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# HOMOGENIZATION OF MONOTONE PARABOLIC PROBLEMS WITH SEVERAL TEMPORAL SCALES

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*Abstract.* In this paper we homogenize monotone parabolic problems with two spatial scales and any number of temporal scales. Under the assumption that the spatial and temporal scales are well-separated in the sense explained in the paper, we show that there is an H-limit defined by at most four distinct sets of local problems corresponding to slow temporal oscillations, slow resonant spatial and temporal oscillations (the "slow" self-similar case), rapid temporal oscillations, and rapid resonant spatial and temporal oscillations (the "rapid" self-similar case), respectively.

*Keywords*: homogenization, H-convergence, multiscale convergence, parabolic, monotone

*MSC 2010*: 35B27

## 1. Introduction

We will give here a brief survey—with some important references—of homogenization theory and two-scale convergence techniques which is followed by a statement of the research objective of the present paper. Finally in this section we give a list of notation employed in the paper.

**1.1. Homogenization theory.** Homogenization theory is the study of the convergence—in some suitable sense—of sequences of equations involving sequences of operators and (possibly) source functions and the responding sequences of solutions. The main applications involve the study of the convergence of sequences of partial differential equations described by heterogeneous coefficients which become more and more refined such that the problem tends to a homogenized limit. In the case of parabolic partial differential equations the convergence modes used to achieve homogenized limits are the so called G- and H-convergences, where the former is employed when the coefficients can be arranged as a symmetric matrix (see [21]), and the latter is the generalization which includes non-symmetric matrices (see [15], [24]) and even non-linear problems (see [23]). "Homogenizing" a problem means in this context to find the limit in the G- or H-convergence process.

**1.2. Two-scale convergence.** The theory of homogenization experienced a quantum leap in the late 1980's when the two-scale convergence technique was introduced (see [16], [1])—effectively replacing Tartar's method of oscillating test functions (see [23], [24]) as the main tool to achieve G- or H-convergence—and the technique has subsequently improved since then. Two-scale convergence (with generalizations such as multiscale convergence [2], "generalized" two-scale convergence [8], scale convergence [14], $\lambda$-scale convergence [10], $\Sigma$-convergence [17] etc.) is today an indispensable tool to the modern homogenization theorist.

**1.3. Objectives and main results of the paper.** The main purpose of this paper is to perform homogenization of monotone, possibly non-linear, parabolic problems of the type

$$
\begin{cases}
\dfrac{\partial}{\partial t}u_\varepsilon(x,t) - \nabla\cdot a\Big(x,t,\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon'_1},\ldots,\dfrac{t}{\varepsilon'_m};\nabla u_\varepsilon\Big) = f(x,t) & \text{in } \Omega\times(0,T),\\
u_\varepsilon(x,0) = u_0(x) & \text{in } \Omega,\\
u_\varepsilon(x,t) = 0 & \text{on } \partial\Omega\times(0,T),
\end{cases}
\tag{1.1}
$$

i.e., having two spatial and $m+1$ temporal scales, where $\Omega$ is an open bounded set in $\mathbb{R}^N$ and $T>0$. As $\varepsilon$ tends to 0 we get a sequence of equations given by (1.1) above and the objective is to find the homogenized problem, i.e., to find the homogenized limit $b$ of the flux $a$ which defines a homogenized equation which admits a limit $u$ of the sequence of solutions $\{u_\varepsilon\}$. In order to homogenize (1.1) we impose a certain separatedness restriction on the scale functions $\varepsilon,\varepsilon'_1,\ldots,\varepsilon'_m$. The homogenized limit $b$ will not contain any fast spatial or temporal oscillations and (if considered as a function of $\nabla u$) is given in terms of an integral over the local variables $y,s_1,\ldots,s_m$ involving the flux $a$ and a function $u_1$ which is the unique solution of some local problems depending on the behaviour of the scale functions. We discern four distinct cases giving different local problems for $u_1$, namely the cases (i) $\varepsilon^2/\varepsilon'_m\to 0$ as $\varepsilon\to 0$, (ii) $\varepsilon'_m\sim\varepsilon^2$, and (iii) $\varepsilon'_i/\varepsilon^2\to 0$ but $\varepsilon'_{i-1}/\varepsilon^2\to\infty$ as $\varepsilon\to 0$ for some $\varepsilon'_i$ tending more rapidly to 0 than $\varepsilon$ does, and (iv) $\varepsilon'_{l^\circ-1}\sim\varepsilon^2$ for some $\varepsilon'_{l^\circ-1}\neq\varepsilon'_m$ tending more rapidly to 0 than $\varepsilon$ does. Case (i) corresponds to slow temporal oscillations (compared to the spatial one), (ii) is the so-called "slow" self-similar case where the spatial and temporal oscillations are in resonance, (iii) corresponds to rapid temporal oscillations, and (iv) is the "rapid" self-similar case.

**1.4. Notation and conventions.** The following notation and conventions are used in this paper.

Throughout the paper, $\Omega$ defining the spatial domain is a non-empty open bounded set in $\mathbb{R}^N$ with Lipschitz boundary, and $T > 0$ is the maximal time defining the temporal domain $(0, T)$.

We introduce the integer sets $[\![i, j]\!] = [i, j] \cap \mathbb{Z}$ for $0 < i \leqslant j$, and $[\![i]\!] = [1, i] \cap \mathbb{Z}$. Furthermore, let $[\![i]\!]_0 = [0, i] \cap \mathbb{Z}$. Note that we naturally interpret, e.g., $[\![0]\!] = \emptyset$.

Let $\mathcal{F}(A)/\mathbb{R}$ denote all functions in $\mathcal{F}(A)$ with mean value zero over $A \subset \mathbb{R}^M$, and let $\mathcal{F}_\#(Z)$ denote all locally $\mathcal{F}$ functions over $\mathbb{R}^M$ that are periodical repetitions of some functions in $\mathcal{F}(Z)$ where $Z = (0,1)^M$.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be some function spaces and introduce the tensor product space $\mathcal{F} = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_k$. We then define the subset $\mathcal{F}_1 \odot \ldots \odot \mathcal{F}_k$ of $\mathcal{F}$ by

$$\mathcal{F}_1 \odot \ldots \odot \mathcal{F}_k = \{f \in \mathcal{F}\colon\ f = f_1 \ldots f_k \text{ for some } f_i \in \mathcal{F}_i,\ i \in [\![k]\!]\}$$

which, in general, is not a subspace of $\mathcal{F}$ though spanning it.

There are two kinds of partial derivatives. The partial derivatives of the first kind, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$ and $\partial/\partial t$, only discern whether one differentiates with respect to the space variable $x = (x_1, \ldots, x_N)$ or the time variable $t$, respectively. The partial derivatives of the second kind, $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_N})$ and $\partial_t$ (i.e., with the variable as a subscript) are proper partial derivatives with respect to space and time, respectively. Note that partial derivatives of the local variables will always be of the proper, second kind. Example: Let $\psi = \psi(x, t, y, s)$ be a weakly differentiable real-valued function with respect to the global space and time variables $x$ and $t$ and the local space and time variables $y$ and $s$. Suppose $y = \eta x$ and $s = \sigma t$ for some real constants $\eta$ and $\sigma$, then the chain rule and the conventions above give

$$\nabla\psi = \nabla_x\psi + \eta\nabla_y\psi \quad \text{and} \quad \frac{\partial}{\partial t}\psi = \partial_t\psi + \sigma\partial_s\psi;$$

these differentiation rules will be important to keep in mind later in this paper.

## 2. Preliminaries

In order to perform the homogenization procedure for monotone parabolic problems with several temporal scales we first need to take a look at the necessary background theory.

**2.1. Multiscale convergence.** The concept of two-scale convergence was introduced in 1989 by Nguetseng (see [16]) and further developed by Allaire in 1992 (see [1]). In words, two-scale convergence is a kind of weak convergence mode for a sequence of functions of a global variable where the limit is a function of both the global (or macroscopic) and the local (or microscopic) variable.

The rigorous definition of two-scale convergence is given below. (If nothing else is stated, in this paper we let $y \in Y$ where $Y = (0,1)^N$.)

**Definition 2.1.** A sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0 \in L^2(\Omega \times Y)$ if, as $\varepsilon \to 0$ (from above),

$$\int_\Omega u_\varepsilon(x) v\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}x \to \int_\Omega \int_Y u_0(x,y) v(x,y)\, \mathrm{d}y\, \mathrm{d}x$$

for all $v \in L^2(\Omega; \mathcal{C}_\#(Y))$, and we write $u_\varepsilon \xrightarrow{2} u_0$ as $\varepsilon \to 0$.

From now on we assume that all limits are taken as $\varepsilon \to 0$ (from above) if nothing else is stated.

In Definition 2.2 below we introduce the notion of scale functions which are functions with respect to the scale parameter.

**Definition 2.2.** A scale function $\varepsilon_*\colon \mathbb{R}_+ \to \mathbb{R}$ is a real-valued function of the scale parameter $\varepsilon$ for which $\varepsilon_*(\varepsilon) \to 0$ (i.e., $\varepsilon_*$ is microscopic), and for which there exists $\delta > 0$ such that $\varepsilon_*(\varepsilon) > 0$ for all $0 < \varepsilon < \delta$ (i.e., $\varepsilon_*$ is ultimately positive).

The concept of scale functions leads to the notion of multiscale convergence which was introduced in 1996 by Allaire and Briane (see [2]) as a generalization of two-scale convergence in order to be able to perform homogenization of problems with multiple scales. This convergence mode is defined below. (If nothing else is stated, in this paper we let $y_i \in Y_i$, where $Y_i = (0,1)^N$, $i \in [\![n]\!]$.)

**Definition 2.3.** A sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ is said to $(n+1)$-scale converge to a limit $u_0 \in L^2(\Omega \times Y_1 \times \ldots \times Y_n)$ if

$$\int_\Omega u_\varepsilon(x) v\Big(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}\Big)\, \mathrm{d}x \\ \to \int_\Omega \int_{Y_1} \cdots \int_{Y_n} u_0(x, y_1, \ldots, y_n) v(x, y_1, \ldots, y_n)\, \mathrm{d}y_n \ldots \mathrm{d}y_1\, \mathrm{d}x$$

for all $v \in L^2(\Omega; \mathcal{C}_\#(Y_1 \times \ldots \times Y_n))$, and we write $u_\varepsilon \xrightarrow{n+1} u_0$.

In order to simplify the notation, from now on we will write $\mathbf{y}_n = (y_1, \ldots, y_n)$ and $Y^n = Y_1 \times \ldots \times Y_n$ so that $\mathbf{y}_n \in Y^n$ which collects the local variables and local sets under one roof. (Naturally, the Lebesgue measure on $Y^n$ is denoted $\mathrm{d}\mathbf{y}_n$.) We also write $\mathbf{x}_n^\varepsilon = (x/\varepsilon_1, \ldots, x/\varepsilon_n)$ in the same spirit where we note that $\mathbf{x}_n^\varepsilon$ actually depends on the particular choice of scale functions $\varepsilon_1, \dots, \varepsilon_n$. Of course, multiscale convergence is highly dependent on the behaviour of the (spatial) scale functions. For ordered lists of scale functions we have the following definitions:

**Definition 2.4.** The list $\{\varepsilon_i\}_{i=1}^n$ of scale functions is said to be separated if $\varepsilon_{k+1}/\varepsilon_k \to 0$ for all $k \in [\![n-1]\!]$.

**Definition 2.5.** The list $\{\varepsilon_i\}_{i=1}^n$ of scale functions is said to be well-separated if there exists a positive integer $l$ such that ${\varepsilon_k}^{-1}(\varepsilon_{k+1}/\varepsilon_k)^l \to 0$ for all $k \in [\![n-1]\!]$.

Remark 2.6. Note that well-separatedness is a stronger requirement than separatedness.

Homogenization for linear parabolic problems with several temporal scales using the multiscale convergence technique was first achieved by Flodén and Olsson in 2007 (see [6]). This was a further development of the work by Holmbom in 1996 (see [8]) where two-scale convergence was employed to homogenize linear parabolic problems with both a spatial and a temporal microscale. General $(n+1, m+1)$-scale convergence can be expressed according to the definition below. (If nothing else is stated, in this paper we let $s_j \in S_j$, where $S_j = (0,1)$, $j \in [\![m]\!]$.)

**Definition 2.7.** A sequence $\{u_\varepsilon\}$ in $L^2(\Omega \times (0,T))$ is said to $(n+1, m+1)$-scale converge to a limit $u_0 \in L^2(\Omega \times (0,T) \times Y^n \times S_1 \times \dots \times S_m)$ if

$$\int_0^T \int_\Omega u_\varepsilon(x,t) v\Big(x, t, \mathbf{x}_n^\varepsilon, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\to \int_0^T \int_\Omega \int_{Y^n} \int_{S_1} \dots \int_{S_m} u_0(x, t, \mathbf{y}_n, s_1, \dots, s_m) \times v(x, t, \mathbf{y}_n, s_1, \dots, s_m)\,\mathrm{d}s_m \dots \mathrm{d}s_1\,\mathrm{d}\mathbf{y}_n\,\mathrm{d}x\,\mathrm{d}t$$

for all $v \in L^2(\Omega \times (0,T); \mathcal{C}_\#(Y^n \times S_1 \times \dots \times S_m))$, and we write $u_\varepsilon \xrightarrow{(n+1,m+1)} u_0$.

Trivially, this definition also works for vector valued functions where the product becomes the dot product, or mixed scalar and vector valued functions which would give vector valued integrals above. In particular, gradient functions will later be of interest.

In order to simplify the notation, from now on we will write $\mathbf{s}_m = (s_1, \dots, s_m)$ and $S^m = S_1 \times \dots \times S_m$ so that $\mathbf{s}_m \in S^m$. (The Lebesgue measure on $S^m$ will of course be denoted $\mathrm{d}\mathbf{s}_m$.) Moreover, $\mathbf{t}_m^\varepsilon = (t/\varepsilon_1', \dots, t/\varepsilon_m')$ which depends on the particular choice of temporal scale functions $\{\varepsilon_j'\}_{j=1}^m$. Furthermore, $\Omega_T = \Omega \times (0,T)$ so that $(x,t) \in \Omega_T$, and $\mathcal{Y}_{nm} = Y^n \times S^m$ so that $(\mathbf{y}_n, \mathbf{s}_m) \in \mathcal{Y}_{nm}$.

It is clear that we need to introduce some convenient restrictions on the spatial and temporal scale functions $\{\varepsilon_i\}_{i=1}^n$ and $\{\varepsilon_j'\}_{j=1}^m$ in order for them to collaborate in a meritorious manner. In Definition 2.8 below we define a certain set of pairs of lists of such meritoriously collaborating spatial and temporal scale functions.

**Definition 2.8.** Suppose we have a list $\{\varepsilon_i\}_{i=1}^n$ of $n$ spatial scale functions and a list $\{\varepsilon'_j\}_{j=1}^m$ of $m$ temporal scale functions. We say that the pair $(\{\varepsilon_i\}_{i=1}^n, \{\varepsilon'_j\}_{j=1}^m)$ belongs to the set $\mathcal{J}_{\text{sep}}^{nm}$ if $\{\varepsilon_i\}_{i=1}^n$ and $\{\varepsilon'_j\}_{j=1}^m$ are both separated and the following two conditions hold:

(i) There exist possibly empty subsets $A \subset [\![n]\!]$ and $A' \subset [\![m]\!]$ with $|A| = |A'| = k$ such that there exist bijections $\beta\colon [\![k]\!] \to A$ and $\beta'\colon [\![k]\!] \to A'$, respectively, such that $\varepsilon_{\beta(i)} = \varepsilon'_{\beta'(i)}$ for all $i \in [\![k]\!]$. (In the empty case $k = 0$ we have no requirement.)

(ii) There exists a permutation $\pi$ on $[\![n+m-2k]\!]$ such that the permutation $\{\varepsilon''_{\pi(l)}\}_{l=1}^{n+m-2k}$ of the list $\{\varepsilon''_l\}_{l=1}^{n+m-2k} = \{\{\varepsilon_i\}_{i\notin A}, \{\varepsilon'_j\}_{j\notin A'}\}$ of the remaining $n+m-2k$ scale functions is separated. (In the empty case $n+m-2k = 0$ we have no requirement.)

If we require well-separatedness instead of mere separatedness we can define the corresponding set $\mathcal{J}_{\text{wsep}}^{nm}$.

Note that $\mathcal{J}_{\text{wsep}}^{nm} \subset \mathcal{J}_{\text{sep}}^{nm}$. The idea of the definition above is that we can localize all the spatial and temporal scale functions in two disjoint categories, (i) and (ii), where the former category consists of those that are equal and the latter category consists of those that are jointly (well-)separated. Note also that since neither $n$ nor $m$ vanishes, it can not be the case that both categories (i) and (ii) of Definition 2.8 are empty.

The details of the rather straightforward proofs of the propositions and theorems below concerning some convergence results in the multiscale setting can be found in [20] by the author which is available at the arXiv e-print database operated by Cornell University. It should be noted that the proofs are similar to the ones for the corresponding results in [6] from 2007 by Flodén and Olsson which in turn utilize the techniques employed in [8] from 1996 by Holmbom.

We have the following important compactness result.

**Theorem 2.9.** *Suppose that the pair $(\{\varepsilon_i\}_{i=1}^n, \{\varepsilon'_j\}_{j=1}^m)$ of lists of spatial and temporal scale functions belongs to $\mathcal{J}_{\text{sep}}^{nm}$. Furthermore, let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$. Then there is a function $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{nm})$ such that, up to a subsequence, $u_\varepsilon \xrightarrow{(n+1,m+1)} u_0$.*

P r o o f. See Theorem 13 in [20]. □

In the remainder of the paper, let $\mathcal{W}_k = H^1_{\#}(Y_k)/\mathbb{R}$, $k \in [\![n]\!]$. For the $(n+1, m+1)$-scale convergence of sequences of gradients we have the important Theorem 2.10 below.

**Theorem 2.10.** *Suppose that the pair* $(\{\varepsilon_i\}_{i=1}^n, \{\varepsilon'_j\}_{j=1}^m)$ *of lists of spatial and temporal scale functions belongs to* $\mathcal{J}^{nm}_{\text{wsep}}$. *Moreover, assume that* $\{u_\varepsilon\}$ *is a bounded sequence in* $H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega))$. *Then, up to a subsequence, we have*

$$\begin{aligned} u_\varepsilon &\to u \quad \text{in } L^2(\Omega_T), \\ u_\varepsilon &\rightharpoonup u \quad \text{in } L^2(0,T; H^1_0(\Omega)), \end{aligned}$$

*and*

$$\nabla u_\varepsilon \xrightarrow{(n+1,m+1)} \nabla u + \sum_{k=1}^n \nabla_{y_k} u_k,$$

*where* $u \in L^2(0,T; H^1_0(\Omega))$ *and* $u_k \in L^2(\Omega_T \times \mathcal{Y}_{(k-1)m}; \mathcal{W}_k)$ *for all* $k \in [\![n]\!]$.

Proof. See Theorem 18 in [20]. □

When performing the homogenization later in this paper we will limit ourselves to two spatial scales, $n = 1$, where the microscale is described by the single spatial scale function $\varepsilon_1$. The scale function $\varepsilon_1$ is, without loss of generality, assumed to coincide with the scale parameter, i.e., $\varepsilon_1(\varepsilon) = \varepsilon$. Note that in what follows, the list $\{\varepsilon\}$ of the single spatial scale function will be written as $\varepsilon$ for brevity. In the remainder of the paper, let $\mathcal{W} = H^1_{\#}(Y)/\mathbb{R}$. In this setting we have Theorem 2.11 below.

**Theorem 2.11.** *Suppose that the pair* $(\varepsilon, \{\varepsilon'_i\}_{i=1}^m)$ *of lists of spatial and temporal scale functions belongs to* $\mathcal{J}^{1m}_{\text{wsep}}$ *and assume that* $\{u_\varepsilon\}$ *is a bounded sequence in* $H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega))$. *Then, up to a subsequence,*

$$\begin{aligned} &\int_{\Omega_T} \frac{1}{\varepsilon} u_\varepsilon(x,t) \varphi\Big(x, t, \frac{x}{\varepsilon}, \mathbf{t}^\varepsilon_m\Big) \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad \to \int_{\Omega_T} \int_{\mathcal{Y}_{1m}} u_1(x,t,y,\mathbf{s}_m) \varphi(x,t,y,\mathbf{s}_m) \, \mathrm{d}\mathbf{s}_m \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

*for all* $\varphi \in \mathcal{D}(\Omega) \odot \mathcal{D}(0,T) \odot (\mathcal{C}^\infty_{\#}(Y)/\mathbb{R}) \odot \prod_{i=1}^m \mathcal{C}^\infty_{\#}(S_i)$, *where* $u_1 \in L^2(\Omega_T \times S^m; \mathcal{W})$ *is as in Theorem* 2.10 *with* $n = 1$.

Proof. See Theorem 20 in [20]. □

R e m a r k 2.12. Theorem 2.11 is a mere varant of Lemma 3.1 in [19] in the special case of periodicity but generalized to include many temporal scales. In its turn, the result in [19] is a mere variation of Corollary 3.3 in [9] generalized to the non-periodic setting and with the sequence $\{\varepsilon^{-1}u_\varepsilon\}$ (as in Theorem 2.11 above) instead of the slightly more complicated sequence $\{\varepsilon^{-1}(u_\varepsilon - u)\}$ found in [8], [9].

The convergence mode in Theorem 2.11 can be regarded as a kind of feeble, or "very weak", $(2, m+1)$-scale convergence of $\{\varepsilon^{-1}u_\varepsilon\}$ since the heavily restricted set of test functions in question is more permissible compared to the larger set of test functions employed in ordinary $(2, m+1)$-scale convergence.

**2.2. H-convergence of monotone parabolic problems.** In 1967 Spagnolo introduced the notion of G-convergence for linear problems governed by symmetric matrices (see [21]). The name "G"-convergence comes from the fact that this convergence mode corresponds to the convergence of the Green functions associated to the sequence of problems. The G-convergence of symmetric matrices is defined via the weak convergence of solutions to the sequence of problems.

The concept of H-convergence—"H" as in "homogenization"—is a generalization of Spagnolo's G-convergence to cover also non-symmetric matrices. It was introduced in 1976 by Tartar (see [24]) and further developed by Murat in 1978 (see [15]), and in 1977 Tartar defined H-convergence for non-linear monotone problems. Early studies of H-convergence for non-linear monotone parabolic problems were conducted by Kun'ch and Pankov in 1986 (see [12]) and Svanstedt in 1992 (see [22]).

We introduce a convenient set of flux functions in the following definition.

**Definition 2.13.** A function $a\colon \Omega_T \times \mathbb{R}^N \to \mathbb{R}^N$ is said to belong to $\mathcal{M}(\Omega_T)$ if the following four structure conditions are satisfied for some $C_0, C_1 > 0$ and $0 < \alpha \leqslant 1$:

- $a(x,t;0) = 0$ a.e. $(x,t) \in \Omega_T$;
- $a(\cdot;k)$ is (Lebesgue) measurable for every $k \in \mathbb{R}^N$;
- $(a(x,t;k) - a(x,t;k'))\cdot(k-k') \geqslant C_0|k-k'|^2$ a.e. $(x,t) \in \Omega_T$ and for all $k, k' \in \mathbb{R}^N$;
- $|a(x,t;k) - a(x,t;k')| \leqslant C_1(1+|k|+|k'|)^{1-\alpha}|k-k'|^\alpha$ a.e. $(x,t) \in \Omega_T$ and for all $k, k' \in \mathbb{R}^N$.

The important concept of H-convergence of monotone parabolic problems—coined $\text{H}_{\text{MP}}$-convergence in this paper for brevity—is introduced in the definition below.

**Definition 2.14.** Suppose $\{a^\varepsilon\}$ is a sequence of fluxes in $\mathcal{M}(\Omega_T)$. We say that $\{a^\varepsilon\}$ $\text{H}_{\text{MP}}$-converges to the flux $b \in \mathcal{M}(\Omega_T)$ if, for any $f \in L^2(0,T;H^{-1}(\Omega))$ and any $u^0 \in L^2(\Omega)$, the weak solutions $u_\varepsilon \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ to the sequence

$$
(2.1) \qquad \begin{cases} \dfrac{\partial}{\partial t} u_\varepsilon(x,t) - \nabla \cdot a^\varepsilon(x,t;\nabla u_\varepsilon) = f(x,t) & \text{in } \Omega_T, \\ u_\varepsilon(x,0) = u^0(x) & \text{in } \Omega, \\ u_\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \end{cases}
$$

of evolution problems satisfy

$$
\begin{cases} u_\varepsilon \rightharpoonup u & \text{in } L^2(0,T; H_0^1(\Omega)), \\ a^\varepsilon(\,\cdot\,; \nabla u_\varepsilon) \rightharpoonup b(\,\cdot\,; \nabla u) & \text{in } L^2(\Omega_T)^N, \end{cases}
$$

where $u \in H^1(0,T; H_0^1(\Omega), H^{-1}(\Omega))$ is the weak unique solution to the evolution problem

$$
(2.2) \qquad \begin{cases} \dfrac{\partial}{\partial t} u(x,t) - \nabla \cdot b(x,t;\nabla u) = f(x,t) & \text{in } \Omega_T, \\ u(x,0) = u^0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}
$$

Moreover, for brevity, we write this convergence $a^\varepsilon \xrightarrow{\mathrm{H_{MP}}} b$, and $b$ is called the $\mathrm{H_{MP}}$-limit of $\{a^\varepsilon\}$.

It is Definition 2.14 above that demarcates what we mean by homogenizing a problem. Let us introduce the following five structure conditions on the function $a\colon \overline{\Omega}_T \times \mathbb{R}^{nN+m} \times \mathbb{R}^N$:

(I) $a(x,t,\mathbf{y}_n,\mathbf{s}_m;0) = 0$ for all $(x,t) \in \overline{\Omega}_T$ and all $(\mathbf{y}_n,\mathbf{s}_m) \in \mathbb{R}^{nN+m}$;

(II) $a(x,t,\cdot\,;k)$ is $\mathcal{Y}_{nm}$-periodic for all $(x,t) \in \overline{\Omega}_T$ and all $k \in \mathbb{R}^N$, and $a(\cdot\,;k)$ is continuous for all $k \in \mathbb{R}^N$;

(III) $a(x,t,\mathbf{y}_n,\mathbf{s}_m;\cdot)$ is continuous for all $(x,t) \in \overline{\Omega}_T$ and all $(\mathbf{y}_n,\mathbf{s}_m) \in \mathbb{R}^{nN+m}$;

(IV) there exists $C_0 > 0$ such that

$$
(a(x,t,\mathbf{y}_n,\mathbf{s}_m;k) - a(x,t,\mathbf{y}_n,\mathbf{s}_m;k')) \cdot (k - k') \geqslant C_0 |k - k'|^2
$$

for all $(x,t) \in \Omega_T$, all $(\mathbf{y}_n,\mathbf{s}_m) \in \mathbb{R}^{nN+m}$ and all $k,k' \in \mathbb{R}^N$;

(V) there exist $C_1 > 0$ and $0 < \alpha \leqslant 1$ such that

$$
|a(x,t,\mathbf{y}_n,\mathbf{s}_m;k) - a(x,t,\mathbf{y}_n,\mathbf{s}_m;k')| \leqslant C_1(1 + |k| + |k'|)^{1-\alpha} |k - k'|^\alpha
$$

for all $(x,t) \in \Omega_T$, all $(\mathbf{y}_n,\mathbf{s}_m) \in \mathbb{R}^{nN+m}$ and all $k,k' \in \mathbb{R}^N$.

We introduce the oscillating fluxes

$$a^{\varepsilon}(x,t;k) = a(x,t,\mathbf{x}_n^{\varepsilon},\mathbf{t}_m^{\varepsilon};k), \quad (x,t) \in \Omega_T, \ \ k \in \mathbb{R}^N. \tag{2.3}$$

Below we have a proposition governing some a priori estimates on the solutions to the sequence of evolution problems.

**Proposition 2.15.** *Suppose that* $a\colon \overline{\Omega}_T \times \mathbb{R}^{nN+m} \times \mathbb{R}^N \to \mathbb{R}^N$ *fulfils the structure conditions* (I)–(V). *Then the sequence* $\{u_\varepsilon\}$ *of weak solutions to the evolution problem* 2.1 *with* $\{a^\varepsilon\}$ *defined through* (2.3) *is uniformly bounded in* $H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$.

Proof. See Proposition 31 in [20]. □

Remark 2.16. The problem (2.1) with $\{a^\varepsilon\}$ defined through (2.3) is the same as (1.1) but generalized to $n+1$ spatial scales. Note that the weak formulation of (2.1) is that, given $f \in X' = L^2(0,T;H^{-1}(\Omega))$ and $u^0 \in L^2(\Omega)$, we want to find $u_\varepsilon \in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ such that

$$\begin{aligned}\Big\langle \frac{\partial}{\partial t} u_\varepsilon, v\Big\rangle_{X',X} &+ \int_{\Omega_T} a(x,t,\mathbf{x}_n^{\varepsilon},\mathbf{t}_m^{\varepsilon};\nabla u_\varepsilon)\cdot\nabla v(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T} f(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.4}$$

for all $v \in X = L^2(0,T;H_0^1(\Omega))$.

## 3. Homogenization

In this section we derive some homogenization results for monotone parabolic problems with several temporal scales.

**3.1. Historical background.** The notion of homogenization of problems with multiple microscales was introduced in 1978 by Bensoussan, Lions and Papanicolaou (see [3]) who homogenized problems with two microscales characterized by the list $\{\varepsilon,\varepsilon^2\}$ of scale functions. In 1996, Allaire and Briane (see [2]) succeeded to generalize this to homogenization of linear elliptic problems with an arbitrary number of microscales—even infinitely many—without even assuming the scale functions to be power functions using the notion of (well-)separatedness instead. This was achieved by introducing the multiscale convergence technique. In 2001, Lions, Lukkassen, Persson, and Wall performed homogenization of non-linear monotone elliptic problems with scale functions $\{\varepsilon, \varepsilon^2\}$ (see [13]), and in 2005 Holmbom, Svanstedt, and Wellander studied homogenization of linear parabolic problems with pairs $(\{\varepsilon, \varepsilon^2\}, \varepsilon^k)$ of lists of scale functions (see [11]). In 2006, Flodén and Olsson generalized to monotone parabolic problems (see [5]; see also [7] by Flodén, Olsson, Holmbom, and Svanstedt for a related study from 2007), and in 2007 Flodén and Olsson achieved homogenization results for linear parabolic problems involving pairs $(\varepsilon, \{\varepsilon, \varepsilon^k\})$ of lists of scale functions (see [6]); this was actually the first time homogenization was performed for problems with more than one temporal microscale. In 2009, Woukeng studied non-linear non-monotone degenerated parabolic problems with the pair $(\varepsilon, \{\varepsilon, \varepsilon^k\})$ of lists of spatial and temporal scale functions (see [25]).

This paper deals with monotone parabolic problems with an arbitrary number of temporal microscales not necessarily characterized by scale functions in the form of power functions but instead using the concept of (well-)separatedness in spirit of [2]. Furthermore—for simplicity—we only consider two spatial scales of which one is microscopical, i.e., henceforth we fix $n = 1$.

**3.2. A special mutually disjoint collection of sets.** Let $k \in [\![m]\!]$. Define $\mathcal{J}_{\text{wsep}}^{m\sim k}$ to be the set of all pairs $(\varepsilon, \{\varepsilon'_j\}_{j=1}^{m})$ in $\mathcal{J}_{\text{wsep}}^{1m}$ such that $\varepsilon'_k \sim \varepsilon$, i.e., $\varepsilon'_k$ asymptotically equals $\varepsilon$; recall that this means that for some $q \to 1$, $\varepsilon'_k = q\varepsilon$. There is no loss of generality to assume mere asymptotic equality rather than the ostensibly more general asymptotic equality modulo a positive constant, i.e., $\varepsilon'_k \sim C\varepsilon$, $C \in \mathbb{R}$. In other words, $\mathcal{J}_{\text{wsep}}^{m\sim k}$ consists of pairs $(\varepsilon, \{\varepsilon'_j\}_{j=1}^{m})$ for which the temporal scale functions are separated and the $k$th temporal scale function coincides asymptotically with the spatial scale function. This clearly explains the convenient notation "$\sim k$" which could be read "the spatial scale is asymptotically equal to the $k$th temporal scale".

Define the collection $\{\mathcal{J}_{\text{wsep},i}^{m\sim k}\}_{i=1}^{1+2(m-k)}$ of $1 + 2(m-k)$ subsets of $\mathcal{J}_{\text{wsep}}^{m\sim k}$ by

$$\mathcal{J}_{\text{wsep},1}^{m\sim k} = \left\{(\varepsilon, \{\varepsilon'_j\}_{j=1}^{m}) \in \mathcal{J}_{\text{wsep}}^{m\sim k} \colon \frac{\varepsilon^2}{\varepsilon'_m} \to 0\right\},$$

$$\mathcal{J}_{\text{wsep},2}^{m\sim k} = \left\{(\varepsilon, \{\varepsilon'_j\}_{j=1}^{m}) \in \mathcal{J}_{\text{wsep}}^{m\sim k} \colon \varepsilon'_m \sim \varepsilon^2\right\},$$

$$\mathcal{J}_{\text{wsep},2+i-k}^{m\sim k} = \left\{(\varepsilon, \{\varepsilon'_j\}_{j=1}^{m}) \in \mathcal{J}_{\text{wsep}}^{m\sim k} \colon \frac{\varepsilon'_i}{\varepsilon^2} \to 0 \text{ but } \frac{\varepsilon'_{i-1}}{\varepsilon^2} \to \infty\right\},$$

and

$$\mathcal{J}_{\text{wsep},1+m+i^\circ-2k}^{m\sim k} = \left\{(\varepsilon, \{\varepsilon'_j\}_{j=1}^{m}) \in \mathcal{J}_{\text{wsep}}^{m\sim k} \colon \varepsilon'_{i^\circ-1} \sim \varepsilon^2\right\}$$

for $i \in [\![k+1, m]\!]$ and $i^\circ \in [\![k+2, m]\!]$. Note that if $k = m$, the collection of subsets of $\mathcal{J}_{\text{wsep}}^{m\sim m}$ reduces to merely $\{\mathcal{J}_{\text{wsep},1}^{m\sim m}\}$. The sets $\mathcal{J}_{\text{wsep},1}^{m\sim k}$, $\mathcal{J}_{\text{wsep},2}^{m\sim k}$ and the collections

$\{\mathcal{J}^{m\sim k}_{\mathrm{wsep},2+i-k}\}_{i=k+1}^{m}$ and $\{\mathcal{J}^{m\sim k}_{\mathrm{wsep},1+m+i^{\circ}-2k}\}_{i^{\circ}=k+2}^{m}$ correspond to slow temporal oscillations, slow resonance (i.e., "slow" self-similar case), rapid temporal oscillations and rapid resonance (i.e., "rapid" self-similar case), respectively.

Remark 3.1. It can be shown that the collection $\{\mathcal{J}^{m\sim k}_{\mathrm{wsep},i}\}_{i=1}^{1+2(m-k)}$ of $1+2(m-k)$ subsets of $\mathcal{J}^{m\sim k}_{\mathrm{wsep}}$ is mutually disjoint for every $k\in[\![m]\!]$; see Theorem 32 in [20].

Remark 3.2. In the "classical" case of temporal scale functions expressed as power functions it can be proven that one does not merely have mutual disjointness but that the subsets form a partition of the universe of "classical" lists. For more details, see Proposition 33 in [20].

Example 3.3. Let $m=5$ and $k=3$, giving the collection $\{\mathcal{J}^{5\sim 3}_{\mathrm{wsep},j}\}_{j=1}^{5}$ of five subsets of $\mathcal{J}^{5\sim 3}_{\mathrm{wsep}}$, and consider the following six pairs of lists of scale functions:

$$
\begin{aligned}
e_1 &= \Big\{\varepsilon, \Big\{\varepsilon^{0.2}, \varepsilon^{0.5}, \varepsilon, \varepsilon^{1.2}, \frac{\varepsilon^{1.5}}{|\log\varepsilon|}\Big\}\Big\},\\
e_2 &= \Big\{\varepsilon, \Big\{\varepsilon^{0.2}, \frac{\varepsilon^{0.5}}{|\log\varepsilon|}, \varepsilon, \frac{\varepsilon^{1.2}}{|\log\varepsilon|}, \varepsilon^{2}\Big\}\Big\},\\
e_3 &= \Big\{\varepsilon, \Big\{\varepsilon^{0.2}, \varepsilon^{0.5}, \varepsilon, \frac{\varepsilon^{2.5}}{|\log\varepsilon|}, \frac{\varepsilon^{3}}{|\log\varepsilon|}\Big\}\Big\},\\
e_4 &= \Big\{\varepsilon, \Big\{\varepsilon^{0.2}, \varepsilon^{0.5}, \varepsilon, \frac{\varepsilon^{1.5}}{|\log\varepsilon|}, \varepsilon^{2.5}\Big\}\Big\},\\
e_5 &= \Big\{\varepsilon, \Big\{\frac{\varepsilon^{0.2}}{|\log\varepsilon|}, \varepsilon^{0.5}, \varepsilon, \varepsilon^{2}, \frac{\varepsilon^{2.5}}{|\log\varepsilon|}\Big\}\Big\},\\
e_6 &= \Big\{\varepsilon, \Big\{\varepsilon^{0.2}, \varepsilon^{0.5}, \varepsilon, \frac{\varepsilon^{2}}{|\log\varepsilon|}, \varepsilon^{2.5}\Big\}\Big\}.
\end{aligned}
$$

We see that $e_j\in\mathcal{J}^{5\sim 3}_{\mathrm{wsep},j}$ for every $j\in[\![5]\!]$ but that $e_6$ does not belong to any $\mathcal{J}^{5\sim 3}_{\mathrm{wsep},j}$, $j\in[\![5]\!]$, even though it belongs to $\mathcal{J}^{5\sim 3}_{\mathrm{wsep}}$.

**3.3. The homogenization procedure.** Let $S=(0,1)$ and define $H^1_{\#}(S;V,V')$, $V$ being any Banach space with topological dual $V'$, as the space of functions $u$ satisfying $u\in L^2_{\#}(S;V)$ and $(\mathrm{d}/\mathrm{d}s)u\in L^2_{\#}(S;V')$. In order to prove Theorem 3.6—our preliminary homogenization result—we first need the lemmas below.

**Lemma 3.4.** *The tensor product space $(\mathcal{C}^{\infty}_{\#}(Y)/\mathbb{R})\otimes\mathcal{C}^{\infty}_{\#}(S)$ is dense in the space $H^1_{\#}(S;\mathcal{W},\mathcal{W}')$.*

Proof. This is just Proposition 4.6 in [19] in which $\mathcal{E}$ and $\mathcal{V}$ correspond to the present paper's $(\mathcal{C}^{\infty}_{\#}(Y)/\mathbb{R})\otimes\mathcal{C}^{\infty}_{\#}(S)$ and $H^1_{\#}(S;\mathcal{W},\mathcal{W}')$, respectively. □

**Lemma 3.5.** *Suppose that $u, v \in H^1_{\#}(S; \mathcal{W}, \mathcal{W}')$. Then*

$$\langle \partial_s u, v\rangle_{L^2_{\#}(S;\mathcal{W}'), L^2_{\#}(S;\mathcal{W})} + \langle \partial_s v, u\rangle_{L^2_{\#}(S;\mathcal{W}'), L^2_{\#}(S;\mathcal{W})} = 0$$

*holds. In particular,*

$$\langle \partial_s u, u\rangle_{L^2_{\#}(S;\mathcal{W}'), L^2_{\#}(S;\mathcal{W})} = 0.$$

Proof. This follows immediately from Corollary 4.1 in [19]. □

Introduce the following notation. We write $S^{[\![j_1,j_2]\!]} = S_{j_1} \times \ldots \times S_{j_2}$ and let $\mathbf{s}_{[\![j_1,j_2]\!]} \in S^{[\![j_1,j_2]\!]}$ be the corresponding local variable. The Lebesgue measures on the introduced local set is written accordingly.

Let us now state the theorem.

**Theorem 3.6.** *Let $k \in [\![m]\!]$. Suppose that the pair $e = (\varepsilon, \{\varepsilon'_j\}_{j=1}^m)$ of lists of spatial and temporal scale functions belongs to $\bigcup_{i=1}^{1+2(m-k)} \mathcal{J}^{m\sim k}_{\mathrm{wsep},i}$. Let $\{u_\varepsilon\}$ be the sequence of weak solutions in $H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega))$ to the evolution problem* (1.1) *with $a\colon \overline{\Omega}_T \times \mathbb{R}^{N+m} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the structure conditions* (I)–(V). *Then*

$$u_\varepsilon \to u \quad \text{in } L^2(\Omega_T), \tag{3.1}$$

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^2(0,T; H^1_0(\Omega)), \tag{3.2}$$

*and*

$$\nabla u_\varepsilon \xrightarrow{(2,m+1)} \nabla u + \nabla_y u_1, \tag{3.3}$$

*where $u \in H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega))$ and $u_1 \in L^2(\Omega_T \times S^m; \mathcal{W})$. Here $u$ is the unique weak solution to the homogenized problem* (2.2) *with the homogenized flux*

$$b\colon \overline{\Omega}_T \times \mathbb{R}^N \to \mathbb{R}^N$$

*given by*

$$b(x,t;\nabla u) = \int_{\mathcal{Y}_{1m}} a(x,t,y,\mathbf{s}_m; \nabla u + \nabla_y u_1)\,\mathrm{d}\mathbf{s}_m\,\mathrm{d}y. \tag{3.4}$$

*Moreover, we have the following characterization of $u_1$:*

- *If $e \in \mathcal{J}^{m\sim k}_{\mathrm{wsep},1}$ then the function $u_1$ is the unique weak solution to the local problem*

$$-\nabla_y \cdot a(x,t,y,\mathbf{s}_m; \nabla u + \nabla_y u_1) = 0. \tag{3.5}$$

- *If* $e \in \mathcal{J}^{m\sim k}_{\mathrm{wsep},2}$, *assuming* $u_1 \in L^2(\Omega_T \times S^{m-1}; H^1_\#(S_m;\mathcal{W},\mathcal{W}'))$, *then the function* $u_1$ *is the unique weak solution to the local problem*

$$\partial_{s_m} u_1(x,t,y,\mathbf{s}_m) - \nabla_y \cdot a(x,t,y,\mathbf{s}_m;\nabla u + \nabla_y u_1) = 0. \tag{3.6}$$

- *If* $e \in \mathcal{J}^{m\sim k}_{\mathrm{wsep},2+\bar{l}-k}$ *for some* $\bar{l} \in [\![k+1,m]\!]$, *provided* $k \in [\![m-1]\!]$, *then the function* $u_1$ *is the unique weak solution to the system of local problems*

$$\begin{cases} -\nabla_y \cdot \int_{S^{[\![\bar{l},m]\!]}} a(x,t,y,\mathbf{s}_m;\nabla u + \nabla_y u_1)\,\mathrm{d}\mathbf{s}_{[\![\bar{l}^\circ,m]\!]} = 0, \\ \quad \forall\, i \in [\![\bar{l}^\circ,m]\!] \quad \partial_{s_i} u_1(x,t,y,\mathbf{s}_m) = 0. \end{cases} \tag{3.7}$$

- *If* $e \in \mathcal{J}^{m\sim k}_{\mathrm{wsep},1+m+l^\circ-2k}$ *for some* $l^\circ \in [\![k+2,m]\!]$, *provided* $k \in [\![m-2]\!]$ *and assuming* $u_1 \in L^2(\Omega_T \times S^{l^\circ-2} \times S^{[\![l^\circ,m]\!]}; H^1_\#(S_{l^\circ-1};\mathcal{W},\mathcal{W}'))$, *then the function* $u_1$ *is the unique weak solution to the system of local problems*

$$\begin{cases} \partial_{s_{l^\circ-1}} u_1(x,t,y,\mathbf{s}_m) - \nabla_y \cdot \int_{S^{[\![l^\circ,m]\!]}} a(x,t,y,\mathbf{s}_m;\nabla u + \nabla_y u_1)\,\mathrm{d}\mathbf{s}_{[\![l^\circ,m]\!]} = 0, \\ \quad \forall\, i \in [\![l^\circ,m]\!] \quad \partial_{s_i} u_1(x,t,y,\mathbf{s}_m) = 0. \end{cases} \tag{3.8}$$

P r o o f. Since $a$ fulfils (I)–(V) we can use Proposition 2.15 for the sequence $\{u_\varepsilon\}$ of weak solutions; we have ensured uniform boundedness in $H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega))$. We can then employ Theorem 2.10 with $n=1$ obtaining, up to a subsequence, exactly the claimed convergences (3.1)–(3.3) where $u \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega))$ and $u_1 \in L^2(\Omega_T \times S^m;\mathcal{W})$. Consider the sequence $\{a_\varepsilon\}$ defined according to

$$a_\varepsilon(x,t) = a^\varepsilon(x,t;\nabla u_\varepsilon) = a\Big(x,t,\frac{x}{\varepsilon},\mathbf{t}^\varepsilon_m;\nabla u_\varepsilon\Big), \quad (x,t) \in \Omega_T.$$

We have that $\{a_\varepsilon\}$ is uniformly bounded in $L^2(\Omega_T)^N$ which can be shown by using (I) and (V), the triangle inequality and Proposition 2.15; see the proof of Theorem 37 in [20] for details. By Theorem 2.9 (with $n=1$) we then know that, up to a subsequence,

$$a_\varepsilon \xrightarrow{(2,m+1)} a_0 \tag{3.9}$$

for some $a_0 \in L^2(\Omega_T \times \mathcal{Y}_{1m})^N$.

Recall the weak form (2.4) with $n=1$ of the evolution problem, i.e.,

$$\Big\langle \frac{\partial}{\partial t} u_\varepsilon, \psi \Big\rangle_{X',X} + \int_{\Omega_T} a_\varepsilon(x,t)\cdot\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{3.10}$$

for every $\psi \in L^2(0,T;H^1_0(\Omega))$.

Choose an arbitrary $\psi \in H_0^1(\Omega) \odot \mathcal{D}(0,T)$. Then we can shift the weak temporal derivative $\partial/\partial t$ in (3.10) from acting on $u_\varepsilon$ to acting on $\psi$ instead, i.e.,

$$\int_{\Omega_T} \Big(-u_\varepsilon(x,t)\frac{\partial}{\partial t}\psi(x,t) + a_\varepsilon(x,t)\cdot\nabla\psi(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.11}$$

Passing to the limit—using (3.2) and (3.9) on the first and second terms on the left-hand side, respectively, and letting $\partial/\partial t$ to act on $u$ again—we obtain, up to a subsequence,

$$\Big\langle\frac{\partial}{\partial t}u,\psi\Big\rangle_{X',X} + \int_{\Omega_T}\int_{\mathcal{Y}_{1m}} a_0(x,t,y,\mathbf{s}_m)\,\mathrm{d}\mathbf{s}_m\,\mathrm{d}y\cdot\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

for any $\psi \in L^2(0,T;H_0^1(\Omega))$ by density. We have obtained the weak form of the homogenized evolution problem (2.2) with the homogenized flux given by

$$b(x,t;\nabla u) = \int_{\mathcal{Y}_{1m}} a_0(x,t,y,\mathbf{s}_m)\,\mathrm{d}\mathbf{s}_m\,\mathrm{d}y.$$

What remains is to find the local problems for $u_1$ and to give the limit $a_0$ in terms of $a$. We will first extract the pre-local-problems, i.e., the problems expressed in terms of $a_0$ which become the local problems once $a_0$ is given in terms of $a$. Introduce arbitrary

$$\omega_l \in \mathcal{D}(\Omega)\odot\mathcal{D}(0,T)\odot(\mathcal{C}^\infty_\#(Y)/\mathbb{R})\odot\prod_{i=1}^{l}\mathcal{C}^\infty_\#(S_i), \quad l\in[\![m]\!].$$

For each $l \in [\![m]\!]$ we define

$$\omega_l^\varepsilon(x,t) = \omega_l(x,t,\mathbf{x}_{1l}^\varepsilon), \quad (x,t)\in\Omega_T.$$

Let $\{r_\varepsilon\}$ be a sequence of positive numbers such that $r_\varepsilon \to 0$. We will now study sequences of test functions $\{\psi^\varepsilon\}$ in (3.11) such that

$$\psi_l^\varepsilon(x,t) = r_\varepsilon\omega_l^\varepsilon(x,t), \quad (x,t)\in\Omega_T,$$

with appropriate choices of $\{r_\varepsilon\}$ and $l$ in order to extract the pre-local-problems. For the sequence $\{\psi_l^\varepsilon\}$, $l \in [\![m]\!]$, of test functions given above, (3.11) becomes in the limit

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-u_\varepsilon(x,t)\sum_{i=1}^{l}\frac{r_\varepsilon}{\varepsilon_i'}\,\partial_{s_i}\omega_l^\varepsilon(x,t) + a_\varepsilon(x,t)\cdot\frac{r_\varepsilon}{\varepsilon_k'}\nabla_y\omega_l^\varepsilon(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.12}$$

recalling that $\varepsilon'_k = \varepsilon$. For the pre-local-problems, (3.12) will be our point of departure.

Suppose that the real sequence $\{r_\varepsilon/\varepsilon'_l\}$ is bounded, then the limit equation becomes

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-u_\varepsilon(x,t)\frac{r_\varepsilon}{\varepsilon'_l}\,\partial_{s_l}\omega_l^\varepsilon(x,t)+a_\varepsilon(x,t)\cdot\frac{r_\varepsilon}{\varepsilon'_k}\nabla_y\omega_l^\varepsilon(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{3.13}$$

Choose $r_\varepsilon = \varepsilon'_k$, which implies that $\{r_\varepsilon/\varepsilon'_l\} = \{\varepsilon'_k/\varepsilon'_l\}$ is bounded for $l \in [\![k]\!]$. Then (3.13) becomes

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-u_\varepsilon(x,t)\frac{\varepsilon'_k}{\varepsilon'_l}\,\partial_{s_l}\omega_l^\varepsilon(x,t)+a_\varepsilon(x,t)\cdot\nabla_y\omega_l^\varepsilon(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t=0. \tag{3.14}$$

If $l \in [\![k-1]\!]$, provided $k \in [\![2,m]\!]$, the first term tends to 0, and we get in this case

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} a_\varepsilon(x,t)\cdot\nabla_y\omega_l^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t=0,$$

which after taking the limit can be written

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1m}} a_0(x,t,y,\mathbf{s}_m)\cdot\nabla_y\omega_l(x,t,y,\mathbf{s}_l)\,\mathrm{d}\mathbf{s}_m\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=0, \tag{3.15}$$

i.e.,

$$\int_{\Omega_T}\int_{\mathcal{Y}_{1l}}\int_{S^{[\![l+1,m]\!]}} a_0(x,t,y,\mathbf{s}_m)\,\mathrm{d}\mathbf{s}_{[\![l+1,m]\!]}\cdot\nabla_y\omega_l(x,t,y,\mathbf{s}_l)\,\mathrm{d}\mathbf{s}_l\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=0.$$

Suppose $v_1 \in \mathcal{C}^\infty_\#(Y)/\mathbb{R}$ is the factor of $\omega_l$ with respect to the $y$ variable. Then,

$$\int_Y\int_{S^{[\![l+1,m]\!]}} a_0(x,t,y,\mathbf{s}_m)\,\mathrm{d}\mathbf{s}_{[\![l+1,m]\!]}\cdot\nabla_y v_1(y)\,\mathrm{d}y=0 \tag{3.16}$$

a.e. on $\Omega_T \times S^l$. If $l = k$ the limit equation (3.14) instead reduces to

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}(-u_\varepsilon(x,t)\partial_{s_k}\omega_k^\varepsilon(x,t)+a_\varepsilon(x,t)\cdot\nabla_y\omega_k^\varepsilon(x,t))\,\mathrm{d}x\,\mathrm{d}t=0,$$

which in the limit becomes

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{1m}}&(-u(x,t)\partial_{s_k}\omega_k(x,t,y,\mathbf{s}_k)\\&+a_0(x,t,y,\mathbf{s}_m)\cdot\nabla_y\omega_k(x,t,y,\mathbf{s}_k))\,\mathrm{d}\mathbf{s}_m\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

The first term gives no contribution, since $\omega_k$ is $S_k$-periodic in the $s_k$ variable. Progressing like in the case $l \in [\![k-1]\!]$ we finally arrive at (3.16) which now also includes $l = k$, i.e., (3.16) holds for all $l \in [\![k]\!]$. But it is clear that (3.16) holding for $l = k$ implies that it holds also for any $l \in [\![k-1]\!]$ provided $k \in [\![2, m]\!]$. Thus, we only have to consider (3.16) for $l = k$, i.e., we have so far obtained

$$\int_Y \int_{S^{[\![k+1,m]\!]}} a_0(x, t, y, \mathbf{s}_m) \, \mathrm{d}\mathbf{s}_{[\![k+1,m]\!]} \cdot \nabla_y v_1(y) \, \mathrm{d}y = 0 \tag{3.17}$$

for all $v_1 \in \mathcal{C}_\#^\infty(Y)/\mathbb{R}$. It should be emphasized here that this equation is always true for $\mathcal{J}_{\mathrm{wsep}}^{m\sim k}$ and is not confined to any particular subset $\mathcal{J}_{\mathrm{wsep},j}^{m\sim k}$, $j \in [\![1 + 2(m-k)]\!]$.

If we study the limit equation (3.12) extracting a factor $\varepsilon^{-1}$ in the first term we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \Big( -\frac{1}{\varepsilon} \, u_\varepsilon(x, t) \sum_{i=1}^{l} \frac{r_\varepsilon \varepsilon'_k}{\varepsilon'_i} \, \partial_{s_i} \omega_l^\varepsilon(x, t) + a_\varepsilon(x, t) \cdot \frac{r_\varepsilon}{\varepsilon'_k} \nabla_y \omega_l^\varepsilon(x, t) \Big) \, \mathrm{d}x \, \mathrm{d}t = 0,$$

where we have recalled $\varepsilon'_k = \varepsilon$. Suppose that $\{r_\varepsilon \varepsilon'_k / \varepsilon'_l\}$ is bounded (in $\mathbb{R}$), it is then clear that the limit equation above reduces to

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \Big( -\frac{1}{\varepsilon} \, u_\varepsilon(x, t) \frac{r_\varepsilon \varepsilon'_k}{\varepsilon'_l} \, \partial_{s_l} \omega_l^\varepsilon(x, t) + a_\varepsilon(x, t) \cdot \frac{r_\varepsilon}{\varepsilon'_k} \nabla_y \omega_l^\varepsilon(x, t) \Big) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{3.18}$$

- Suppose $(\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}_{\mathrm{wsep},1}^{m\sim k}$. By definition this means that $(\varepsilon, \{\varepsilon'_j\}) \in \mathcal{J}_{\mathrm{wsep}}^{m\sim k}$ and $\varepsilon'^2_k / \varepsilon'_m$ must tend to zero. Consider first $\varepsilon'_m \sim \varepsilon'_k$, i.e., $k = m$. We have already extracted (3.17) which in this case, $k = m$, is merely

$$\int_Y a_0(x, t, y, \mathbf{s}_m) \cdot \nabla_y v_1(y) \, \mathrm{d}y = 0; \tag{3.19}$$

  this is the pre-local-problem. Consider now the situation $\varepsilon'_m \not\sim \varepsilon'_k$, i.e., $k \in [\![m-1]\!]$ requiring $m > 1$. We first note that we have already extracted (3.17). We want to employ (3.18) for $l \in [\![k+1, m]\!]$. Choose $r_\varepsilon = \varepsilon'_k$, and we get that

$$\frac{r_\varepsilon \varepsilon'_k}{\varepsilon'_l} = \frac{\varepsilon'^2_k}{\varepsilon'_l} = \frac{\varepsilon'^2_k}{\varepsilon'_m} \frac{\varepsilon'_m}{\varepsilon'_l} \to 0. \tag{3.20}$$

  We can now use (3.18) yielding

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \Big( -\frac{1}{\varepsilon} \, u_\varepsilon(x, t) \frac{\varepsilon'^2_k}{\varepsilon'_l} \, \partial_{s_l} \omega_l^\varepsilon(x, t) + a_\varepsilon(x, t) \cdot \nabla_y \omega_l^\varepsilon(x, t) \Big) \, \mathrm{d}x \, \mathrm{d}t = 0,$$

which in the limit becomes (3.15) due to Theorem 2.11 and (3.20). Hence, we have again (3.16) but for $l \in [\![k+1, m]\!]$. Apparently we end up at the pre-local-problem (3.19) again since (3.16) in the case $l = m$ implies that (3.16) holds automatically for any $l \in [\![m-1]\!]$.

- Suppose $(\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}_{\mathrm{wsep},2}^{m\sim k}$. By definition this means that $(\varepsilon, \{\varepsilon'_j\}) \in \mathcal{J}_{\mathrm{wsep}}^{m\sim k}$ and $\varepsilon'_m \sim \varepsilon'^2_k$. Let $l = m$ in (3.18). Choose $r_\varepsilon = \varepsilon'_k$ again, giving $r_\varepsilon \varepsilon'_k / \varepsilon'_l = \varepsilon'^2_k / \varepsilon'_m$. We can then write (3.18) as

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \Big( -\frac{1}{\varepsilon}\, u_\varepsilon(x,t) \frac{\varepsilon'^2_k}{\varepsilon'_m}\, \partial_{s_l} \omega_l^\varepsilon(x,t) + a_\varepsilon(x,t) \cdot \nabla_y \omega_l^\varepsilon(x,t) \Big)\, \mathrm{d}x\, \mathrm{d}t = 0,$$

and by Theorem 2.11 together with the assumption $\varepsilon'_m \sim \varepsilon'^2_k$ we arrive at the pre-local-problem, a.e. on $\Omega_T \times S^{m-1}$,

$$\int_Y \int_{S_m} (-u_1(x,t,y,\mathbf{s}_m) v_1(y) \partial_{s_m} c_m(s_m) \\ + a_0(x,t,y,\mathbf{s}_m) \cdot \nabla_y v_1(y) c_m(s_m))\, \mathrm{d}s_m\, \mathrm{d}y = 0$$

for all $v_1 \in \mathcal{C}_\#^\infty(Y)/\mathbb{R}$ and all $c_m \in \mathcal{C}_\#^\infty(S_m)$.

- Suppose $(\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}_{\mathrm{wsep},2+\bar{l}-k}^{m\sim k}$ for some $\bar{l} \in [\![k+1, m]\!]$ where $k \in [\![m-1]\!]$ is required. By definition this means that $(\varepsilon, \{\varepsilon'_j\}) \in \mathcal{J}_{\mathrm{wsep}}^{m\sim k}$ and $\varepsilon'_{\bar{l}}/\varepsilon'^2_k \to 0$ but $\varepsilon'_{\bar{l}-1}/\varepsilon'^2_k \to \infty$. We first note that we have already extracted (3.17) which at this point carries at least one integral, independent of $\bar{l}$. Choose $r_\varepsilon = \varepsilon'_i/\varepsilon'_k$, $i \in [\![\bar{l}, m]\!]$. Apparently, $r_\varepsilon \to 0$ is guaranteed since $i \in [\![k+1, m]\!]$. Trivially, $\{r_\varepsilon \varepsilon'_k/\varepsilon'_i\}$ is bounded. Finally,

$$\frac{r_\varepsilon}{\varepsilon'_k} = \frac{\varepsilon'_i}{\varepsilon'_{\bar{l}}} \frac{\varepsilon'_{\bar{l}}}{\varepsilon'^2_k} \to 0 \tag{3.21}$$

by assumption and separatedness. Hence, we can utilize (3.18) with $l = i$ giving

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \Big( -\frac{1}{\varepsilon}\, u_\varepsilon(x,t) \partial_{s_i} \omega_i^\varepsilon(x,t) + a_\varepsilon(x,t) \cdot \frac{\varepsilon'_i}{\varepsilon'^2_k} \nabla_y \omega_i^\varepsilon(x,t) \Big)\, \mathrm{d}x\, \mathrm{d}t = 0,$$

and taking the limit by using Theorem 2.11 together with (3.21) we arrive at the pre-local-problem

$$-\int_{S_i} u_1(x,t,y,\mathbf{s}_m) \partial_{s_i} c_i(s_i)\, \mathrm{d}s_i = 0 \quad \text{for all } c_i \in \mathcal{C}_\#^\infty(S_i),\ \ i \in [\![\bar{l}, m]\!]. \tag{3.22}$$

Choose now $r_\varepsilon = \varepsilon'_k$ and let $i \in [\![k+1, \bar{l}-1]\!]$ which requires $\bar{l} \in [\![k+2, m]\!]$, which in turn requires $k \in [\![m-2]\!]$. Then, by assumption and separatedness,

$$\frac{r_\varepsilon \varepsilon'_k}{\varepsilon'_i} = \frac{\varepsilon'^2_k}{\varepsilon'_{\bar{l}-1}} \frac{\varepsilon'_{\bar{l}-1}}{\varepsilon'_i} \to 0. \tag{3.23}$$

We have shown that we can employ (3.18) with $l = i$, giving

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-\frac{1}{\varepsilon}\,u_\varepsilon(x,t)\frac{\varepsilon_k'^2}{\varepsilon_{\bar{l}-1}'}\frac{\varepsilon_{\bar{l}-1}'}{\varepsilon_i'}\,\partial_{s_i}\omega_i^\varepsilon(x,t)+a_\varepsilon(x,t)\cdot\nabla_y\omega_i^\varepsilon(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t=0$$

for $i \in [\![k+1, \bar{l}-1]\!]$. Taking the limit by using Theorem 2.11 and (3.23), we get the second pre-local-problem

$$\int_Y\int_{S^{[\![\bar{l},m]\!]}} a_0(x,t,y,\mathbf{s}_m)\,\mathrm{d}\mathbf{s}_{[\![\bar{l},m]\!]}\cdot\nabla_y v_1(y)\,\mathrm{d}y=0, \tag{3.24}$$

since the case $\bar{l} = k+1$ is taken care of by (3.17). The extracted pre-local-problems are (3.22) and (3.24) in this case.

- Suppose $(\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}^{m\sim k}_{\mathrm{wsep},1+m+l^\circ-2k}$ for some $l^\circ \in [\![k+2, m]\!]$ where $k \in [\![m-2]\!]$ is required. By definition this means that $(\varepsilon, \{\varepsilon'_j\}) \in \mathcal{J}^{m\sim k}_{\mathrm{wsep}}$ and $\varepsilon'_{l^\circ-1} \sim \varepsilon'^2_k$. Choose $r_\varepsilon = \varepsilon'_i/\varepsilon'_k$, $i \in [\![l^\circ, m]\!]$. It is clearly guaranteed that $r_\varepsilon \to 0$, since $i \in [\![k+2, m]\!]$. Moreover, it is trivial that $\{r_\varepsilon\varepsilon'_k/\varepsilon'_i\}$ is bounded. Finally,

$$\frac{r_\varepsilon}{\varepsilon_k'}=\frac{\varepsilon_i'}{\varepsilon_k'^2}=\frac{\varepsilon_i'}{\varepsilon_{l^\circ-1}'}\frac{\varepsilon_{l^\circ-1}'}{\varepsilon_k'^2}\to 0 \tag{3.25}$$

by assumption and separatedness. Hence, we can utilize (3.18) with $l = i$ giving

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-\frac{1}{\varepsilon}\,u_\varepsilon(x,t)\partial_{s_i}\omega_i^\varepsilon(x,t)+a_\varepsilon(x,t)\cdot\frac{\varepsilon_i'}{\varepsilon_k'^2}\nabla_y\omega_i^\varepsilon(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t=0,$$

and taking the limit by using Theorem 2.11 and (3.25), we arrive at the pre-local-problem

$$-\int_{S_i} u_1(x,t,y,\mathbf{s}_m)\partial_{s_i}c_i(s_i)\,\mathrm{d}s_i=0\quad\text{for all } c_i\in\mathcal{C}^\infty_\#(S_i),\quad i\in[\![l^\circ,m]\!]. \tag{3.26}$$

In particular, the $\mathbf{s}_{[\![l^\circ,m]\!]}$ independence property (3.26) implies that

$$\int_{S^{[\![l^\circ,m]\!]}} u_1(x,t,y,\mathbf{s}_m)\,\mathrm{d}\mathbf{s}_{[\![l^\circ,m]\!]}=u_1(x,t,y,\mathbf{s}_m) \tag{3.27}$$

holds a.e. on $\Omega_T \times Y \times S^m$. For the second pre-local-problem, choose $r_\varepsilon = \varepsilon'_k$ and let $i = l^\circ - 1$. Then, by assumption,

$$\frac{r_\varepsilon\varepsilon_k'}{\varepsilon_i'}=\frac{\varepsilon_k'^2}{\varepsilon_{l^\circ-1}'}\to 1. \tag{3.28}$$

We have shown that we can employ (3.18), giving

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-\frac{1}{\varepsilon}\,u_\varepsilon(x,t)\frac{\varepsilon_k'^2}{\varepsilon_{l^\circ-1}'}\,\partial_{s_{l^\circ-1}}\omega_{l^\circ-1}^\varepsilon(x,t)+a_\varepsilon(x,t)\cdot\nabla_y\omega_{l^\circ-1}^\varepsilon(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t=0.$$

Taking the limit by using Theorem 2.11 and (3.28) and then utilizing (3.27), we get

$$\begin{aligned}\text{(3.29)}\quad &\int_Y\int_{S_{l^\circ-1}}\Big(-u_1(x,t,y,\mathbf{s}_m)v_1(y)\partial_{s_{l^\circ-1}}c_{l^\circ-1}(s_{l^\circ-1})\\&\qquad+\int_{S^{[\![l^\circ,m]\!]}}a_0(x,t,y,\mathbf{s}_m)\,\mathrm{d}\mathbf{s}_{[\![l^\circ,m]\!]}\nabla_y v_1(y)c_{l^\circ-1}(s_{l^\circ-1})\Big)\,\mathrm{d}s_{l^\circ-1}\,\mathrm{d}y=0\end{aligned}$$

a.e. on $\Omega_T\times S^{l^\circ-2}\times S^{[\![l^\circ,m]\!]}$, which is our second pre-local-problem. Concluding the present case, the extracted pre-local-problems are (3.26) and (3.29).

What is left to do is to characterize $a_0$ in terms of $a$ such that the pre-local-problems become true local problems, i.e., to show that

$$\text{(3.30)}\qquad a_0(x,t,y,\mathbf{s}_m)=a(x,t,y,\mathbf{s}_m;\nabla u+\nabla_y u_1),\quad \text{a.e. on }\ \Omega_T\times\mathcal{Y}_{1m}.$$

The characterization (3.30) would clearly follow from the inequality

$$\begin{aligned}\text{(3.31)}\qquad\int_{\Omega_T}\int_{\mathcal{Y}_{1m}}&((-a_0(x,t,y,\mathbf{s}_m)+a(x,t,y,\mathbf{s}_m;\nabla u+\nabla_y u_1+\delta c))\\&\times\delta c(x,t,y,\mathbf{s}_m))\,\mathrm{d}\mathbf{s}_m\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\geqslant 0,\end{aligned}$$

holding for every $\delta>0$ and every $c\in\mathcal{D}(\Omega_T;\mathcal{C}_\#^\infty(\mathcal{Y}_{1m})^N)$, by first dividing (3.31) with respect to $\delta$, then letting $\delta\to 0$, and finally using the Variational Lemma. Equation (3.30) establishes an $\mathrm{H}_{\mathrm{M}P}$-limit $b$ of the form (3.4). Since $u$ is the unique solution to the homogenized equation and $u_1$ is the unique solution to the local problems, the convergences (3.1)–(3.3) hold not only for the extracted subsequence but for the whole sequence as well.

In order to prove (3.31) and thus complete the proof, we introduced a sequence $\{p_\mu\}_{\mu=1}^\infty$ in $\mathcal{D}(\Omega_T;\mathcal{C}_\#^\infty(\mathcal{Y}_{1m})^N)$ of Evans's perturbed test functions (see [4]) according to $p_\mu=\pi_\mu+\pi_{1\mu}+\delta c$, $\mu\in\mathbb{Z}_+$, where $\delta$ and $c$ are as above. For each $\mu\in\mathbb{Z}_+$, the functions $\pi_\mu$ and $\pi_{1\mu}$ belong to $\mathcal{D}(\Omega_T)^N$ and $\mathcal{D}(\Omega_T;\mathcal{C}_\#^\infty(\mathcal{Y}_{1m})^N)$, respectively. The sequences $\{\pi_\mu\}_{\mu=1}^\infty$ and $\{\pi_{1\mu}\}_{\mu=1}^\infty$ are assumed to tend to $\nabla u$ and $\nabla_y u_1$, respectively, both in $L^2$ and pointwise. We then consider $\{p_\mu^\varepsilon\}$ defined by $p_\mu^\varepsilon(x,t)=p_\mu(x,t,x/\varepsilon,\mathbf{t}_m^\varepsilon)$. By structure condition (IV),

$$\text{(3.32)}\qquad\Big(a\Big(x,t,\frac{x}{\varepsilon},\mathbf{t}_m^\varepsilon;\nabla u_\varepsilon\Big)-a\Big(x,t,\frac{x}{\varepsilon},\mathbf{t}_m^\varepsilon;p_\mu^\varepsilon\Big)\Big)\cdot(\nabla u_\varepsilon(x,t)-p_\mu^\varepsilon(x,t))\geqslant 0$$

for $(x,t)\in\Omega_T$. By first integrating (3.32) over $\Omega_T$, then utilizing (3.10) followed by letting $\varepsilon\to 0$ and using the inequality

$$\Big\langle\frac{\partial}{\partial t}u,u\Big\rangle_{X',X}\leqslant\liminf_{\varepsilon\to0}\Big\langle\frac{\partial}{\partial t}u_\varepsilon,u_\varepsilon\Big\rangle_{X',X}$$

(see, e.g., the end of the proof of Theorem 3.1 in [18]), then letting $\mu\to\infty$ and finally employing Lebesgue's Generalized Dominated Convergence Theorem, we arrive at

$$\begin{aligned}(3.33)\quad \int_{\Omega_T}\int_{\mathcal{Y}_{1m}}&(-a_0(x,t,y,\mathbf{s}_m)\cdot\nabla_y u_1(x,t,y,\mathbf{s}_m)-a_0(x,t,y,\mathbf{s}_m)\cdot\delta c(x,t,y,\mathbf{s}_m)\\&+a(x,t,y,\mathbf{s}_m;\nabla u+\nabla_y u_1+\delta c)\delta c(x,t,y,\mathbf{s}_m))\,\mathrm{d}\mathbf{s}_m\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\geqslant 0.\end{aligned}$$

In order to lose the first term in the integrand of (3.33), we simply employ the pre-local-problems. Note that in the resonant cases we need to employ the density result of Lemma 3.4 and the duality pairing result of Lemma 3.5 using the special assumptions on $u_1$. Hence, we have shown (3.31) and the proof is complete. □

For the details that have been left out in the proof above—mainly in the characterization of $a_0$ in terms of $a$—see the proof of Theorem 37 in the detailed e-print version [20] of this paper.

R e m a r k 3.7. We have two remarks concerning the theorem.

(i) The assumption $u_1\in L^2(\Omega_T\times S^{m-1};H^1_{\#}(S_m;\mathcal{W},\mathcal{W}'))$ in the slow resonant case merely amounts to the hypothesis $\partial_{s_m}u_1\in L^2(\Omega_T\times S^{m-1};L^2_{\#}(S_m;\mathcal{W}'))$, since we already know $u_1\in L^2(\Omega_T\times S^{m-1};L^2_{\#}(S_m;\mathcal{W}))$ as a fact due to Theorem 2.10 (with $n=1$). Of course, we can make a similar remark concerning $u_1$ in the rapid resonant case $\mathcal{J}^{m\sim k}_{\mathrm{wsep},1+m+l^\circ-2k}$, $l^\circ\in[\![k+2,m]\!]$.

(ii) Note that in the formulation of the theorem we employ strongly rather than weakly formulated versions of the local problems. This convention will be used in the remaining homogenization result, Theorem 3.8, as well.

Fix $k\in[\![m]\!]_0$. Let $\mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep}}$ be the set of all pairs $(\varepsilon,\{\varepsilon'_j\}_{j=1}^m)$ of lists in $\mathcal{J}^{1m}_{\mathrm{wsep}}$ such that it either holds that $\varepsilon'_k\sim\varepsilon$ as before, or that

$$\begin{cases}\{\varepsilon,\varepsilon'_1,\dots,\varepsilon'_m\} & \text{if } k=0,\\ \{\varepsilon'_1,\dots,\varepsilon'_k,\varepsilon,\varepsilon'_{k+1},\dots,\varepsilon'_m\} & \text{if } k\in[\![m-1]\!],\\ \{\varepsilon'_1,\dots,\varepsilon'_m,\varepsilon\} & \text{if } k=m\end{cases}$$

is a well-separated list of scale functions. Hence, in the latter case $\varepsilon'_k\not\sim\varepsilon$, $\varepsilon<\varepsilon'_k$ for small enough $\varepsilon$, motivating the notation "$\preccurlyeq k$". This could be read as "the spatial scale is asymptotically equal to or less than the $k$th temporal scale". Introduce the collection $\{\mathcal{J}^{m\preccurlyeq k}_{\text{wsep},i}\}_{i=1}^{1+2(m-k)}$ of $1+2(m-k)$ subsets of $\mathcal{J}^{m\preccurlyeq k}_{\text{wsep}}$ according to

$$\mathcal{J}^{m\preccurlyeq k}_{\text{wsep},1} = \Big\{(\varepsilon,\{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}^{m\preccurlyeq k}_{\text{wsep}}\colon\ \frac{\varepsilon^2}{\varepsilon'_m} \to 0\Big\},$$

$$\mathcal{J}^{m\preccurlyeq k}_{\text{wsep},2} = \Big\{(\varepsilon,\{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}^{m\preccurlyeq k}_{\text{wsep}}\colon\ \varepsilon'_m \sim \varepsilon^2\Big\},$$

$$\mathcal{J}^{m\preccurlyeq k}_{\text{wsep},2+i-k} = \Big\{(\varepsilon,\{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}^{m\preccurlyeq k}_{\text{wsep}}\colon\ \frac{\varepsilon'_i}{\varepsilon^2} \to 0 \text{ but } \frac{\varepsilon'_{i-1}}{\varepsilon^2} \to \infty\Big\}$$

and

$$\mathcal{J}^{m\preccurlyeq k}_{\text{wsep},1+m+i^\circ-2k} = \Big\{(\varepsilon,\{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}^{m\preccurlyeq k}_{\text{wsep}}\colon\ \varepsilon'_{i^\circ-1} \sim \varepsilon^2\Big\}$$

for $i \in [\![k+1,m]\!]$, $(k,i) \neq (0,1)$, and $i^\circ \in [\![k+2,m]\!]$; for $(k,i)=(0,1)$ we define

$$\mathcal{J}^{m\preccurlyeq 0}_{\text{wsep},3} = \Big\{(\varepsilon,\{\varepsilon'_j\}_{j=1}^m) \in \mathcal{J}^{m\preccurlyeq 0}_{\text{wsep}}\colon\ \frac{\varepsilon'_1}{\varepsilon^2} \to 0\Big\}. \tag{3.34}$$

Actually, $\mathcal{J}^{m\preccurlyeq k}_{\text{wsep},3}$ does not really need the second condition, i.e. the non-convergence to 0, since it is already implied by the fact that we are in $\mathcal{J}^{m\preccurlyeq k}_{\text{wsep}}$. Since there does not exist any "$\varepsilon'_0$", we note that we need to impose a special definition (3.34) for $\mathcal{J}^{m\preccurlyeq 0}_{\text{wsep},3}$ without the extra condition. The collection $\{\mathcal{J}^{m\preccurlyeq k}_{\text{wsep},i}\}_{i=1}^{1+2(m-k)}$ of subsets of $\mathcal{J}^{m\preccurlyeq k}_{\text{wsep}}$ is clearly mutually disjoint. Note that if $k=m$, the introduced collection of subsets of $\mathcal{J}^{m\preccurlyeq m}_{\text{wsep}}$ reduces to merely $\{\mathcal{J}^{m\preccurlyeq m}_{\text{wsep},1}\}$.

In Theorem 3.8 below, the main result of this paper and appearing as Corollary 40 in [20], we have a straightforward generalization of Theorem 3.6 where $k$ may be zero and the rather restrictive assumption of asymptotic equality is everywhere replaced by the relaxed assumption of asymptotic inequality as defined above.

**Theorem 3.8.** *Let $k \in [\![m]\!]_0$. Suppose that the pair $e = (\varepsilon,\{\varepsilon'_j\}_{j=1}^m)$ of lists of spatial and temporal scale functions belongs to $\bigcup_{i=1}^{1+2(m-k)} \mathcal{J}^{m\preccurlyeq k}_{\text{wsep},i}$. Let $\{u_\varepsilon\}$ be the sequence of weak solutions in $H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega))$ to the evolution problem* (1.1) *with $a\colon \overline{\Omega}_T \times \mathbb{R}^{N+m} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the structure conditions* (I)–(V). *Then convergences on the form* (3.1)–(3.3) *hold, where $u \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega))$ and $u_1 \in L^2(\Omega_T \times S^m;\mathcal{W})$. Here $u$ is the unique weak solution to the homogenized problem* (2.2) *with the homogenized flux $b\colon \overline{\Omega}_T \times \mathbb{R}^N \to \mathbb{R}^N$ given in the form* (3.4). *Moreover, we have the following characterization of $u_1$:*

- *If $e \in \mathcal{J}^{m\preccurlyeq k}_{\text{wsep},1}$ then the function $u_1$ is the unique weak solution to a local problem of the form* (3.5).

- *If* $e \in \mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep},2}$, *assuming* $u_1 \in L^2(\Omega_T \times S^{m-1}; H^1_\#(S_m; \mathcal{W}, \mathcal{W}'))$, *then the function* $u_1$ *is the unique weak solution to a local problem of the form* (3.6).
- *If* $e \in \mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep},2+\bar{l}-k}$ *for some* $\bar{l} \in [\![k+1, m]\!]$, *provided* $k \in [\![m-1]\!]_0$, *then the function* $u_1$ *is the unique weak solution to a system of local problems of the form* (3.7).
- *If* $e \in \mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep},1+m+l^\circ-2k}$ *for some* $l^\circ \in [\![k+2, m]\!]$, *provided* $k \in [\![m-2]\!]_0$ *and assuming* $u_1 \in L^2(\Omega_T \times S^{l^\circ-2} \times S^{[\![l^\circ,m]\!]}; H^1_\#(S_{l^\circ-1}; \mathcal{W}, \mathcal{W}'))$, *then the function* $u_1$ *is the unique weak solution to a system of local problems of the form* (3.8).

Proof. To prove the theorem we first have to consider the case of strict asymptotic inequality. We then introduce an extra temporal scale function coinciding with the spatial scale function $\varepsilon$ in order to transform the problem to the same form as in Theorem 3.6 which is then applied. For details, see the proof of Theorem 39 in [20]. Theorem 3.8 follows directly from this result by taking into consideration also asymptotic equality employing Theorem 3.6 again. $\square$

Remark 3.9. Theorem 3.8 can only handle the subset $\bigcup_{i=1}^{1+2(m-k)} \mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep},i}$ of $\mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep}}$. The conclusion of Remark 3.2 is true also in the setting of Theorem 3.8 though, i.e., the collection $\{\mathcal{P}^{m\preccurlyeq k}_i\}_{i=1}^{1+2(m-k)}$ forms a partition of $\mathcal{P}^{m\preccurlyeq k}$, where $\mathcal{P}^{m\preccurlyeq k}$ is the subset of $\mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep}}$ with temporal scale functions expressed as power functions, and $\mathcal{P}^{m\preccurlyeq k}_i$ is the corresponding subset of $\mathcal{J}^{m\preccurlyeq k}_{\mathrm{wsep},i}$ for every $i \in [\![1+2(m-k)]\!]$.

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