0\},\\
I_t^{+}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ (\boldsymbol{\lambda}_t^0 - r\boldsymbol{B}_t\boldsymbol{u}^0)_i < -|\lambda_{\nu,i}^0|\},\\
I_t^{-}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ (\boldsymbol{\lambda}_t^0 - r\boldsymbol{B}_t\boldsymbol{u}^0)_i > |\lambda_{\nu,i}^0|\},\\
I_t^{\rm s}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ |(\boldsymbol{\lambda}_t^0 - r\boldsymbol{B}_t\boldsymbol{u}^0)_i| < |\lambda_{\nu,i}^0|\},\\
I_t^{w+}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ (\boldsymbol{\lambda}_t^0 - r\boldsymbol{B}_t\boldsymbol{u}^0)_i = \lambda_{\nu,i}^0\},\\
I_t^{w-}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ (\boldsymbol{\lambda}_t^0 - r\boldsymbol{B}_t\boldsymbol{u}^0)_i = -\lambda_{\nu,i}^0\},\\
J^{-}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ \lambda_{\nu,i}^0 < 0\},\\
J^{0}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ \lambda_{\nu,i}^0 = 0\},\\
J^{+}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,p\}\colon\ \lambda_{\nu,i}^0 > 0\}.
\end{aligned}$$

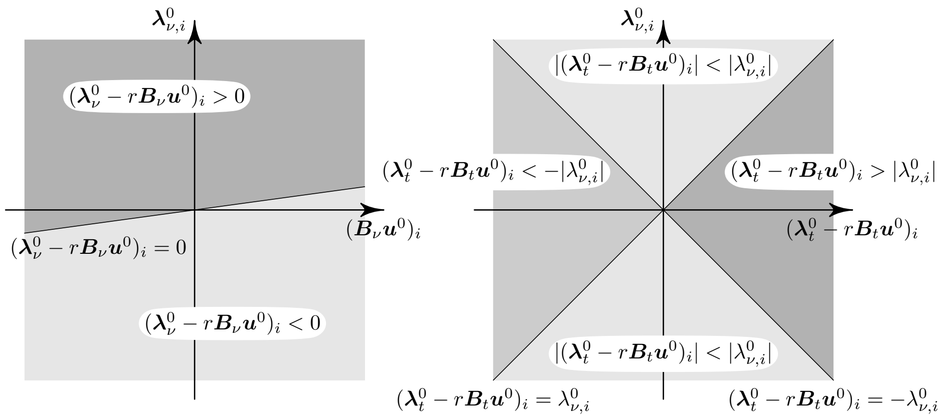


Figure 1. Partitions corresponding to the index sets.

R e m a r k 3.1. To interpret the sets defined above, suppose for a moment that $\boldsymbol{y}^0 \in \boldsymbol{S}_{\boldsymbol{\mathcal{F}}}^*(\boldsymbol{f}^0)$. Then

$$\begin{aligned}
i \in I_\nu^{\rm s}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\nu\boldsymbol{u}^0)_i = 0 \ \&\ \lambda_{\nu,i}^0 < 0 \quad \text{(strong contact)},\\
i \in I_\nu^{0}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\nu\boldsymbol{u}^0)_i < 0 \ \&\ \lambda_{\nu,i}^0 = 0 \quad \text{(no contact)},\\
i \in I_\nu^{w}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\nu\boldsymbol{u}^0)_i = \lambda_{\nu,i}^0 = 0 \qquad\ \ \text{(weak contact)}.
\end{aligned}$$

Analogously,

$$\left.\begin{aligned}
i \in I_t^{+}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_t\boldsymbol{u}^0)_i > 0 \ \&\ \lambda_{t,i}^0 = \lambda_{\nu,i}^0\\
i \in I_t^{-}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_t\boldsymbol{u}^0)_i < 0 \ \&\ \lambda_{t,i}^0 = -\lambda_{\nu,i}^0
\end{aligned}\right\} \quad \text{(slip)},$$

$$
\begin{aligned}
i \in I_t^{\mathrm{s}}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_t\boldsymbol{u}^0)_i = 0 \ \&\ |\lambda_{t,i}^0| < -\lambda_{\nu,i}^0 \qquad \text{(strong stick)},\\
i \in I_t^{w+}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_t\boldsymbol{u}^0)_i = 0 \ \&\ \lambda_{t,i}^0 = \lambda_{\nu,i}^0 \\
i \in I_t^{w-}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_t\boldsymbol{u}^0)_i = 0 \ \&\ \lambda_{t,i}^0 = -\lambda_{\nu,i}^0
\end{aligned}
\Bigg\} \quad \text{(weak stick)}.
$$

Let $I_\nu^{w-} \subseteq I_\nu^w(\boldsymbol{y}^0)$, $I_t^{w++} \subseteq I_t^{w+}(\boldsymbol{y}^0)$ and $I_t^{w--} \subseteq I_t^{w-}(\boldsymbol{y}^0)$ be arbitrary sets. For such sets we shall denote

$$
I_\nu^{w+} := I_\nu^w(\boldsymbol{y}^0) \setminus I_\nu^{w-}, \quad I_t^{w+-} := I_t^{w+}(\boldsymbol{y}^0) \setminus I_t^{w++}, \quad I_t^{w-+} := I_t^{w-}(\boldsymbol{y}^0) \setminus I_t^{w--}
$$

here and in what follows. These index sets will be associated with the set

$$
\begin{aligned}
(3.16)\quad \boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} = \{&(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}\colon\\
&(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i \leqslant 0 \ \forall\, i \in I_\nu^{w-},\ (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i \geqslant 0 \ \forall\, i \in I_\nu^{w+},\\
&(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \geqslant \lambda_{\nu,i} \ \forall\, i \in I_t^{w++},\ (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \leqslant \lambda_{\nu,i} \ \forall\, i \in I_t^{w+-},\\
&(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \leqslant -\lambda_{\nu,i} \ \forall\, i \in I_t^{w--},\ (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \geqslant -\lambda_{\nu,i} \ \forall\, i \in I_t^{w-+}\}
\end{aligned}
$$

and the function $\boldsymbol{\mathcal{H}}^{*(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}\colon \mathbb{R}^n \times \mathbb{R}^{n+2p} \to \mathbb{R}^{n+2p}$ whose components are defined by

$$
\begin{aligned}
(3.17)\quad &\mathcal{H}_i^{*(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}(\boldsymbol{f}, \boldsymbol{y}) = (\boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^\top\boldsymbol{\lambda}_\nu - \boldsymbol{B}_t^\top\boldsymbol{F}\boldsymbol{\lambda}_t - \boldsymbol{f})_i,\\
&\qquad i = 1, \ldots, n,\ (\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p},\\
&\mathcal{H}_{n+i}^{*(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}(\boldsymbol{f}, \boldsymbol{y}) = \begin{cases} r(\boldsymbol{B}_\nu\boldsymbol{u})_i & \text{if } i \in I_\nu^{\mathrm{s}}(\boldsymbol{y}^0) \cup I_\nu^{w-},\\ \lambda_{\nu,i} & \text{if } i \in I_\nu^0(\boldsymbol{y}^0) \cup I_\nu^{w+}, \end{cases}\\
&\qquad i = 1, \ldots, p,\ (\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p},\\
&\mathcal{H}_{n+p+i}^{*(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}(\boldsymbol{f}, \boldsymbol{y})\\
&= \begin{cases}
r(\boldsymbol{B}_t\boldsymbol{u})_i & \text{if } i \in \big((I_t^{\mathrm{s}}(\boldsymbol{y}^0) \cup I_t^{w++} \cup I_t^{w--}) \cap J^-(\boldsymbol{y}^0)\big)\\
& \quad \cup (I_t^{w++} \cap I_t^{w--} \cap J^0(\boldsymbol{y}^0)),\\
(2\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i & \text{if } i \in \big((I_t^{\mathrm{s}}(\boldsymbol{y}^0) \cup I_t^{w+-} \cup I_t^{w-+}) \cap J^+(\boldsymbol{y}^0)\big)\\
& \quad \cup (I_t^{w+-} \cap I_t^{w-+} \cap J^0(\boldsymbol{y}^0)),\\
(\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_\nu)_i & \text{if } i \in I_t^+(\boldsymbol{y}^0) \cup (I_t^{w+-} \cap J^-(\boldsymbol{y}^0)) \cup (I_t^{w--} \cap J^+(\boldsymbol{y}^0))\\
& \quad \cup (I_t^{w+-} \cap I_t^{w--} \cap J^0(\boldsymbol{y}^0)),\\
(\boldsymbol{\lambda}_t + \boldsymbol{\lambda}_\nu)_i & \text{if } i \in I_t^-(\boldsymbol{y}^0) \cup (I_t^{w-+} \cap J^-(\boldsymbol{y}^0)) \cup (I_t^{w++} \cap J^+(\boldsymbol{y}^0))\\
& \quad \cup (I_t^{w-+} \cap I_t^{w++} \cap J^0(\boldsymbol{y}^0)),
\end{cases}\\
&\qquad i = 1, \ldots, p,\ (\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}.
\end{aligned}
$$

Then one can easily verify that there exists a neighbourhood $\boldsymbol{W}$ of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$ such that

$$\begin{aligned}\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}, \boldsymbol{y}) = \boldsymbol{\mathcal{H}}^{*(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}(\boldsymbol{f}, \boldsymbol{y}) \\ \forall (\boldsymbol{f}, \boldsymbol{y}) \in \boldsymbol{W} \cap (\{(\boldsymbol{f}^0, \boldsymbol{y}^0)\} + \boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}).\end{aligned}$$

Now consider all possible combinations of $I_\nu^{w-} \subseteq I_\nu^w(\boldsymbol{y}^0)$, $I_t^{w++} \subseteq I_t^{w+}(\boldsymbol{y}^0)$ and $I_t^{w--} \subseteq I_t^{w-}(\boldsymbol{y}^0)$ and denote their total number by l. One obtains the collections $\boldsymbol{\Pi}$ and $\{\boldsymbol{\mathcal{H}}^{*(1)}, \ldots, \boldsymbol{\mathcal{H}}^{*(l)}\}$ of subsets of $\mathbb{R}^n \times \mathbb{R}^{n+2p}$ and functions from $\mathbb{R}^n \times \mathbb{R}^{n+2p}$ into $\mathbb{R}^{n+2p}$, respectively:

$$\begin{aligned}&\forall \boldsymbol{\pi} \in \boldsymbol{\Pi}\ \exists I_\nu^{w-} \subseteq I_\nu^w(\boldsymbol{y}^0),\ I_t^{w++} \subseteq I_t^{w+}(\boldsymbol{y}^0),\ I_t^{w--} \subseteq I_t^{w-}(\boldsymbol{y}^0): \\ &\qquad \boldsymbol{\pi} = \boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}, \\ &\forall j \in \{1, \ldots, l\}\ \exists I_\nu^{w-} \subseteq I_\nu^w(\boldsymbol{y}^0),\ I_t^{w++} \subseteq I_t^{w+}(\boldsymbol{y}^0),\ I_t^{w--} \subseteq I_t^{w-}(\boldsymbol{y}^0): \\ &\qquad \boldsymbol{\mathcal{H}}^{*(j)} = \boldsymbol{\mathcal{H}}^{*(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \quad \text{in } \mathbb{R}^n \times \mathbb{R}^{n+2p}.\end{aligned} \tag{3.18}$$

From the construction it immediately follows that there exists a neighbourhood $\boldsymbol{W}$ of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$ such that:

$$\begin{aligned}&\forall \boldsymbol{\pi} \in \boldsymbol{\Pi}\ \exists j_{\boldsymbol{\pi}} \in \{1, \ldots, l\}: \\ &\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}, \boldsymbol{y}) = \boldsymbol{\mathcal{H}}^{*(j_{\boldsymbol{\pi}})}(\boldsymbol{f}, \boldsymbol{y}) \quad \forall (\boldsymbol{f}, \boldsymbol{y}) \in \boldsymbol{W} \cap (\{(\boldsymbol{f}^0, \boldsymbol{y}^0)\} + \boldsymbol{\pi}).\end{aligned} \tag{3.19}$$

This implies that $\boldsymbol{\mathcal{H}}^*$ is a continuous selection of $\boldsymbol{\mathcal{H}}^{*(1)}, \ldots, \boldsymbol{\mathcal{H}}^{*(l)}$ and consequently a piecewise-differentiable function in a sufficiently small neighbourhood of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$. Let us note that if $\boldsymbol{y}^0$ is such that $I_\nu^w(\boldsymbol{y}^0)$, $I_t^{w+}(\boldsymbol{y}^0)$ as well as $I_t^{w-}(\boldsymbol{y}^0)$ are empty sets then $l = 1$, $\boldsymbol{\Pi} = \{\mathbb{R}^n \times \mathbb{R}^{n+2p}\}$ and $\boldsymbol{\mathcal{H}}^{*(1)} = \boldsymbol{\mathcal{H}}^*$ in a neighbourhood of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$, i.e. $\boldsymbol{\mathcal{H}}^*$ is even differentiable therein. Otherwise, we claim that $\boldsymbol{\Pi}$ is a conical subdivision of $\mathbb{R}^n \times \mathbb{R}^{n+2p}$.

Indeed, let $\boldsymbol{\pi} = \boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \in \boldsymbol{\Pi}$ be given. Introduce functions $\Theta_1^{(I_\nu^w(\boldsymbol{y}^0))}\colon \{1, \ldots, |I_\nu^w(\boldsymbol{y}^0)|\} \to I_\nu^w(\boldsymbol{y}^0)$, $\Theta_2^{(I_t^{w+}(\boldsymbol{y}^0))}\colon \{1, \ldots, |I_t^{w+}(\boldsymbol{y}^0)|\} \to I_t^{w+}(\boldsymbol{y}^0)$ and $\Theta_3^{(I_t^{w-}(\boldsymbol{y}^0))}\colon \{1, \ldots, |I_t^{w-}(\boldsymbol{y}^0)|\} \to I_t^{w-}(\boldsymbol{y}^0)$ such that

$$\begin{aligned}\forall i \in I_\nu^w(\boldsymbol{y}^0)\ \exists j \in \{1, \ldots, |I_\nu^w(\boldsymbol{y}^0)|\}:\ \Theta_1^{(I_\nu^w(\boldsymbol{y}^0))}(j) = i, \\ \forall i \in I_t^{w+}(\boldsymbol{y}^0)\ \exists j \in \{1, \ldots, |I_t^{w+}(\boldsymbol{y}^0)|\}:\ \Theta_2^{(I_t^{w+}(\boldsymbol{y}^0))}(j) = i, \\ \forall i \in I_t^{w-}(\boldsymbol{y}^0)\ \exists j \in \{1, \ldots, |I_t^{w-}(\boldsymbol{y}^0)|\}:\ \Theta_3^{(I_t^{w-}(\boldsymbol{y}^0))}(j) = i,\end{aligned}$$

where $|K|$ stands for the cardinality of a set K. With the aid of these functions define the matrix $\boldsymbol{B}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \in \mathbb{R}^{(|I_\nu^w(\boldsymbol{y}^0)|+|I_t^{w+}(\boldsymbol{y}^0)|+|I_t^{w-}(\boldsymbol{y}^0)|)\times(2n+2p)}$ by

$$
\begin{aligned}
\boldsymbol{B}_j^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} &= \begin{cases} \big(\boldsymbol{0},\, (-r\boldsymbol{B}_\nu)_i,\, (\boldsymbol{I}_p)_i,\, \boldsymbol{0}\big) & \text{if } i \in I_\nu^{w-}, \\ \big(\boldsymbol{0},\, (r\boldsymbol{B}_\nu)_i,\, (-\boldsymbol{I}_p)_i,\, \boldsymbol{0}\big) & \text{if } i \in I_\nu^{w+}, \end{cases} \\
& \qquad j = 1, \ldots, |I_\nu^w(\boldsymbol{y}^0)|,\ i = \Theta_1^{(I_\nu^w(\boldsymbol{y}^0))}(j), \\
\boldsymbol{B}_j^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} &= \begin{cases} \big(\boldsymbol{0},\, (r\boldsymbol{B}_t)_i,\, (\boldsymbol{I}_p)_i,\, (-\boldsymbol{I}_p)_i\big) & \text{if } i \in I_t^{w++}, \\ \big(\boldsymbol{0},\, (-r\boldsymbol{B}_t)_i,\, (-\boldsymbol{I}_p)_i,\, (\boldsymbol{I}_p)_i\big) & \text{if } i \in I_t^{w+-}, \end{cases} \\
& \qquad j = |I_\nu^w(\boldsymbol{y}^0)| + 1, \ldots, |I_\nu^w(\boldsymbol{y}^0)| + |I_t^{w+}(\boldsymbol{y}^0)|,\ i = \Theta_2^{(I_t^{w+}(\boldsymbol{y}^0))}(j - |I_\nu^w(\boldsymbol{y}^0)|), \\
\boldsymbol{B}_j^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} &= \begin{cases} \big(\boldsymbol{0},\, (-r\boldsymbol{B}_t)_i,\, (\boldsymbol{I}_p)_i,\, (\boldsymbol{I}_p)_i\big) & \text{if } i \in I_t^{w--}, \\ \big(\boldsymbol{0},\, (r\boldsymbol{B}_t)_i,\, (-\boldsymbol{I}_p)_i,\, (-\boldsymbol{I}_p)_i\big) & \text{if } i \in I_t^{w-+}, \end{cases} \\
& \qquad j = |I_\nu^w(\boldsymbol{y}^0)| + |I_t^{w+}(\boldsymbol{y}^0)| + 1, \ldots, |I_\nu^w(\boldsymbol{y}^0)| + |I_t^{w+}(\boldsymbol{y}^0)| + |I_t^{w-}(\boldsymbol{y}^0)|, \\
& \qquad i = \Theta_3^{(I_t^{w-}(\boldsymbol{y}^0))}(j - |I_\nu^w(\boldsymbol{y}^0)| - |I_t^{w+}(\boldsymbol{y}^0)|).
\end{aligned}
$$

Here $\boldsymbol{C}_i$ denotes the ith row vector of a matrix $\boldsymbol{C}$ and $\boldsymbol{I}_p$ represents the identity matrix of order p. Then we have

$$
\boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} = \left\{(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p} \colon \boldsymbol{B}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{y} \end{pmatrix} \leqslant \boldsymbol{0}\right\}, \tag{3.20}
$$

which shows that $\boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}$ is a polyhedral cone with vertex at $\boldsymbol{0}$.

By the assumption (2.2), $\boldsymbol{B}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}$ is a full-row-rank matrix and one can find a vector $(\overline{\boldsymbol{f}}, \overline{\boldsymbol{y}}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}$ with

$$
\boldsymbol{B}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \begin{pmatrix} \overline{\boldsymbol{f}} \\ \overline{\boldsymbol{y}} \end{pmatrix} < \boldsymbol{0}.
$$

Hence the dimension of the linear hull of $\boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}$ equals $(2n + 2p)$. The union of all cones in $\boldsymbol{\Pi}$ covers $\mathbb{R}^n \times \mathbb{R}^{n+2p}$ as we consider all possible choices of I_ν^{w-}, I_t^{w++} and I_t^{w--}. Finally, the intersection of any two cones $\boldsymbol{\pi} = \boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}, \tilde{\boldsymbol{\pi}} = \tilde{\boldsymbol{\pi}}^{(\tilde{I}_\nu^{w-}, \tilde{I}_t^{w++}, \tilde{I}_t^{w--})} \in \boldsymbol{\Pi}$ takes the form

$$
\begin{aligned}
&\boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \cap \tilde{\boldsymbol{\pi}}^{(\tilde{I}_\nu^{w-}, \tilde{I}_t^{w++}, \tilde{I}_t^{w--})} = \{(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p} \colon \\
&\quad (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i = 0\ \forall\, i \in (I_\nu^{w-} \cap \tilde{I}_\nu^{w+}) \cup (I_\nu^{w+} \cap \tilde{I}_\nu^{w-}), \\
&\quad (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i \leqslant 0\ \forall\, i \in I_\nu^{w-} \cap \tilde{I}_\nu^{w-},\ (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i \geqslant 0\ \forall\, i \in I_\nu^{w+} \cap \tilde{I}_\nu^{w+},
\end{aligned}
$$

$$
\begin{aligned}
&(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i = \lambda_{\nu,i} \ \forall\, i \in (I_t^{w++} \cap \tilde{I}_t^{w+-}) \cup (I_t^{w+-} \cap \tilde{I}_t^{w++}),\\
&(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \geqslant \lambda_{\nu,i} \ \forall\, i \in I_t^{w++} \cap \tilde{I}_t^{w++},\ (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \leqslant \lambda_{\nu,i} \ \forall\, i \in I_t^{w+-} \cap \tilde{I}_t^{w+-},\\
&(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i = -\lambda_{\nu,i} \ \forall\, i \in (I_t^{w--} \cap \tilde{I}_t^{w-+}) \cup (I_t^{w-+} \cap \tilde{I}_t^{w--}),\\
&(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \leqslant -\lambda_{\nu,i} \ \forall\, i \in I_t^{w--} \cap \tilde{I}_t^{w--},\\
&(\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \geqslant -\lambda_{\nu,i} \ \forall\, i \in I_t^{w-+} \cap \tilde{I}_t^{w-+}\}.
\end{aligned}
$$

Whenever $\boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}$ and $\tilde{\boldsymbol{\pi}}^{(\tilde{I}_\nu^{w-}, \tilde{I}_t^{w++}, \tilde{I}_t^{w--})}$ are distinct, at least one of the sets I_ν^{w-}, I_t^{w++} or I_t^{w--} does not coincide with $\tilde{I}_\nu^{w-}$, $\tilde{I}_t^{w++}$, $\tilde{I}_t^{w--}$, respectively, and the set above forms a common proper face of both cones.

Next, let q denote the dimension of the lineality space of $\boldsymbol{\Pi}$. According to the assumptions of the previously mentioned implicit function theorem for piecewise-differentiable equations, either $(2n + 2p - q) \leqslant 1$ needs to be satisfied or there has to exist a number $m \in \{2, \ldots, (2n + 2p - q)\}$ such that the mth branching number of $\boldsymbol{\Pi}$ does not exceed $2m$.

The lineality space of any cone $\boldsymbol{\pi} = \boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \in \boldsymbol{\Pi}$ is the subspace

$$
\left\{(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p} :\ \boldsymbol{B}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{y} \end{pmatrix} = \boldsymbol{0}\right\}
$$

with an appropriate $\boldsymbol{B}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})} \in \mathbb{R}^{(|I_\nu^{w}(\boldsymbol{y}^0)| + |I_t^{w+}(\boldsymbol{y}^0)| + |I_t^{w-}(\boldsymbol{y}^0)|) \times (2n+2p)}$ (cf. (3.20)). The full row rank of any $\boldsymbol{B}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}$ under consideration guaranteed by (2.2) yields that the dimension of the lineality space of $\boldsymbol{\pi}^{(I_\nu^{w-}, I_t^{w++}, I_t^{w--})}$ (and of $\boldsymbol{\Pi}$, as well) is equal to $(2n+2p-(|I_\nu^{w}(\boldsymbol{y}^0)|+|I_t^{w+}(\boldsymbol{y}^0)|+|I_t^{w-}(\boldsymbol{y}^0)|))$. Consequently, the condition $(2n+2p-q) \leqslant 1$ is equivalent to $|I_\nu^{w}(\boldsymbol{y}^0)|+|I_t^{w+}(\boldsymbol{y}^0)|+|I_t^{w-}(\boldsymbol{y}^0)| \leqslant 1$. If it is not satisfied, we assert that the other condition holds with $m = 2$. Indeed, the 2nd branching number of $\boldsymbol{\Pi}$ is the maximal number of cones in $\boldsymbol{\Pi}$ containing a common face of dimension $(2n + 2p - 2)$. Having in mind (2.2), each such face can be written as

$$
\begin{aligned}
&\{(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}:\\
&\quad (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i = 0 \ \forall\, i \in I_\nu^{w0},\ (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i \leqslant 0 \ \forall\, i \in I_\nu^{w-} \setminus I_\nu^{w0},\\
&\quad (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i \geqslant 0 \ \forall\, i \in I_\nu^{w+} \setminus I_\nu^{w0},\ (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i = \lambda_{\nu,i} \ \forall\, i \in I_t^{w+0},\\
&\quad (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \geqslant \lambda_{\nu,i} \ \forall\, i \in I_t^{w++} \setminus I_t^{w+0},\ (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \leqslant \lambda_{\nu,i} \ \forall\, i \in I_t^{w+-} \setminus I_t^{w+0},\\
&\quad (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i = -\lambda_{\nu,i} \ \forall\, i \in I_t^{w-0},\ (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \leqslant -\lambda_{\nu,i} \ \forall\, i \in I_t^{w--} \setminus I_t^{w-0},\\
&\quad (\boldsymbol{\lambda}_t - r\boldsymbol{B}_t\boldsymbol{u})_i \geqslant -\lambda_{\nu,i} \ \forall\, i \in I_t^{w-+} \setminus I_t^{w-0}\}
\end{aligned}
$$

for some $I_\nu^{w-}, I_\nu^{w0} \subseteq I_\nu^{w}(\boldsymbol{y}^0)$, $I_t^{w++}, I_t^{w+0} \subseteq I_t^{w+}(\boldsymbol{y}^0)$ and $I_t^{w--}, I_t^{w-0} \subseteq I_t^{w-}(\boldsymbol{y}^0)$ with $|I_\nu^{w0}| + |I_t^{w+0}| + |I_t^{w-0}| = 2$. From this it easily follows that the 2nd branching number of $\boldsymbol{\Pi}$ is equal to 4.

To conclude, the following two theorems are valid (cf. Theorem 4.2.2 and Proposition 4.2.2 in [16]).

Theorem 3.6. *Let* $(\boldsymbol{f}^0, \boldsymbol{y}^0) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}$ *be a vector with* $\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}^0, \boldsymbol{y}^0) = \mathbf{0}$. *If all matrices* $\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{*(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0)$, $j = 1, \ldots, l$, *where* $\boldsymbol{\mathcal{H}}^{*(j)}$ *are given by* (3.18) *have the same nonvanishing determinant sign then*

1. *the equation* $\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}, \boldsymbol{y}) = \mathbf{0}$ *determines an implicit* PC^1*-function at* $(\boldsymbol{f}^0, \boldsymbol{y}^0)$, *i.e. there exist neighbourhoods* $\boldsymbol{O}$, $\hat{\boldsymbol{V}}$ *of* $\boldsymbol{f}^0$, $\boldsymbol{y}^0$, *respectively, and a* PC^1*-function* $\boldsymbol{\varphi}_{\boldsymbol{\mathcal{F}}} \colon \boldsymbol{O} \to \hat{\boldsymbol{V}}$ *such that*
$$\boldsymbol{\varphi}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}^0) = \boldsymbol{y}^0 \qquad \text{and} \qquad \boldsymbol{\varphi}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}) = \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}) \cap \hat{\boldsymbol{V}} \quad \forall\, \boldsymbol{f} \in \boldsymbol{O};$$
2. *the implicit functions determined by the equations* $\boldsymbol{\mathcal{H}}^{*(j)}(\boldsymbol{f}, \boldsymbol{y}) = \mathbf{0}$, $j = 1, \ldots, l$, *form a collection of selection functions for the* PC^1*-function* $\boldsymbol{\varphi}_{\boldsymbol{\mathcal{F}}}$ *at* $\boldsymbol{f}^0$;
3. *for every* $\boldsymbol{h} \in \mathbb{R}^n$ *the identity* $\boldsymbol{\xi} = \boldsymbol{\varphi}'_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}^0; \boldsymbol{h})$ *holds if and only if* $\boldsymbol{\xi}$ *satisfies the piecewise-linear equation* $\boldsymbol{\mathcal{H}}^{*\prime}((\boldsymbol{f}^0, \boldsymbol{y}^0); (\boldsymbol{h}, \boldsymbol{\xi})) = \mathbf{0}$.

Theorem 3.7. *Suppose that the assumptions of the previous theorem are satisfied and* $\boldsymbol{h} \in \mathbb{R}^n$ *is arbitrary.*

1. *Then there exists a cone* $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ *such that*
$$\begin{pmatrix} \boldsymbol{h} \\ \mathbf{0} \end{pmatrix} \in \begin{pmatrix} \boldsymbol{I}_n & \mathbf{0}_{n,(n+2p)} \\ \boldsymbol{\nabla}_{\boldsymbol{f}} \boldsymbol{\mathcal{H}}^{*(j_{\boldsymbol{\pi}})}(\boldsymbol{f}^0, \boldsymbol{y}^0) & \boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{*(j_{\boldsymbol{\pi}})}(\boldsymbol{f}^0, \boldsymbol{y}^0) \end{pmatrix} \boldsymbol{\pi} \tag{3.21}$$
with $j_{\boldsymbol{\pi}}$ *being given by* (3.19).
2. *The inclusion* (3.21) *holds if and only if*
$$\begin{pmatrix} \boldsymbol{h} \\ -\big(\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{*(j_{\boldsymbol{\pi}})}(\boldsymbol{f}^0, \boldsymbol{y}^0)\big)^{-1} \boldsymbol{\nabla}_{\boldsymbol{f}} \boldsymbol{\mathcal{H}}^{*(j_{\boldsymbol{\pi}})}(\boldsymbol{f}^0, \boldsymbol{y}^0) \boldsymbol{h} \end{pmatrix} \in \boldsymbol{\pi}.$$
3. *If* $\boldsymbol{h}$ *satisfies* (3.21) *then*
$$\boldsymbol{\varphi}'_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}^0; \boldsymbol{h}) = -\big(\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{*(j_{\boldsymbol{\pi}})}(\boldsymbol{f}^0, \boldsymbol{y}^0)\big)^{-1} \boldsymbol{\nabla}_{\boldsymbol{f}} \boldsymbol{\mathcal{H}}^{*(j_{\boldsymbol{\pi}})}(\boldsymbol{f}^0, \boldsymbol{y}^0) \boldsymbol{h},$$
where $\boldsymbol{\varphi}_{\boldsymbol{\mathcal{F}}}$ *is the implicit* PC^1*-function determined by the equation* $\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}, \boldsymbol{y}) = \mathbf{0}$ *at* $(\boldsymbol{f}^0, \boldsymbol{y}^0)$.

Applying Corollary 4.1.1 in [16], which tells us that every piecewise-differentiable function is locally Lipschitz-continuous, we get the following consequence of Theorems 3.4 and 3.6.

Corollary 3.3. *If $\boldsymbol{\mathcal{F}} \in \boldsymbol{\mathcal{A}}$ and $(\boldsymbol{f}^0, \boldsymbol{y}^0) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}$ are such that the assumptions of Theorem 3.6 are fulfilled then there are neighbourhoods $\boldsymbol{U}, \boldsymbol{V}$ of $\boldsymbol{\mathcal{F}}$, $\boldsymbol{y}^0$, respectively, and a single-valued Lipschitz-continuous function $\boldsymbol{\sigma}_{\boldsymbol{f}^0}\colon \boldsymbol{U} \to \boldsymbol{V}$ satisfying*

$$\boldsymbol{\sigma}_{\boldsymbol{f}^0}(\boldsymbol{\mathcal{F}}) = \boldsymbol{y}^0 \qquad \textit{and} \qquad \boldsymbol{\sigma}_{\boldsymbol{f}^0}(\boldsymbol{\xi}_{\mathcal{F}}) = \boldsymbol{\mathcal{S}}^*_{\boldsymbol{f}^0}(\boldsymbol{\xi}_{\mathcal{F}}) \cap \boldsymbol{V} \quad \forall \boldsymbol{\xi}_{\mathcal{F}} \in \boldsymbol{U}.$$

Let us note that the assertion of the corollary is close to Theorem 1 in [11], which concerns discrete contact problems with Coulomb friction and a coefficient of friction represented by one real. However, the latter result was obtained from the version of the implicit function theorem involving Clarke's gradient and one has to deal with generally infinite number of matrices included in the respective generalized Jacobian to verify its assumptions.

At the end of this section we shall analyse the cases when the assumption concerning the determinant signs in Theorem 3.6 is not satisfied.

1. There exists an index $j \in \{1, \dots, l\}$ such that

$$\boldsymbol{\mathcal{H}}^{*(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0) = \boldsymbol{0}, \tag{3.22}$$

$$\operatorname{rank}\bigl(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{*(j)}\bigr) = n + 2p - s, \quad s > 0. \tag{3.23}$$

Here we denote $\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{*(j)} := \boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{*(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0)$ because $\boldsymbol{\mathcal{H}}^{*(j)}$, $j = 1, \dots, l$, are affine functions.

From (3.17) and (3.18) it is readily seen that $\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{*(j)}$ satisfies

$$\begin{aligned}
\nabla_{\boldsymbol{y}}\mathcal{H}^{*(j)}_i &= \bigl(\boldsymbol{A}_i,\, (-\boldsymbol{B}_\nu^\top)_i,\, (-\boldsymbol{B}_t^\top \boldsymbol{F})_i\bigr), \quad i = 1, \dots, n,\\
\nabla_{\boldsymbol{y}}\mathcal{H}^{*(j)}_{n+i} &\in \{((r\boldsymbol{B}_\nu)_i,\, \boldsymbol{0},\, \boldsymbol{0}),\, (\boldsymbol{0},\, (\boldsymbol{I}_p)_i,\, \boldsymbol{0})\}, \quad i = 1, \dots, p,\\
\nabla_{\boldsymbol{y}}\mathcal{H}^{*(j)}_{n+p+i} &\in \{((r\boldsymbol{B}_t)_i,\, \boldsymbol{0},\, \boldsymbol{0}),\, ((-r\boldsymbol{B}_t)_i,\, \boldsymbol{0},\, (2\boldsymbol{I}_p)_i),\, (\boldsymbol{0},\, (-\boldsymbol{I}_p)_i,\, (\boldsymbol{I}_p)_i),\\
&\qquad (\boldsymbol{0},\, (\boldsymbol{I}_p)_i,\, (\boldsymbol{I}_p)_i)\}, \quad i = 1, \dots, p,
\end{aligned}$$

for an arbitrary j.

Taking into account that $\boldsymbol{\mathcal{H}}^{*(j)}$ is an affine function, we see that (3.22) is equivalent to

$$\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{*(j)}\boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}.$$

Making use of (2.2) one can eliminate $2p$ columns with the aid of the last $2p$ rows of the matrix $\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{*(j)}$ and one can arrive at an equivalent system of the type

$$\boldsymbol{M}\boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}_u \\ \boldsymbol{M}_\nu \\ \boldsymbol{M}_t \end{pmatrix}, \quad \begin{aligned} &\boldsymbol{M}_u \in \mathbb{R}^{n\times(n+2p)},\\ &\boldsymbol{M}_\nu, \boldsymbol{M}_t \in \mathbb{R}^{p\times(n+2p)}, \end{aligned} \tag{3.24}$$

where the rows of the matrix $\begin{pmatrix} \boldsymbol{M}_\nu \\ \boldsymbol{M}_t \end{pmatrix}$ are linearly independent not only to each other but also to the rows of $\boldsymbol{M}_u$. This and (3.23) yield that $\operatorname{rank}(\boldsymbol{M}_u) = n - s$. Moreover, the system in (3.24) is solvable if and only if $\boldsymbol{f}^0$ is contained in the range of $\boldsymbol{M}_u$. Therefore, (3.22) and (3.23) restrict $\boldsymbol{f}^0$ to some $(n-s)$-dimensional subspace of $\mathbb{R}^n$.

Since the number of all possible selection functions of $\boldsymbol{\mathcal{H}}^*$ is finite, the presented situation occurs generally only for $(\boldsymbol{f}^0, \boldsymbol{y}^0)$ such that $\boldsymbol{f}^0$ is from a union of some lower-dimensional subspaces of $\mathbb{R}^n$.

2. Two or more selection functions with nonsingular Jacobians are active at $(\boldsymbol{f}^0, \boldsymbol{y}^0)$, satisfying $\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}^0, \boldsymbol{y}^0) = \boldsymbol{0}$.

Taking one such selection function, say $\boldsymbol{\mathcal{H}}^{*(j)}$, it follows that $\boldsymbol{\mathcal{H}}^{*(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0) = \boldsymbol{0}$, i.e.

$$\nabla_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{*(j)} \boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}. \tag{3.25}$$

In addition to this, $|I_\nu^w(\boldsymbol{y}^0) \cup I_t^{w+}(\boldsymbol{y}^0) \cup I_t^{w-}(\boldsymbol{y}^0)| > 0$ (which means that at least one contact node is in weak contact or in weak stick) and the following $(|I_\nu^w(\boldsymbol{y}^0)| + |I_t^{w+}(\boldsymbol{y}^0)| + |I_t^{w-}(\boldsymbol{y}^0)|)$ conditions have to be satisfied:

$$\begin{cases} (\boldsymbol{\lambda}_\nu^0 - r\boldsymbol{B}_\nu \boldsymbol{u}^0)_i = 0 & \forall\, i \in I_\nu^w(\boldsymbol{y}^0), \\ (\boldsymbol{\lambda}_t^0 - r\boldsymbol{B}_t \boldsymbol{u}^0)_i = \lambda_{\nu,i}^0 & \forall\, i \in I_t^{w+}(\boldsymbol{y}^0), \\ (\boldsymbol{\lambda}_t^0 - r\boldsymbol{B}_t \boldsymbol{u}^0)_i = -\lambda_{\nu,i}^0 & \forall\, i \in I_t^{w-}(\boldsymbol{y}^0). \end{cases} \tag{3.26}$$

Notice that if $i \in I_\nu^0(\boldsymbol{y}^0) \cap I_t^{w+}(\boldsymbol{y}^0) \cap I_t^{w-}(\boldsymbol{y}^0)$ then the $(n+i)$th equation in (3.25) is $\lambda_{\nu,i}^0 = 0$, which together with the two corresponding conditions from $(3.26)_2$ and $(3.26)_3$ yields only two linearly independent equations with respect to $\boldsymbol{y}^0$. Furthermore, if $i \in I_\nu^w(\boldsymbol{y}^0) \cap I_t^{w+}(\boldsymbol{y}^0) \cap I_t^{w-}(\boldsymbol{y}^0)$ then the $(n+i)$th equation in (3.25) and the corresponding equation in $(3.26)_1$ are equivalent to $\lambda_{\nu,i}^0 = (\boldsymbol{B}_\nu \boldsymbol{u}^0)_i = 0$, which added to the two corresponding conditions in $(3.26)_2$ and $(3.26)_3$ leads only to three linearly independent equations. As a consequence, we can leave out one of the equations in $(3.26)_2$ or $(3.26)_3$ for any such i and (3.26) reduces in this way to a system of s equations with

$$\begin{aligned} s :=\ & |I_\nu^w(\boldsymbol{y}^0)| + |I_t^{w+}(\boldsymbol{y}^0)| + |I_t^{w-}(\boldsymbol{y}^0)| \\ & - |(I_\nu^0(\boldsymbol{y}^0) \cup I_\nu^w(\boldsymbol{y}^0)) \cap I_t^{w+}(\boldsymbol{y}^0) \cap I_t^{w-}(\boldsymbol{y}^0)| > 0. \end{aligned}$$

This system extended by (3.25) can be transformed similarly to the previous case into an equivalent system of the form

$$\boldsymbol{M}\boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}_u \\ \boldsymbol{M}_\nu \\ \boldsymbol{M}_t \\ \boldsymbol{M}_c \end{pmatrix}, \quad \begin{array}{l} \boldsymbol{M}_u \in \mathbb{R}^{n\times(n+2p)}, \ \boldsymbol{M}_\nu, \boldsymbol{M}_t \in \mathbb{R}^{p\times(n+2p)}, \\ \boldsymbol{M}_c \in \mathbb{R}^{s\times(n+2p)}, \end{array}$$

in which the rows of the matrix $\begin{pmatrix} \boldsymbol{M}_\nu \\ \boldsymbol{M}_t \\ \boldsymbol{M}_c \end{pmatrix}$ are linearly independent to each other and also to the rows of $\boldsymbol{M}_u$.

Arguing in the same way as previously, one can show that (3.25) and (3.26) confine $\boldsymbol{f}^0$ to some subspace of $\mathbb{R}^n$ of dimension $(n-s)$ and that the set of all $\boldsymbol{f}^0$ corresponding to this case forms a union of some lower-dimensional subspaces of $\mathbb{R}^n$ again.

We get the following remark.

Remark 3.2. All vectors $(\boldsymbol{f}^0, \boldsymbol{y}^0) \in \mathbb{R}^n \times \mathbb{R}^{n+2p}$ with $\boldsymbol{\mathcal{H}}^*(\boldsymbol{f}^0, \boldsymbol{y}^0) = \boldsymbol{0}$ which do not satisfy the assumption on the determinant sign of the Jacobians in Theorem 3.6 are such that $\boldsymbol{y}^0 \in \boldsymbol{S}^*_{\boldsymbol{\mathcal{F}}}(\boldsymbol{f}^0)$ and $\boldsymbol{f}^0$ is an element from a union of subspaces of dimension strictly lower than n.

4. An elementary example

This section presents an elementary discrete contact problem for one contact node (see Fig. 2). This example is taken from [11] and is nothing else than a special case of the model studied in [12].

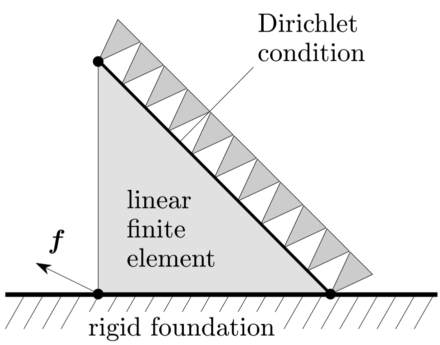


Figure 2. Geometry of the elementary example.

Denoting $\boldsymbol{u} := (u_\nu, u_t)$ and $\boldsymbol{f} := (f_\nu, f_t)$, an alternative of the projection formulation (3.15) of this problem reads as follows:

$$
(4.1) \qquad \begin{cases} \text{Find } \boldsymbol{y} := (u_\nu, u_t, \lambda_\nu, \lambda_t) \in \mathbb{R}^4 \text{ such that} \\ \boldsymbol{\mathcal{H}}(\boldsymbol{y}) := \begin{pmatrix} a u_\nu - b u_t - \lambda_\nu - f_\nu \\ -b u_\nu + a u_t - \lambda_t - f_t \\ \lambda_\nu - P_{(-\infty,0]}(\lambda_\nu - r u_\nu) \\ \lambda_t - P_{[-\mathcal{F}|\lambda_\nu|, \mathcal{F}|\lambda_\nu|]}(\lambda_t - r u_t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \end{cases}
$$

where the constants $a := (\lambda + 3\mu)/2$ and $b := (\lambda + \mu)/2$ depend on the Lamé coefficients $\lambda \geqslant 0$ and $\mu > 0$ characterizing the isotropic and homogeneous material of the body.

We derive exact solutions of this problem by considering all possible contact modes.

First, let there be no contact forces between the body and the rigid foundation, i.e. $\lambda_\nu = 0$. Then the fourth equation in (4.1) implies that $\lambda_t = 0$. Substituting these values of λ_ν and λ_t into the first and the second equation in (4.1), we obtain a system of two linear equations with the solution

$$
u_\nu = \frac{a f_\nu + b f_t}{a^2 - b^2}, \quad u_t = \frac{a f_t + b f_\nu}{a^2 - b^2}.
$$

In addition, from $\lambda_\nu = 0$ and the third equation in (4.1) it is readily seen that $u_\nu \leqslant 0$ so that

$$
a f_\nu + b f_t \leqslant 0.
$$

Secondly, suppose that there is a stick contact between the body and the rigid foundation, i.e. $u_\nu = u_t = 0$. Consequently, $(4.1)_{1,2}$ yield

$$
\lambda_\nu = -f_\nu, \quad \lambda_t = -f_t.
$$

Since $\lambda_\nu \leqslant 0$, $\mathcal{F}\lambda_\nu \leqslant \lambda_t \leqslant -\mathcal{F}\lambda_\nu$ by $(4.1)_3$ and $(4.1)_4$, respectively, one has

$$
f_\nu \geqslant 0, \quad -\mathcal{F} f_\nu \leqslant f_t \leqslant \mathcal{F} f_\nu.
$$

Finally, consider a slip contact, i.e. $u_\nu = 0$, $u_t \neq 0$.

If $u_t > 0$ then $\lambda_t = \mathcal{F}\lambda_\nu$ by virtue of $(4.1)_{3,4}$, and $(4.1)_{1,2}$ give

$$
\lambda_\nu = -\frac{a f_\nu + b f_t}{a + b\mathcal{F}}, \quad u_t = \frac{f_t - \mathcal{F} f_\nu}{a + b\mathcal{F}}.
$$

From the conditions $\lambda_\nu \leqslant 0$ and $u_t > 0$ it follows that

$$
a f_\nu + b f_t \geqslant 0, \quad f_t - \mathcal{F} f_\nu > 0.
$$

If $u_t < 0$ then $\lambda_t = -\mathcal{F}\lambda_\nu$, and $(4.1)_{1,2}$ are equivalent to

$$\begin{cases} -bu_t - \lambda_\nu = f_\nu, \\ (a - b\mathcal{F})u_t = f_t + \mathcal{F}f_\nu. \end{cases} \tag{4.2}$$

Assuming $\mathcal{F} \neq a/b$ this system has a unique solution

$$\lambda_\nu = -\frac{af_\nu + bf_t}{a - b\mathcal{F}}, \quad u_t = \frac{f_t + \mathcal{F}f_\nu}{a - b\mathcal{F}},$$

whose constraints are

$$\Big(\mathcal{F} < \frac{a}{b} \ \&\ af_\nu + bf_t \geqslant 0 \ \&\ f_t + \mathcal{F}f_\nu < 0 \ \&\ f_\nu \geqslant 0\Big)$$
$$\vee \Big(\mathcal{F} > \frac{a}{b} \ \&\ af_\nu + bf_t \leqslant 0 \ \&\ f_t + \mathcal{F}f_\nu > 0 \ \&\ f_\nu \geqslant 0\Big).$$

If $\mathcal{F} = a/b$ then (4.2) is solvable if and only if

$$f_t + \mathcal{F}f_\nu = 0$$

and its solutions form the set

$$\{(u_t, \lambda_\nu) \in \mathbb{R}^2\colon\ \lambda_\nu = -bu_t - f_\nu,\ u_t \in \mathbb{R}^1\}.$$

Due to the conditions $\lambda_\nu \leqslant 0$ and $u_t < 0$, u_t has to satisfy

$$-\frac{f_\nu}{b} \leqslant u_t < 0.$$

Consequently,

$$f_\nu > 0.$$

Introduce linear functions $\boldsymbol{S}_\mathcal{F}^{(i)}\colon\ \mathbb{R}^2 \to \mathbb{R}^4$, $i = 1, \dots, 5$, and a multi-valued function $\boldsymbol{S}_\mathcal{F}^{(6)}\colon\ \mathbb{R}^2 \rightrightarrows \mathbb{R}^4$ by

$$\boldsymbol{S}_\mathcal{F}^{(1)}(\boldsymbol{f}) = \Big(\frac{af_\nu + bf_t}{a^2 - b^2}, \frac{bf_\nu + af_t}{a^2 - b^2}, 0, 0\Big), \quad \boldsymbol{f} \in \mathbb{R}^2, \mathcal{F} \in \mathbb{R}^1_+,$$
$$\boldsymbol{S}_\mathcal{F}^{(2)}(\boldsymbol{f}) = (0, 0, -f_\nu, -f_t), \quad \boldsymbol{f} \in \mathbb{R}^2, \mathcal{F} \in \mathbb{R}^1_+,$$
$$\boldsymbol{S}_\mathcal{F}^{(3)}(\boldsymbol{f}) = \Big(0, \frac{f_t - \mathcal{F}f_\nu}{a + b\mathcal{F}}, -\frac{af_\nu + bf_t}{a + b\mathcal{F}}, -\mathcal{F}\frac{af_\nu + bf_t}{a + b\mathcal{F}}\Big), \quad \boldsymbol{f} \in \mathbb{R}^2, \mathcal{F} \in \mathbb{R}^1_+,$$
$$\boldsymbol{S}_\mathcal{F}^{(4)}(\boldsymbol{f}) = \Big(0, \frac{f_t + \mathcal{F}f_\nu}{a - b\mathcal{F}}, -\frac{af_\nu + bf_t}{a - b\mathcal{F}}, \mathcal{F}\frac{af_\nu + bf_t}{a - b\mathcal{F}}\Big), \quad \boldsymbol{f} \in \mathbb{R}^2, \mathcal{F} \in \Big[0, \frac{a}{b}\Big),$$
$$\boldsymbol{S}_\mathcal{F}^{(5)}(\boldsymbol{f}) = \Big(0, \frac{f_t + \mathcal{F}f_\nu}{a - b\mathcal{F}}, -\frac{af_\nu + bf_t}{a - b\mathcal{F}}, \mathcal{F}\frac{af_\nu + bf_t}{a - b\mathcal{F}}\Big), \quad \boldsymbol{f} \in \mathbb{R}^2, \mathcal{F} \in \Big(\frac{a}{b}, \infty\Big),$$
$$\boldsymbol{S}_\mathcal{F}^{(6)}(\boldsymbol{f}) = \Big\{(u_\nu, u_t, \lambda_\nu, \lambda_t) \in \mathbb{R}^4\colon$$
$$u_\nu = 0, -\frac{f_\nu}{b} \leqslant u_t \leqslant 0, \lambda_\nu = -(f_\nu + bu_t), \lambda_t = \mathcal{F}(f_\nu + bu_t)\Big\}, \quad \boldsymbol{f} \in \mathbb{R}^2, \mathcal{F} = \frac{a}{b}.$$

Moreover, let $\mathcal{F} \in \mathbb{R}^1_+$ and let us define sets:

$$
(4.3)\quad
\begin{cases}
\boldsymbol{\varrho}^{(1)}_{\mathcal{F}} = \{\boldsymbol{f} \in \mathbb{R}^2 \colon\ af_\nu + bf_t \leqslant 0\} \quad \text{(no contact)}, \\
\boldsymbol{\varrho}^{(2)}_{\mathcal{F}} = \{\boldsymbol{f} \in \mathbb{R}^2 \colon\ f_\nu \geqslant 0,\, -\mathcal{F}f_\nu \leqslant f_t \leqslant \mathcal{F}f_\nu\} \quad \text{(contact and stick)}, \\
\boldsymbol{\varrho}^{(3)}_{\mathcal{F}} = \{\boldsymbol{f} \in \mathbb{R}^2 \colon\ af_\nu + bf_t \geqslant 0,\, f_t - \mathcal{F}f_\nu \geqslant 0\} \\
\qquad\qquad \text{(contact and non-negative slip)}, \\
\boldsymbol{\varrho}^{(4)}_{\mathcal{F}} = \{\boldsymbol{f} \in \mathbb{R}^2 \colon\ af_\nu + bf_t \geqslant 0,\, f_t + \mathcal{F}f_\nu \leqslant 0,\, f_\nu \geqslant 0\} \\
\qquad\qquad \text{(contact and non-positive slip, } \mathcal{F} < a/b\text{)}, \\
\boldsymbol{\varrho}^{(5)}_{\mathcal{F}} = \{\boldsymbol{f} \in \mathbb{R}^2 \colon\ af_\nu + bf_t \leqslant 0,\, f_t + \mathcal{F}f_\nu \geqslant 0,\, f_\nu \geqslant 0\} \\
\qquad\qquad \text{(contact and non-positive slip, } \mathcal{F} > a/b\text{)}.
\end{cases}
$$

Observe that only $\boldsymbol{\varrho}^{(1)}_{\mathcal{F}}$ does not depend on $\mathcal{F}$. One can easily verify that $\boldsymbol{S}^{(i)}_{\mathcal{F}}(\boldsymbol{f})$ solves (4.1) for $\boldsymbol{f} \in \boldsymbol{\varrho}^{(i)}_{\mathcal{F}}$, $i = 1, 2, 3$, $\mathcal{F} \in \mathbb{R}^1_+$, $\boldsymbol{S}^{(4)}_{\mathcal{F}}(\boldsymbol{f})$ solves (4.1) for $\boldsymbol{f} \in \boldsymbol{\varrho}^{(4)}_{\mathcal{F}}$, $\mathcal{F} \in [0, a/b)$, $\boldsymbol{S}^{(5)}_{\mathcal{F}}(\boldsymbol{f})$ is a solution for $\boldsymbol{f} \in \boldsymbol{\varrho}^{(5)}_{\mathcal{F}}$, $\mathcal{F} \in (a/b, \infty)$, and $\boldsymbol{S}^{(6)}_{\mathcal{F}}(\boldsymbol{f})$ is a set of solutions for $\boldsymbol{f} \in \boldsymbol{\varrho}^{(4)}_{\mathcal{F}} = \boldsymbol{\varrho}^{(5)}_{\mathcal{F}}$ valid when $\mathcal{F} = a/b$.

Denote by $\mathring{\boldsymbol{\varrho}}^{(i)}_{\mathcal{F}}$ the interior of $\boldsymbol{\varrho}^{(i)}_{\mathcal{F}}$, $i = 1, \ldots, 5$. From (4.3) it is readily seen that $\mathring{\boldsymbol{\varrho}}^{(3)}_{\mathcal{F}}$ is disjoint with $\mathring{\boldsymbol{\varrho}}^{(i)}_{\mathcal{F}}$, $i \neq 3$, for any $\mathcal{F} > 0$. Hence, the structure of the solution set to (4.1) will be determined by the mutual position of $\boldsymbol{\varrho}^{(1)}_{\mathcal{F}}$, $\boldsymbol{\varrho}^{(2)}_{\mathcal{F}}$, and $\boldsymbol{\varrho}^{(4)}_{\mathcal{F}}$ or $\boldsymbol{\varrho}^{(5)}_{\mathcal{F}}$, which depends on the magnitude of $\mathcal{F}$. We shall distinguish 3 cases.

Case 1: $\mathcal{F} \in [0, a/b)$.

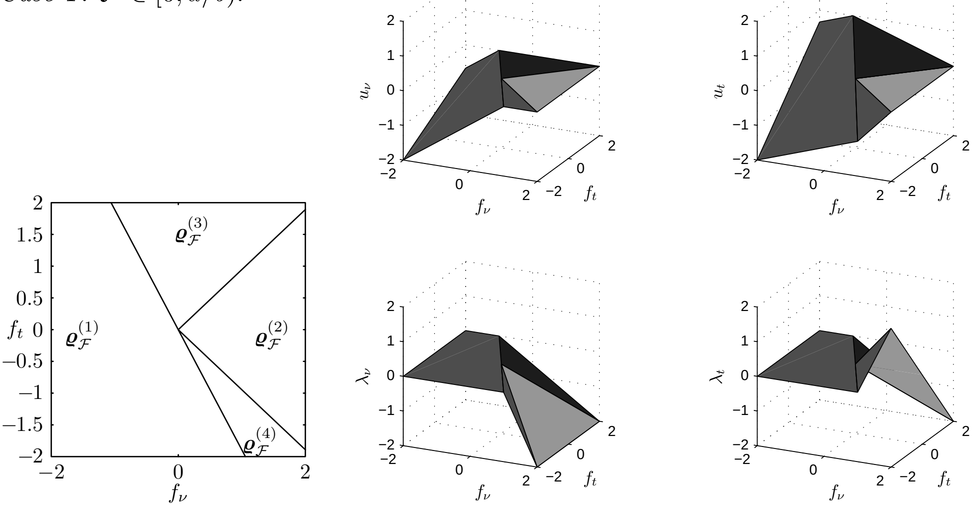


Figure 3. Solution for $0 < \mathcal{F} < a/b$.

Suppose first that $\mathcal{F} \neq 0$. Then $\boldsymbol{\varrho}_{\mathcal{F}}^{(5)} = \{\mathbf{0}\}$, $\mathring{\boldsymbol{\varrho}}_{\mathcal{F}}^{(1)} \cap \mathring{\boldsymbol{\varrho}}_{\mathcal{F}}^{(2)} = \emptyset$ and $\boldsymbol{\varrho}_{\mathcal{F}}^{(4)}$ is the wedge between $\boldsymbol{\varrho}_{\mathcal{F}}^{(1)}$ and $\boldsymbol{\varrho}_{\mathcal{F}}^{(2)}$ (see Fig. 3). The system $\{\boldsymbol{\varrho}_{\mathcal{F}}^{(1)}, \boldsymbol{\varrho}_{\mathcal{F}}^{(2)}, \boldsymbol{\varrho}_{\mathcal{F}}^{(3)}, \boldsymbol{\varrho}_{\mathcal{F}}^{(4)}\}$ defines the partition of $\mathbb{R}^2$, i.e. $\mathbb{R}^2 = \bigcup_{i=1}^{4} \boldsymbol{\varrho}_{\mathcal{F}}^{(i)}$, $\mathring{\boldsymbol{\varrho}}_{\mathcal{F}}^{(i)} \cap \mathring{\boldsymbol{\varrho}}_{\mathcal{F}}^{(j)} = \emptyset$, $i \neq j$, $i, j = 1, \ldots, 4$.

The solution map $\boldsymbol{S}_{\mathcal{F}} : \mathbb{R}^2 \to \mathbb{R}^4$ is defined by

$$\boldsymbol{S}_{\mathcal{F}} = \boldsymbol{S}_{\mathcal{F}}^{(i)} \quad \text{in } \boldsymbol{\varrho}_{\mathcal{F}}^{(i)},\ i = 1, \ldots, 4.$$

It is easy to verify that

$$\boldsymbol{S}_{\mathcal{F}}^{(i)}{}_{|\Gamma_{ij}} = \boldsymbol{S}_{\mathcal{F}}^{(j)}{}_{|\Gamma_{ij}} \quad \forall\, i, j = 1, \ldots, 4,$$

where Γ_{ij} is the common side of two neighbouring sectors $\boldsymbol{\varrho}_{\mathcal{F}}^{(i)}$, $\boldsymbol{\varrho}_{\mathcal{F}}^{(j)}$. Hence, $\boldsymbol{S}_{\mathcal{F}}$ is a single-valued function in the whole $\mathbb{R}^2$.

If $\mathcal{F} = 0$ then $\boldsymbol{\varrho}_{\mathcal{F}}^{(2)} = \boldsymbol{\varrho}_{\mathcal{F}}^{(3)} \cap \boldsymbol{\varrho}_{\mathcal{F}}^{(4)}$ and the partition is realized by $\{\boldsymbol{\varrho}_{\mathcal{F}}^{(1)}, \boldsymbol{\varrho}_{\mathcal{F}}^{(3)}, \boldsymbol{\varrho}_{\mathcal{F}}^{(4)}\}$. Adapting $\boldsymbol{S}_{\mathcal{F}}$ to this case we see that it is again single-valued in $\mathbb{R}^2$.

Consequently, if $\mathcal{F} \in [0, a/b)$ *then* (4.1) *has a unique solution for any* $\boldsymbol{f} \in \mathbb{R}^2$.

Case 2: $\mathcal{F} > a/b$.

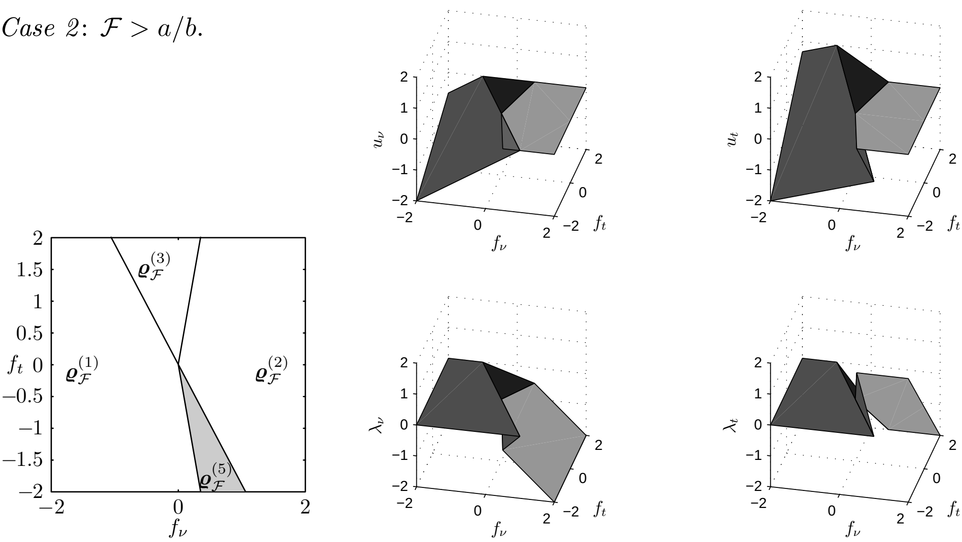


Figure 4. Solution for $\mathcal{F} > a/b$.

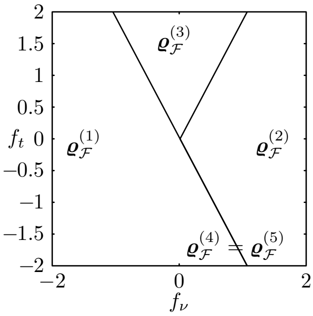
In this case $\boldsymbol{\varrho}_{\mathcal{F}}^{(4)} = \{\mathbf{0}\}$, $\boldsymbol{\varrho}_{\mathcal{F}}^{(5)} = \boldsymbol{\varrho}_{\mathcal{F}}^{(1)} \cap \boldsymbol{\varrho}_{\mathcal{F}}^{(2)}$ and $\mathring{\boldsymbol{\varrho}}_{\mathcal{F}}^{(5)} \neq \emptyset$ (see Fig. 4). Introduce the function $\boldsymbol{S}_{\mathcal{F}}$ as before:

$$\boldsymbol{S}_{\mathcal{F}} = \boldsymbol{S}_{\mathcal{F}}^{(i)} \quad \text{in } \boldsymbol{\varrho}_{\mathcal{F}}^{(i)},\ i = 1, 2, 3, 5,$$

which is multi-valued in $\boldsymbol{\varrho}_{\mathcal{F}}^{(5)}$.

We conclude that there exists a unique solution to (4.1) *if* $\boldsymbol{f}$ *belongs to* $((\boldsymbol{\varrho}_{\mathcal{F}}^{(1)} \cup \boldsymbol{\varrho}_{\mathcal{F}}^{(2)} \cup \boldsymbol{\varrho}_{\mathcal{F}}^{(3)}) \setminus \boldsymbol{\varrho}_{\mathcal{F}}^{(5)}) \cup \{\mathbf{0}\}$*, there are two solutions on* $\partial \boldsymbol{\varrho}_{\mathcal{F}}^{(5)} \setminus \{\mathbf{0}\}$ *and three solutions in* $\mathring{\boldsymbol{\varrho}}_{\mathcal{F}}^{(5)}$.

Case 3: $\mathcal{F} = a/b$.



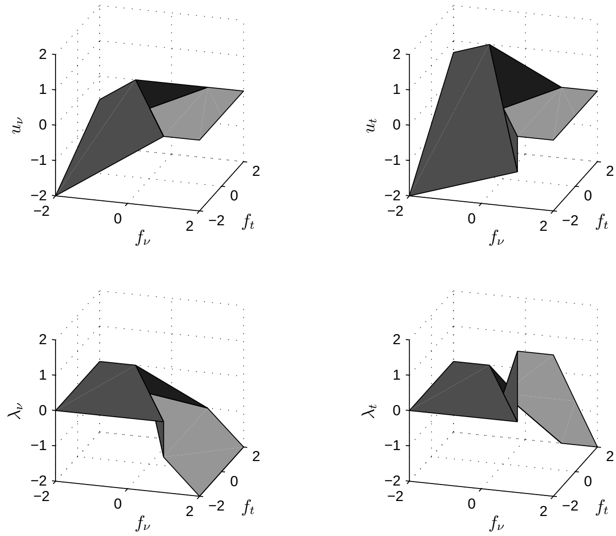


Figure 5. Solution for $\mathcal{F} = a/b$.

This is the limit case, in which $\boldsymbol{\varrho}_{\mathcal{F}}^{(4)} = \boldsymbol{\varrho}_{\mathcal{F}}^{(5)} = \boldsymbol{\varrho}_{\mathcal{F}}^{(1)} \cap \boldsymbol{\varrho}_{\mathcal{F}}^{(2)}$ is the ray emanating from the origin and separating $\boldsymbol{\varrho}_{\mathcal{F}}^{(1)}$ and $\boldsymbol{\varrho}_{\mathcal{F}}^{(2)}$ (see Fig. 5).

If $\boldsymbol{f} \in (\mathbb{R}^2 \setminus \boldsymbol{\varrho}_{\mathcal{F}}^{(5)}) \cup \{\mathbf{0}\}$*, there exists a unique solution to* (4.1)*. For* $\boldsymbol{f} \in \boldsymbol{\varrho}_{\mathcal{F}}^{(5)} \setminus \{\mathbf{0}\}$ *the continuous branch* $\boldsymbol{S}_{\mathcal{F}}^{(6)}(\boldsymbol{f})$ *of solutions connects* $\boldsymbol{S}_{\mathcal{F}}^{(1)}(\boldsymbol{f})$ *and* $\boldsymbol{S}_{\mathcal{F}}^{(2)}(\boldsymbol{f})$.

From the above analysis we see that the solution of (4.1) is a PC^1-function of $\mathcal{F} \in [0, a/b)$ for an arbitrary $\boldsymbol{f} \in \mathbb{R}^2$ fixed. Therefore, it is Lipschitz-continuous with respect to $\mathcal{F}$ in $[0, \overline{\mathcal{F}}_{\max}]$ for any $\overline{\mathcal{F}}_{\max} \in [0, a/b)$. On the other hand, we have proved the global uniqueness as well as the Lipschitz continuity of the solutions with respect to $\mathcal{F}$ in $[0, \mathcal{F}_{\max}]$ with $\mathcal{F}_{\max} \in [0, \beta\gamma/\|\boldsymbol{A}\|_n)$ in Section 3. After some computations one obtains $\beta\gamma/\|\boldsymbol{A}\|_n = (a-b)/(a+b)$, which is strictly less than a/b. Since the situation concerning the Lipschitz continuity with respect to $\boldsymbol{f}$ is analogous, one can see that the bounds derived before are pessimistic.

Nevertheless, this example shows that unicity of solutions depends not only on $\mathcal{F}$ but also on $\boldsymbol{f}$. Even if one takes $\mathcal{F}$ so large that there are non-unique solutions for some $\boldsymbol{f}$, for the same $\mathcal{F}$ there still exist such $\boldsymbol{f}$ that the corresponding solution is unique. Furthermore, one can easily verify that in this particular example Theorem 3.6 guarantees local uniqueness of solutions precisely except the cases where it is actually lost. Hence, the presented local approach seems to be better suited for studying the behaviour of solutions than the global one.

Finally, let us mention that if one introduces selection functions $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(l)}$ of the PC^1-function $\boldsymbol{\mathcal{H}}$ in a way analogous to (3.18) for an appropriate l, each function $\boldsymbol{f} \mapsto \boldsymbol{S}_{\mathcal{F}}^{(i)}(\boldsymbol{f})$, $i = 1, \ldots, 6$, is nothing else than a mapping associating $\boldsymbol{f}$ with the solution of the equation $\boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{y}) = \boldsymbol{0}$ for some particular $\boldsymbol{\mathcal{H}}^{(j)}$. Since $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(l)}$ are piecewise-linear functions of the load vector $\boldsymbol{f}$, the structure of solutions to (4.1) as functions of $\boldsymbol{f}$ is quite simple. On the other hand, dependence of the solutions on the coefficient $\mathcal{F}$ is substantially more complicated, as exhibited in [11].

5. Conclusions

Theoretical analysis of discrete contact problems with Coulomb friction in which the coefficient of friction $\boldsymbol{\mathcal{F}}$ is assumed to be a vector was presented. The existence result is obtained for any coefficient $\boldsymbol{\mathcal{F}}$ whereas to get the global uniqueness result one needs the norm of $\boldsymbol{\mathcal{F}}$ to be sufficiently small. Moreover, the unique solution is a Lipschitz-continuous function of $\boldsymbol{\mathcal{F}}$ as well as of the load vector $\boldsymbol{f}$. Local analysis of potentially non-unique solutions is based on two different but equivalent formulations of the problem—the former consists of generalized equations, the latter of non-smooth equations. For the first formulation we showed that the study of local behaviour of solutions as functions of $\boldsymbol{\mathcal{F}}$ can be replaced by the study of local behaviour of the solutions as functions of $\boldsymbol{f}$. For the second one we got that the solutions are locally unique and Lipschitz-continuous with respect to $\boldsymbol{f}$ if particular Jacobian matrices depending on the contact status of the solutions have the same nonvanishing determinant sign. Results determining directional derivatives to these locally Lipschitz-continuous branches were also achieved. In the end, benefits of the proposed approach are illustrated by a simple example.

Appendix A. Piecewise-differentiable functions

For the sake of completeness we give here a brief introduction to the theory of piecewise-differentiable functions. The exposition is extracted from [16].

We start with some basic notions. Let $\boldsymbol{\pi} := \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{B}\boldsymbol{x} \leqslant \boldsymbol{0}\}$ with $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ be a polyhedral cone with vertex at $\boldsymbol{0}$. Recall that *the dimension* of $\boldsymbol{\pi}$ is defined as the dimension of its linear hull and nonempty faces of $\boldsymbol{\pi}$ can be represented as the sets

$$\{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{B}_i\boldsymbol{x} = 0 \ \forall\, i \in I,\ \boldsymbol{B}_j\boldsymbol{x} \leqslant 0 \ \forall\, j \in \{1, \ldots, m\} \setminus I\}$$

for some index set $I \in \mathcal{I}(\boldsymbol{B}, \boldsymbol{0})$, where

$$\mathcal{I}(\boldsymbol{B}, \boldsymbol{0}) = \{I \subseteq \{1, \ldots, m\} \colon \\ \exists\, \boldsymbol{x} \in \mathbb{R}^n \colon\ \boldsymbol{B}_i \boldsymbol{x} = 0\ \forall i \in I,\ \boldsymbol{B}_j \boldsymbol{x} < 0\ \forall j \in \{1, \ldots, m\} \setminus I\}$$

([16, Proposition 2.1.3]). Here $\boldsymbol{B}_i$ is the ith row vector of the matrix $\boldsymbol{B}$. A nonempty face of $\boldsymbol{\pi}$ which does not coincide with $\boldsymbol{\pi}$ is called *a proper face*. Further, the lineality space of $\boldsymbol{\pi}$ is the linear subspace $\{\boldsymbol{x} \in \mathbb{R}^n \colon\ \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}\}$.

A finite collection $\boldsymbol{\Pi}$ of convex polyhedral cones in $\mathbb{R}^n$ is called *a conical subdivision* of a polyhedral cone $\boldsymbol{\varrho} \subseteq \mathbb{R}^n$ if

1. all polyhedral cones in $\boldsymbol{\Pi}$ are subsets of $\boldsymbol{\varrho}$;
2. the dimension of the cones in $\boldsymbol{\Pi}$ coincides with the dimension of $\boldsymbol{\varrho}$;
3. the union of all cones in $\boldsymbol{\Pi}$ covers $\boldsymbol{\varrho}$;
4. the intersection of any two distinct cones in $\boldsymbol{\Pi}$ is either empty or a common proper face of both cones.

It holds that if $\boldsymbol{\Pi}$ is a conical subdivision of a polyhedral cone then all polyhedral cones $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ have the same lineality space ([16, Proposition 2.2.4]). Hence, *the lineality space* of $\boldsymbol{\Pi}$ is introduced as the common lineality space of the polyhedral cones of $\boldsymbol{\Pi}$.

The pth branching number of a conical subdivision $\boldsymbol{\Pi}$ of a polyhedral cone $\boldsymbol{\varrho}$ is defined as the maximal number of cones in $\boldsymbol{\Pi}$ containing a common face of dimension $(\dim \boldsymbol{\varrho} - p)$, where $p \in \{1, \ldots, \dim \boldsymbol{\varrho} - q\}$ and q is the dimension of the lineality space of $\boldsymbol{\Pi}$.

Finally, let $\boldsymbol{U}$ be a subset of $\mathbb{R}^n$ and let $\boldsymbol{\mathcal{H}}^{(j)} \colon \boldsymbol{U} \to \mathbb{R}^k$, $j = 1, \ldots, l$, be a collection of continuous functions. A function $\boldsymbol{\mathcal{H}} \colon \boldsymbol{U} \to \mathbb{R}^k$ is said to be *a continuous selection* of the functions $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(l)}$ on the set $\boldsymbol{W} \subseteq \boldsymbol{U}$ if it is continuous on $\boldsymbol{W}$ and $\boldsymbol{\mathcal{H}}(\boldsymbol{x}) \in \{\boldsymbol{\mathcal{H}}^{(1)}(\boldsymbol{x}), \ldots, \boldsymbol{\mathcal{H}}^{(l)}(\boldsymbol{x})\}$ for every $\boldsymbol{x} \in \boldsymbol{W}$. A function $\boldsymbol{\mathcal{H}} \colon \boldsymbol{U} \to \mathbb{R}^k$ defined on an open set $\boldsymbol{U} \subseteq \mathbb{R}^n$ is called *a PC^r-function* for some $r \in \{1, 2, \ldots\} \cup \{\infty\}$ if for every $\boldsymbol{x}^0 \in \boldsymbol{U}$ there exist an open neighbourhood $\boldsymbol{W} \subseteq \boldsymbol{U}$ of $\boldsymbol{x}^0$ and a finite number, say l, of C^r-functions $\boldsymbol{\mathcal{H}}^{(j)} \colon \boldsymbol{W} \to \mathbb{R}^k$ such that $\boldsymbol{\mathcal{H}}$ is a continuous selection of $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(l)}$ on $\boldsymbol{W}$. A set of C^r-functions $\boldsymbol{\mathcal{H}}^{(j)} \colon \boldsymbol{W} \to \mathbb{R}^k$, $j = 1, \ldots, l$, defined on an open neighbourhood $\boldsymbol{W} \subseteq \boldsymbol{U}$ of $\boldsymbol{x}^0$ is called a set of *selection functions* for the PC^r-function $\boldsymbol{\mathcal{H}}$ at $\boldsymbol{x}^0$ if $\boldsymbol{\mathcal{H}}(\boldsymbol{x}) \in \{\boldsymbol{\mathcal{H}}^{(1)}(\boldsymbol{x}), \ldots, \boldsymbol{\mathcal{H}}^{(l)}(\boldsymbol{x})\}$ for every $\boldsymbol{x} \in \boldsymbol{W}$. The selection functions $\boldsymbol{\mathcal{H}}^{(j)}$ such that $\boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{x}^0) = \boldsymbol{\mathcal{H}}(\boldsymbol{x}^0)$ are called *active selection functions* at $\boldsymbol{x}^0$. PC^1-functions are also called *piecewise-differentiable* functions.

Theorem A.1 ([16, Theorem 4.2.2])**.** *Let $\boldsymbol{U} \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be open, let $\boldsymbol{\mathcal{H}} \colon \boldsymbol{U} \to \mathbb{R}^k$ be a PC^r-function and let $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in \boldsymbol{U}$ be a vector with $\boldsymbol{\mathcal{H}}(\boldsymbol{x}^0, \boldsymbol{y}^0) = \boldsymbol{0}$. Further, let $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(l)} \colon \boldsymbol{W} \to \mathbb{R}^k$ be a collection of selection functions for $\boldsymbol{\mathcal{H}}$ at $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in$*

$\boldsymbol{W} \subseteq \boldsymbol{U}$, *and* $\boldsymbol{\Pi}$ *a conical subdivision of* $\mathbb{R}^n \times \mathbb{R}^k$ *with a lineality space of dimension* q. *If*

1. *for every* $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ *there exists an index* $j_{\boldsymbol{\pi}} \in \{1, \ldots, l\}$ *such that* $\boldsymbol{\mathcal{H}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}, \boldsymbol{y})$ *for every* $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{W} \cap (\{(\boldsymbol{x}^0, \boldsymbol{y}^0)\} + \boldsymbol{\pi})$;
2. *either* $(n + k - q) \leqslant 1$ *or there exists a number* $p \in \{2, \ldots, (n + k - q)\}$ *such that the pth branching number of* $\boldsymbol{\Pi}$ *does not exceed* $2p$;
3. *all matrices* $\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}^0, \boldsymbol{y}^0)$, $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, *have the same nonvanishing determinant sign,*

then

1. *the equation* $\boldsymbol{\mathcal{H}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ *determines an implicit* PC^r*-function* $\boldsymbol{y}(\boldsymbol{x})$ *at* $(\boldsymbol{x}^0, \boldsymbol{y}^0)$;
2. *the implicit functions* $\boldsymbol{y}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x})$ *determined by the equations* $\boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$, $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, *form a collection of selection functions for the* PC^r*-function* $\boldsymbol{y}(\boldsymbol{x})$ *at* $\boldsymbol{x}^0$;
3. *for every* $\boldsymbol{h} \in \mathbb{R}^n$ *the identity* $\boldsymbol{\xi} = \boldsymbol{y}'(\boldsymbol{x}^0; \boldsymbol{h})$ *holds if and only if* $\boldsymbol{\xi}$ *satisfies the piecewise-linear equation* $\boldsymbol{\mathcal{H}}'((\boldsymbol{x}^0, \boldsymbol{y}^0); (\boldsymbol{h}, \boldsymbol{\xi})) = \boldsymbol{0}$.

Theorem A.2 ([16, Proposition 4.2.2]). *Suppose that the assumptions of the previous theorem are satisfied and* $\boldsymbol{h} \in \mathbb{R}^n$ *is arbitrary.*

1. *Then there exists a cone* $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ *such that*

$$\begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{0} \end{pmatrix} \in \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{0}_{n,k} \\ \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}^0, \boldsymbol{y}^0) & \boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}^0, \boldsymbol{y}^0) \end{pmatrix} \boldsymbol{\pi}. \tag{A.1}$$

2. *The inclusion* (A.1) *holds if and only if*

$$\begin{pmatrix} \boldsymbol{h} \\ -\big(\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}^0, \boldsymbol{y}^0)\big)^{-1} \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}^0, \boldsymbol{y}^0) \boldsymbol{h} \end{pmatrix} \in \boldsymbol{\pi}.$$

3. *If* $\boldsymbol{h}$ *satisfies* (A.1) *then*

$$\boldsymbol{y}'(\boldsymbol{x}^0; \boldsymbol{h}) = -\big(\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}^0, \boldsymbol{y}^0)\big)^{-1} \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{\mathcal{H}}^{(j_{\boldsymbol{\pi}})}(\boldsymbol{x}^0, \boldsymbol{y}^0) \boldsymbol{h}.$$

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