Nazim I. Mahmudov

Approximation properties of bivariate complex q-Bernstein polynomials in the case $q > 1$

Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 2, 557–566

Persistent URL: http://dml.cz/dmlcz/142846

Czechoslovak Mathematical Journal, 62 (137) (2012), 557–566

APPROXIMATION PROPERTIES OF BIVARIATE COMPLEX q-BERNSTEIN POLYNOMIALS IN THE CASE $q > 1$

Nazim I. Mahmudov, Gazimagusa

(Received March 20, 2011)

Abstract. In the paper, we discuss convergence properties and Voronovskaja type theorem for bivariate q-Bernstein polynomials for a function analytic in the polydisc $D_{R_1} \times D_{R_2} = \{z \in C\colon |z| < R_1\} \times \{z \in C\colon |z| < R_1\}$ for arbitrary fixed $q > 1$. We give quantitative Voronovskaja type estimates for the bivariate q-Bernstein polynomials for $q > 1$. In the univariate case the similar results were obtained by S. Ostrovska: q-Bernstein polynomials and their iterates. J. Approximation Theory 123 (2003), 232–255, and S. G. Gal: Approximation by Complex Bernstein and Convolution Type Operators. Series on Concrete and Applicable Mathematics 8. World Scientific, New York, 2009.

Keywords: q-Bernstein polynomials, modulus of continuity, Voronovskaja type theorem

MSC 2010: 41A35, 33D15, 41A10

1. Introduction and Main results

For each integer $k \geqslant 0$, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q} & \text{if } q \in \mathbb{R}^+ \setminus \{1\}, \\ k & \text{if } q = 1 \end{cases} \qquad \text{for } k \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[k]_q! := [1]_q[2]_q \dots [k]_q \quad \text{for } k \in \mathbb{N} \text{ and } [0]! = 1.$$

For integers $0 \leqslant k \leqslant n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For fixed $q > 1$, we denote the q-derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

Let $1 \leqslant r < R/q$, $q > 1$, $\mathbb{D}_R := \{z \in \mathbb{C}\colon |z| < R\}$, $H(\mathbb{D}_R)$ denote the set of all analytic functions on $\mathbb{D}_R$. For $f \in H(\mathbb{D}_R)$ and $|z| < R/q$ we introduce an operator L_q defined in [8]

$$L_q(f;z) := \frac{(1-z)(D_q f(z) - f'(z))}{q-1} \quad \text{for } q > 1.$$

It is clear that for $f \in H(\mathbb{D}_R)$

$$L_q(f;z) = \sum_{m=2}^{\infty} a_m \frac{[m]_q - m}{q-1} z^{m-1}(1-z) = \sum_{m=2}^{\infty} a_m \left(\sum_{i=1}^{m-1} [i]_q\right) z^{m-1}(1-z),$$

$$L_q(e_m;z) = \sum_{i=1}^{m-1} [i]_q z^{m-1}(1-z),$$

where $f(z) = \sum\limits_{m=0}^{\infty} a_m z^m$, $e_m(z) = z^m$.

In several recent papers, convergence properties of complex q-Bernstein polynomials, proposed by Phillips [7] for real variables, defined by

$$B_{n,q}(f;z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1} (1 - q^j z)$$

and attached to an analytic function f in a closed disk, were intensively studied by many authors, see [2] and the references theirin. It is known that the cases $0 < q < 1$ and $q > 1$ are not similar to each other. This difference is caused by the fact that, for $0 < q < 1$, $B_{n,q}$ are positive linear operators on $C[0,1]$ while for $q > 1$, the positivity fails ([5], [4]). The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than for $0 < q < 1$. There are few papers ([5], [6], [8], and [9]) studying systematically the convergence in the case $q > 1$. If $q \geqslant 1$ then qualitative Voronovskaja-type and saturation results for complex q-Bernstein polynomials were obtained in Wang-Wu [8]. Wu [9] studied saturation of convergence on the interval $[0,1]$ for the q-Bernstein polynomials of a continuous function f for arbitrary fixed $q > 1$. Notice that the results for the complex univariate q-Bernstein operators can be extended to the case of several complex variables. For simplicity, the results are presented for bivariate case, but from the proofs it is easy to see that they remain valid for several complex variables.

We consider the bivariate complex q-Bernstein polynomials of tensor product kind given by

$$B_{n,m,q}(f;(z_1,z_2)) = \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{[k]_q}{[n]_q}, \frac{[j]_q}{[m]_q}\right) p_{n,k}(q;z_1) p_{m,j}(q;z_2),$$

where $p_{n,k}(q;z) = \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1}(1-q^j z)$. If $f \in H(\mathbb{D}_{R_1} \times \mathbb{D}_{R_2})$ we assume that $f(z_1,z_2) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j$ for all $(z_1,z_2) \in \mathbb{D}_{R_1} \times \mathbb{D}_{R_2}$.

Note that in the case of several real variables, the tensor product Bernstein polynomial was first introduced and studied in Hildebrandt-Schoenberg [3] and Butzer [1]. In the next theorem, approximation properties of the above bivariate complex polynomials will be proved.

Theorem 1. *Let $q > 1$, $f \in H(\mathbb{D}_{R_1} \times \mathbb{D}_{R_2})$. For all $|z_1| < r_1$, $|z_2| < r_2$, with $1 \leqslant r_1 < R_1/q$, $1 \leqslant r_2 < R_2/q$ and $n, m \in \mathbb{N}$ we have*

$$|B_{n,m,q}(f;(z_1,z_2)) - f(z_1,z_2)| \leqslant C_{r_1,r_2,n,m}(f),$$

where

$$C_{r_1,r_2,n,m}(f) = \frac{3r_2(1+r_2)}{2[m]_q}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|c_{k,j}|j(j-1)q^{j-2}r_2^{j-2}r_1^k + \frac{3r_1(1+r_1)}{2[n]_q}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|c_{k,j}|k(k-1)q^{k-2}r_1^{k-2}r_2^j.$$

Remark 2. It is worth mentioning that in the univariate case the results similar to Theorem 1 were obtained by Ostrovska [5] and Gal [2]. On the other hand in the classical case ($q = 1$) the similar result is given in [2].

Next a Voronovskaja type result for $B_{n,m,q}$ is presented. It is the product of the parametric extensions generated by the Voronovskaja formula in the univariate case. Indeed, for $f(z_1,z_2)$ defining the parametric extensions of the Voronovskaja formula by

$$L_{n,q}^1(f(\cdot,z_2);z_1) := B_{n,q}(f(\cdot,z_2);z_1) - f(z_1,z_2) - \frac{1}{[n]_q}L_q(f(\cdot,z_2);z_1),$$
$$L_{m,q}^2(f(z_1,\cdot);z_2) := B_{m,q}(f(z_1,\cdot);z_2) - f(z_1,z_2) - \frac{1}{[m]_q}L_q(f(z_1,\cdot);z_2),$$

their composition gives

$$(1) \quad (L_{m,q}^2 \circ L_{n,q}^1)(f;(z_1,z_2)) := B_{m,q}(L_{n,q}^1(f(\cdot,z_2);z_1);z_2) - L_{n,q}^1(f(\cdot,z_2);z_1) - \frac{1}{[m]_q}L_q(L_{n,q}^1(f;(z_1,\cdot));z_2) =: E_1 - E_2 - E_3.$$

Theorem 3. *Let $f \in H(\mathbb{D}_{R_1} \times \mathbb{D}_{R_2})$, $q > 1$, $1 \leqslant r_1 < R_1/q^2$, $1 \leqslant r_2 < R_2/q^2$. The following Voronovskaja-type result holds*

$$|(L^2_{m,q} \circ L^1_{n,q})(f;(z_1,z_2))| \leqslant C_{r_1,r_2}(f)\Big(\frac{1}{[n]_q^2} + \frac{1}{[m]_q^2}\Big),$$

where

$$C_{r_1,r_2}(f) :=$$
$$\sum_{k=0}^{\infty}\bigg(2\sum_{j=0}^{\infty}|c_{k,j}|r_2^j + \sum_{j=0}^{\infty}2|c_{k,j}|\sum_{i=1}^{j-1}[i]_q r_2^j\bigg)\frac{3r_1(1+qr_1)^2}{2}\,(q^2r_1)^{k-3}(k-1)^2(k-2)^2,$$

$|z_1| < r_1$, $|z_2| < r_2$ *and* $n, m \in \mathbb{N}$, $m \geqslant 3$.

Remark 4. It should be mentioned that in the univariate case the Voronovskaja type results were obtained in [8]. Our result is new even for the univariate case since the estimate given in Theorem 3 has quantitative character. The classical case was studied by Gal [2].

2. Proofs of the main results

Lemma 5. *Let $f \in H(\mathbb{D}_{R_1})$, $1 \leqslant r < R_1/q$ and $q > 1$. Then we have*

$$|B_{n,q}(e_k;z) - e_k(z)| \leqslant \frac{3r(1+r)}{2[n]_q}\,k(k-1)(qr)^{k-2}$$

for all $n \in \mathbb{N}$, $|z| \leqslant r$.

Proof. Let us start with the recurrence formulas

$$B_{n,q}(e_k;z) = \frac{z(1-z)}{[n]_q}\,D_qB_{n,q}(e_{k-1};z) + z_1B_{n,q}(e_{k-1};z), \tag{2}$$

$$B_{n,q}(e_k;z) - e_k(z) = \frac{z(1-z)}{[n]_q}\,D_q(B_{n,q}(e_{k-1};z) - e_{k-1}(z)$$
$$+ z(B_{n,q}(e_{k-1};z) - e_{k-1}(z)) + \frac{[k-1]_q}{[n]_q}\,z^{k-1}(1-z).$$

It is known that by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$|P_k'(z)| \leqslant \frac{k}{qr}\,\|P_k\|_{qr}, \quad \text{for all } |z| \leqslant qr,\ r \geqslant 1,$$

which combined with the mean value theorem in complex analysis implies

$$|D_q P_k(z)| \leqslant \|P_k'\|_{qr} \leqslant \frac{k}{qr}\|P_k\|_{qr},$$

for all $|z| \leqslant r$, where $P_k(z)$ is a complex polynomial of degree $\leqslant k$. From the above recurrence formula we get

$$\begin{aligned}
|B_{n,q}(e_k;z) - e_k(z)| &\leqslant \frac{r(1+r)}{[n]_q}\,\frac{k-1}{qr}\,\|B_{n,q}(e_{k-1}) - e_{k-1}\|_{qr} \\
&\quad + r|B_{n,q}(e_{k-1};z) - e_{k-1}(z)| + \frac{[k-1]_q}{[n]_q}\,r^{k-1}(1+r) \\
&\leqslant \frac{r(1+r)}{[n]_q}\,\frac{k-1}{qr}(\|B_{n,q}(e_{k-1})\|_{qr} + \|e_{k-1}\|_{qr}) \\
&\quad + r|B_{n,q}(e_{k-1};z) - e_{k-1}(z)| + \frac{[k-1]_q}{[n]_q}\,r^{k-1}(1+r) \\
&\leqslant 2(k-1)\frac{r(1+r)}{[n]_q}\,(qr)^{k-2} + r|B_{n,q}(e_{k-1};z) - e_{k-1}(z)| \\
&\quad + \frac{[k-1]_q}{[n]_q}\,r^{k-1}(1+r) \\
&\leqslant r|B_{n,q}(e_{k-1};z) - e_{k-1}(z)| + \frac{3(k-1)}{[n]_q}\,r(1+r)(qr)^{k-2}.
\end{aligned}$$

By writing the last inequality for $k = 1, 2, \ldots$, we easily obtain, step by step, the following

$$\begin{aligned}
|B_{n,q}(e_k;z) - e_k(z)| &\leqslant \frac{3r(1+r)}{[n]_q}\,(qr)^{k-2}(k-1+k-2+\ldots+1) \\
&= \frac{3r(1+r)}{2[n]_q}\,k(k-1)(qr)^{k-2}.
\end{aligned}$$

□

P r o o f of Theorem 1. Denote $e_{k,j}(z_1,z_2) = e_k(z_1)e_j(z_2) = z_1^k z_2^j$. Clearly we get

$$\begin{aligned}
&|B_{n,m,q}(f;(z_1,z_2)) - f(z_1,z_2)| \\
&\qquad \leqslant \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} |c_{k,j}||B_{n,m,q}(e_{k,j};(z_1,z_2)) - e_{k,j}(z_1,z_2)|.
\end{aligned}$$

But taking into account the estimate in Lemma 5, for all $|z_1| < r_1$, $|z_2| < r_2$ we obtain

$$\begin{aligned}
&|B_{n,m,q}(e_{k,j};(z_1,z_2)) - e_{k,j}(z_1,z_2)| \\
&\qquad = |B_{n,q}(e_k;z_1)B_{m,q}(e_j;z_2) - z_1^k z_2^j| \\
&\qquad \leqslant |B_{n,q}(e_k;z_1)B_{m,q}(e_j;z_2) - B_{n,q}(e_k;z_1)z_2^j| + |B_{n,q}(e_k;z_1)z_2^j - z_1^k z_2^j|
\end{aligned}$$

$$\leqslant |B_{n,q}(e_k;z_1)||B_{m,q}(e_j;z_2)-z_2^j|+|z_2^j||B_{n,q}(e_k;z_1)-z_1^k|$$
$$\leqslant r_1^k\frac{3r_2(1+r_2)}{2[m]_q}\,j(j-1)q^{j-2}r_2^{j-2}+r_2^j\frac{3r_1(1+r_1)}{2[n]_q}\,k(k-1)q^{k-2}r_1^{k-2}$$

which immediately implies the estimate in Theorem 1. □

Lemma 6. *We have*

$$\text{(3)}\qquad (L^2_{m,q}\circ L^1_{n,q})(f;(z_1,z_2))$$
$$=\sum_{k=0}^{\infty}\Bigl(\sum_{j=0}^{\infty}c_{k,j}B_{m,q}(e_j;z_2)-f_k(z_2)-\frac{1}{[m]_q}\sum_{j=2}^{\infty}c_{k,j}L_q(e_j;z_2)\Bigr)$$
$$\times\Bigl[B_{n,q}(e_k;z_1)-e_k(z_1)-\frac{1}{[n]_q}L_q(e_k;z_1)\Bigr].$$

Proof. By the hypothesis we can write

$$f(z_1,z_2)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c_{k,j}z_1^kz_2^j=\sum_{k=0}^{\infty}f_k(z_2)z_1^k,$$

where $f_k(z_2)=\sum\limits_{j=0}^{\infty}c_{k,j}z_2^j$. The second term of (1) can be written as follows.

$$E_2=L^1_{n,q}(f(\cdot,z_2);z_1)$$
$$=B_{n,q}(f(\cdot,z_2);z_1)-f(z_1,z_2)-\frac{1}{[n]_q}\,L_q(f(\cdot,z_2);z_1)$$
$$=\sum_{k=0}^{\infty}f_k(z_2)\Bigl[B_{n,q}t(e_k;z_1)-e_k(z_1)-\frac{1}{[n]_q}\,L_q(e_k;z_1)\Bigr].$$

Applying now $B_{m,q}$ to the last expression with respect to z_2, we obtain the expression for the first term of (1)

$$E_1=B_{m,q}(L^1_{n,q}(f(\cdot,z_2);z_1);z_2)$$
$$=\sum_{k=0}^{\infty}B_{m,q}(f_k;z_2)\Bigl[B_{n,q}(e_k;z_1)-e_k(z_1)-\frac{1}{[n]_q}\,L_q(e_k;z_1)\Bigr]$$
$$=\sum_{k=0}^{\infty}\Bigl(\sum_{j=0}^{\infty}c_{k,j}B_{m,q}(e_j;z_2)\Bigr)\Bigl[B_{n,q}(e_k;z_1)-e_k(z_1)-\frac{1}{[n]_q}\,L_q(e_k;z_1)\Bigr].$$

For the third term we have

$$\begin{aligned}[m]_q E_3 &= L_q(L^1_{n,q}(f;(z_1,\cdot));z_2)\\ &= \frac{(1-z_2)\big(D_q L^1_{n,q}(f;(z_1,z_2)) - (L^1_{n,q}(f;(z_1,z_2)))'\big)}{q-1}\\ &= B_{n,q}(L_q(f(\cdot,z_2);z_2);z_1) - L_q(f(z_1,z_2);z_2) - \frac{1}{[n]_q} L_q\circ L_q(f(\cdot,z_2);z_1),\end{aligned}$$

where

$$\begin{aligned}L_q(f(\cdot,z_2);z_2) &= \sum_{k=0}^{\infty} L_q(f_k(z_2);z_2)z_1^k = \sum_{k=0}^{\infty}\sum_{j=2}^{\infty} c_{k,j} L_q(e_j;z_2)z_1^k,\\ B_{n,q}(L_q(f(\cdot,z_2);z_2);z_1) &= \sum_{k=0}^{\infty}\sum_{j=2}^{\infty} c_{k,j} L_q(e_j;z_2)B_{n,q}(e_k;z_1),\\ (L_q\circ L_q)(f(\cdot,z_2);z_1) &= L_q\bigg(\sum_{k=0}^{\infty}\sum_{j=2}^{\infty} c_{k,j} L_q(e_j;z_2)z_1^k;z_1\bigg)\\ &= \sum_{k=0}^{\infty}\sum_{j=2}^{\infty} c_{k,j} L_q(e_j;z_2)L_q(e_k;z_1).\end{aligned}$$

Introducing the expressions of E_1, E_2 and E_3 in (1), we arrive at (3). □

Lemma 7. *Let $f \in H(\mathbb{D}_{R_1})$, $1 \leqslant r_1 < R_1/q^2$ and $q > 1$. Then we have*

$$\begin{aligned}&\Big|B_{n,q}(e_m;z_1) - e_m(z_1) - \frac{1}{[n]_q} L_q(e_m;z_1)\Big|\\ &\qquad\leqslant \frac{3r_1(1+qr_1)^2}{2[n]_q^2}(q^2r_1)^{m-3}(m-1)^2(m-2)^2\end{aligned}$$

for all $n\in\mathbb{N}$, $m \geqslant 3$, $|z_1| \leqslant r_1$.

P r o o f. Let us consider the relationship given by

$$\text{(4)}\quad E_{n,m}(z_1) := B_{n,q}(e_m;z_1) - e_m(z_1) - \frac{1}{[n]_q}\sum_{i=1}^{m-1}[i]_q\, z_1^{m-1}(1-z_1),$$

$$\begin{aligned}E_{n,m}(z_1) &= \frac{z_1(1-z_1)}{[n]_q} D_q(B_{n,q}(e_{m-1};z_1) - e_{m-1}(z_1))\\ &\quad + z_1\bigg(B_{n,q}(e_{m-1};z_1) - e_{m-1}(z_1) - \frac{1}{[n]_q}\sum_{i=1}^{m-2}[i]_q\, z_1^{m-2}(1-z_1)\bigg),\\ E_{n,m}(z_1) &= \frac{z_1(1-z_1)}{[n]_q} D_q(B_{n,q}(e_{m-1};z_1) - e_{m-1}(z_1)) + z_1E_{n,m-1}(z_1),\end{aligned}$$

for all $m \geqslant 2$, $n \in \mathbb{N}$ and $z_1 \in \mathbb{C}$. Lemma 5 with $r = qr_1$ and (4) imply for $|z_1| \leqslant r_1$

$$\begin{aligned}|E_{n,m}(z_1)| &\leqslant r_1|E_{n,m-1}(z_1)| + \frac{r_1(1+r_1)}{[n]_q}\,\frac{m-1}{qr_1}\,\|B_{n,q}(e_{m-1}) - e_{m-1}\|_{qr_1}\\ &\leqslant r_1|E_{n,m-1}(z_1)| + \frac{3r_1(1+qr_1)^2}{2[n]_q^2}(m-1)^2(m-2)(q^2r_1)^{m-3}.\end{aligned}$$

By writing the last inequality for $m = 3, 4, \ldots$ we easily obtain, step by step, the following

$$\begin{aligned}|E_{n,m}(z_1)| &\leqslant \frac{3r_1(1+qr_1)^2}{2[n]_q^2}\,(q^2r_1)^{m-3}\sum_{j=3}^{m}(j-1)^2(j-2)\\ &\leqslant \frac{3r_1(1+qr_1)^2}{2[n]_q^2}\,(q^2r_1)^{m-3}(m-1)^2(m-2)^2.\end{aligned}$$

□

P r o o f of Theorem 3. By the hypothesis we can write

$$f(z_1, z_2) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j = \sum_{k=0}^{\infty} f_k(z_2) z_1^k,$$

where $f_k(z_2) = \sum\limits_{j=0}^{\infty} c_{k,j} z_2^j$. It follows, by Lemma 6 that

$$\begin{aligned}&|(L^2_{m,q} \circ L^1_{n,q})(f;(z_1,z_2))|\\ &\quad\leqslant \sum_{k=0}^{\infty}\Bigl|\sum_{j=0}^{\infty} c_{k,j} B_{m,q}(e_j;z_2) - f_k(z_2) - \frac{1}{[m]_q}\sum_{j=2}^{\infty} c_{k,j} L_q(e_j;z_2)\Bigr|\\ &\quad\times \Bigl|B_{n,q}(e_k;z_1) - e_k(z_1) - \frac{1}{[n]_q}\,L_q(e_k;z_1)\Bigr|.\end{aligned}$$

Using now Lemma 7 with $|z_1| < r_1 < R_1/q^2$, $|z_2| < r_2 < R_2/q^2$ and the inequalities

$$\begin{aligned}|B_{m,q}(e_j;z_2)| &\leqslant r_2^j,\\ |f_k(z_2)| &\leqslant \sum_{j=0}^{\infty}|c_{k,j}|r_2^j,\\ |c_{k,j}L_q(e_j;z_2)| &\leqslant 2|c_{k,j}|\sum_{i=1}^{j-1}[i]_q r_2^j\end{aligned}$$

we have

$$
\begin{aligned}
&|(L^2_{m,q} \circ L^1_{n,q})(f;(z_1,z_2))| \\
&\quad\leqslant \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{\infty}|c_{k,j}|r_2^j + \sum_{j=0}^{\infty}|c_{k,j}|r_2^j + \frac{1}{[m]_q}\sum_{j=0}^{\infty}2|c_{k,j}|\sum_{i=1}^{j-1}[i]_q r_2^j\Big) \\
&\qquad\times\Big|B_{n,q}(e_k;z_1) - e_k(z_1) - \frac{1}{[n]_q}\,L_q(e_k;z_1)\Big| \\
&\quad\leqslant \sum_{k=0}^{\infty}\Big(2\sum_{j=0}^{\infty}|c_{k,j}|r_2^j + \frac{1}{[m]_q}\sum_{j=0}^{\infty}2|c_{k,j}|\sum_{i=1}^{j-1}[i]_q r_2^j\Big) \\
&\qquad\times\frac{3r_1(1+qr_1)^2}{2[n]_q^2}\,(q^2r_1)^{k-3}(k-1)^2(k-2)^2.
\end{aligned}
$$

Note that if we estimate now $|(L^2_{m,q} \circ L^1_{n,q})(f;(z_1,z_2))|$, then by reasons of symmetry we get a similar order of approximation, simply interchanging above the places of n and m. In conclusion,

$$|(L^2_{m,q} \circ L^1_{n,q})(f;(z_1,z_2))| \leqslant C_{r_1,r_2}(f)\Big(\frac{1}{[n]_q^2} + \frac{1}{[m]_q^2}\Big).$$

□

Acknowledgement. The author is grateful to the reviewers for the careful reading of the paper and for pointing out an insufficiency in the proof of Theorem 3 in the draft of this paper.

References

[1] *P. L. Butzer*: On two-dimensional Bernstein polynomials. Can. J. Math. *5* (1953), 107–113.
[2] *S. G. Gal*: Approximation by Complex Bernstein and Convolution Type Operators. Series on Concrete and Applicable Mathematics 8. World Scientific, New York, 2009.
[3] *T. H. Hildebrandt, I. J. Schoenberg*: On linear functional operations and the moment problem for a finite interval in one or several dimensions. Ann. Math. *34* (1933), 317–328.
[4] *N. I. Mahmudov*: Korovkin-type theorems and applications. Cent. Eur. J. Math. *7* (2009), 348–356.
[5] *S. Ostrovska*: q-Bernstein polynomials and their iterates. J. Approximation Theory *123* (2003), 232–255.
[6] *S. Ostrovska*: The sharpness of convergence results for q-Bernstein polynomials in the case $q > 1$. Czech. Math. J. *58* (2008), 1195–1206.
[7] *G. M. Phillips*: Bernstein polynomials based on the q-integers. Ann. Numer. Math. *4* (1997), 511–518.
[8] *H. Wang, X. Wu*: Saturation of convergence for q-Bernstein polynomials in the case $q > 1$. J. Math. Anal. Appl. *337* (2008), 744–750.

[9] *Z. Wu*: The saturation of convergence on the interval $[0;1]$ for the q-Bernstein polynomials in the case $q > 1$. J. Math. Anal. Appl. *357* (2009), 137–141.

Author's address: N. I. Mahmudov, Department of Mathematics, Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, Turkey, e-mail: `nazim.mahmudov@emu.edu.tr`.