Czechoslovak Mathematical Journal

Dharmanand Baboolal

Conditions under which the least compactification of a regular continuous frame is perfect

Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 2, 505–515

Persistent URL: `http://dml.cz/dmlcz/142842`

Czechoslovak Mathematical Journal, 62 (137) (2012), 505–515

CONDITIONS UNDER WHICH THE LEAST COMPACTIFICATION OF A REGULAR CONTINUOUS FRAME IS PERFECT

Dharmanand Baboolal, Durban

(Received February 21, 2011)

Abstract. We characterize those regular continuous frames for which the least compactification is a perfect compactification. Perfect compactifications are those compactifications of frames for which the right adjoint of the compactification map preserves disjoint binary joins. Essential to our characterization is the construction of the frame analog of the two-point compactification of a locally compact Hausdorff space, and the concept of remainder in a frame compactification. Indeed, one of the characterizations is that the remainder of the regular continuous frame in each of its compactifications is compact and connected.

Keywords: regular continuous frame, perfect compactification

MSC 2010: 06D20, 06B35, 54D35

Introduction

The purpose of this paper is to characterize those regular continuous frames whose least compactifications are perfect. The motivation for investigating this problem comes from the work of Jung [5] who considered this problem in the setting of Hausdorff spaces. Specifically he was concerned with characterizing internally those locally compact Hausdorff spaces whose Alexandroff one-point compactifications are perfect.

It is well known that for spaces, a space has a smallest compactification if and only if it is locally compact Hausdorff. In [3] Banaschewski showed that a frame has a smallest compactification if and only if it is regular continuous. It is natural therefore to speak of regular continuous frames as the frame analogue of locally compact Hausdorff spaces, and the smallest compactification of a frame as the analogue of the Alexandroff one-point compactification. As perfect compactifications have been

A grant from the NRF is gratefully acknowledged.

introduced into the theory of frames (see [2]), this immediately raises the problem of characterizing those regular continuous frames whose least compactifications are perfect.

Before we address this problem let us recall the statement of Jung's characterization result:

Theorem 0.1. *Let X be locally compact, non-compact and let $Y = X \cup \{\infty\}$ be its Alexandroff one-point compactification. The following conditions are equivalent:*
(1) *Y is a perfect compactification of X.*
(2) *The set $Y \setminus X = \{\infty\}$ does not split Y at the point ∞.*
(3) *No compact subset of X splits X at infinity.*
(4) *For any compactification Z of X, the remainder $Z\setminus X$ is compact and connected.*

We remind the reader of some of the terminology used in the above theorem. First, the notion of a perfect compactification goes back to the work of Sklyarenko [6] who defined a compactification Y of X to be perfect if for any open subset U of X we have $\mathrm{Fr}_Y(Y \setminus \mathrm{Cl}_Y(X \setminus U)) = \mathrm{Cl}_Y(\mathrm{Fr}_X U)$ where Fr denotes the usual frontier or boundary operator and Cl the closure operator. In his paper Sklyarenko shows that the Stone-Čech compactification of a Tychonoff space and the Freudenthal compactification of a rim-compact Hausdorff space are examples of perfect compactifications.

A set N is said to split the space Y at a point x of N (Sklyarenko [6]) if there is an open set U of Y containing x such that $U \cap (Y \setminus N) = V \cup W$, where V and W are disjoint non-empty open subsets of $Y \setminus N$ with $x \in \mathrm{Cl}_Y V \cap \mathrm{Cl}_Y W$.

A compact subset C of a space X splits the space at infinity if $X \setminus C = U \cup V$ where U and V are non-empty disjoint open subsets of X such that $C \cup U$ and $C \cup V$ are non-compact (Aarts and Van Emde Boas [1]).

1. Preliminaries

We recall that a *frame* L is a complete lattice satisfying the infinite distributive law:

$$x \wedge \bigvee S = \bigvee x \wedge s (s \in S)$$

for any subset S of L. A *homomorphism* $h\colon M \to L$ between frames is a map that preserves finite meets (including the top element e) and arbitrary joins (including the bottom element 0). The resulting category is denoted by **Frm**. A frame L is said to be *regular* if for each $a \in L$, $a = \bigvee x(x \prec a)$. Here $x \prec a$ means that there exists an element $u \in L$ such that $x \wedge u = 0$ and $u \vee a = e$. The frame L is said to be *compact* if whenever $e = \bigvee S$, then there exists a finite $F \subseteq S$ such that $e = \bigvee F$.

A complete lattice is called *continuous* if for each $a \in L$, $a = \bigvee x(x \ll a)$, where $x \ll a$ (read as x is *way below* a) means that for any $S \subseteq L$ such that $a \leqslant \bigvee S$ there exists a finite $F \subseteq S$ such that $x \leqslant \bigvee F$. Banaschewski [3] showed that if a frame L is regular and continuous, then it has a smallest strong inclusion $\blacktriangleleft$ on L given by $a \blacktriangleleft b \Leftrightarrow a \prec b$ and either $\uparrow a^*$ *or* $\uparrow b$ is compact, where $\uparrow x = \{t \in L\colon\ t \geqslant x\}$, and a^* is the *pseudocomplement* of a, i.e. the largest element of L whose meet with a is 0. Recall that a *strong inclusion* $\triangleleft$ on L (Banaschewski [3]) is a binary relation on L being the frame counterpart of the well known *Efremovič proximity* relation for spaces. The set of all strong inclusions on a frame is in a one-to-one correspondence with the set of all compactifications of L. By a *compactification* of L we mean a compact regular frame M together with a dense onto map $h\colon M \to L$, where a *dense* map h is one satisfying the condition that $h(x) = 0 \Longrightarrow x = 0$. A map h is said to be *codense* if $h(x) = e \Longrightarrow x = e$. The correspondence between strong inclusions and compactifications is described by Banaschewski [3] as follows: If $\triangleleft$ is a strong inclusion on L, we consider $\gamma L = \{J\colon\ J$ is a strongly regular ideal of $L\}$ where an ideal J is said to be *strongly regular* if $x \in J$ implies $y \in J$ for some $x \triangleleft y$. Then $(\gamma L, \vee)$ is a compactification of L with $\vee\colon \gamma L \to L$ being the join map. Its right adjoint $k\colon L \to \gamma L$ is described as $k(a) = \{x \in L\colon\ x \triangleleft a\}$. On the other hand, if (M, h) is a compactification of L, we define $\triangleleft$ on L by: $x \triangleleft y \Leftrightarrow r(x) \prec r(y)$ where $r\colon L \to M$ is the right adjoint of $h\colon M \to L$. Then $\triangleleft$ is a strong inclusion on L.

A congruence on a frame L is an equivalence relation on L which is also a subframe of $L \times L$. The *congruence lattice* $\mathcal{C}L$ of L consists of all the congruences on L. It is a frame with bottom element $\Delta = \{(x, x)\colon\ x \in L\}$ and top element $\nabla = L \times L$. Two particular congruences associated with each $a \in L$ are $\Delta_a = \{(x, y) \in L \times L\colon\ x \wedge a = y \wedge a\}$ and $\nabla_a = \{(x, y) \in L \times L\colon\ x \vee a = y \vee a\}$. These members of $\mathcal{C}L$ are complementary to each other in the sense that their meet is the bottom and their join is the top element. In general, a congruence Θ does not necessarily have a complement in $\mathcal{C}L$ though of course it must have a pseudocomplement which we will denote by Θ^*.

It was shown by Banaschewski [3] that the corresponding compactification of the regular continuous frame L arising from the smallest strong inclusion on it as described above, is the smallest compactification of L. This is the analog in frames of the Alexandroff one-point compactification of a locally compact non-compact Hausdorff space. We recall from Banaschewski [3] also that if L is non-compact and compactifiable with a least strong inclusion, then its corresponding least compactification $h\colon M \to L$ satisfies the following: There exists a unique $a \in M$, $a < e$ such that $h(a) = e$. The map $\downarrow a \to L$ by h is then codense, hence one-to-one, and since it is onto, this makes $\downarrow a \cong L$ via the map h. Furthermore, for each $x \in M$ either $x \leqslant a$ or $x \vee a = e$.

The reader is referred to the text of Johnstone [4] for a general reference on frames.

2. Remainder of a frame

We need the concept of the remainder of a frame L in its compactification $h\colon M \to L$ specifically for the class of regular continuous frames L. As a prelude to the discussion on remainders and for the purpose of obtaining our main characterization result later, we note the following theorem proved by Aarts and Van Emde Boas [1]:

Theorem 2.1. *Suppose X is a locally compact, non-compact metric space. Then each remainder of X in a compactification of X is a continuum if and only if no compact subset of X splits X at infinity.*

They remark that the above result is also true in the non-metric case, a remark that prompted our investigation as to whether the analog of this result holds for regular continuous frames in general.

Before we embark on this we draw attention to the following well-known result for spaces: A dense subset of a compact Hausdorff space is locally compact if and only if it is open.

The analogous result for frames does indeed hold as we show below.

Theorem 2.2. *Let $h\colon M \to L$ be any compactification of L, where L is non-compact. Then L is regular continuous if and only if $L \cong {\downarrow}a$ for some $a \in M$.*

Proof. "$\Leftarrow$" This was shown by Banaschewski in [3]

"$\Rightarrow$" Now assume L is regular continuous. Put $a = \bigvee r(x)(x \ll e)$ where r is the right adjoint of h. Now if $a = e$, then by the compactness of M, there exists x_i, $i = 1, 2, \ldots, n$ say, such that $x_i \ll e$ for all i and $r(x_1) \vee \ldots \vee r(x_n) = e$. Hence $e = h(r(x_1) \vee \ldots \vee r(x_n)) = hr(x_1) \vee \ldots \vee hr(x_n) = x_1 \vee \ldots \vee x_n$. Since $x_i \ll e$ for all i, we have $x_1 \vee \ldots \vee x_n \ll e$ and so $e \ll e$, contradicting the fact that L is not compact. Thus $a < e$. Furthermore $h(a) = \bigvee hr(x)(x \ll e) = \bigvee x(x \ll e) = e$. We claim that ${\downarrow}a \cong L$ via the map h: To see this note firstly that h is dense. It is also onto since $h\colon M \to L$ is onto and $h(a) = e$.

Also $h\colon {\downarrow}a \to L$ is codense. For take $b \leqslant a$ such that $h(b) = e$. We shall show $a \leqslant b$. For this take any $x \ll e$. We show $r(x) \leqslant b$. Now $b = \bigvee z(z \prec b)$, so that $e = h(b) = \bigvee h(z)(z \prec b)$. Now $\bigvee(x^* \vee h(z))(z \prec b) = e$. Since $x \ll e$ we have ${\uparrow}x^*$ is compact, and hence we can find $z \prec b$ such that $x^* \vee h(z) = e$. Now $z \prec b$ implies $rh(z) \leqslant b$ by using the fact that h is dense and onto. Also $x \leqslant h(z)$ implies $r(x) \leqslant rh(z) \leqslant b$ as required. Thus ${\downarrow}a \cong L$ via the map h. □

Remark 2.3. We have mentioned at the end of Section 1 the result of Banaschewski [3] that for a non-compact frame L with least compactification $h\colon M \to L$ there is a unique $a \in M$ such that $a < e$ and $h(a) = e$. Moreover, ${\downarrow}a \cong L$ (via h) is an isomorphism and a is a maximal element in M. The uniqueness of $a \in M$ with the stated properties, if one studies the proof of Banaschewski [3], is a consequence of $h\colon M \to L$ being the least compactification of L, and the maximality of a comes from the uniqueness of a.

An examination of the above theorem reveals that for a regular continuous L and any compactification $h\colon M \to L$, whether least or not, there exists $a \in M$, $a < e$, $h(a) = e$ such that $\downarrow a \cong L$ (via h).

Proposition 2.4. *If L is non-compact regular continuous and $h\colon M \to L$ is a compactification of L, then there exists $a \in M$, $a < e$, $h(a) = e$ such that $\downarrow a \cong L$ (via h). Moreover, $a \in M$ is unique with respect to these properties.*

Proof. The existence of $a \in M$ with the stated properties follows from the above theorem. To show uniqueness, assume there exists $b \in M$ with the stated properties. We have $\downarrow a \xrightarrow{h} L$ is an isomorphism, so it is codense. Now $a \wedge b \in \downarrow a$ and $h(a \wedge b) = h(a) \wedge h(b) = e$, so by codenseness, $a \wedge b = a$ so $a \leqslant b$. Also $\downarrow b \xrightarrow{h} L$ is an isomorphism so it is codense. By the same argument we have $b \leqslant a$. Thus $a = b$, showing uniqueness. □

If L is non-compact regular continuous and $h\colon M \to L$ is any compactification of L, denote this unique $a \in M$ by a_L. Thus $\downarrow a_L \cong L$ (via h). We define the *remainder* of L in the compactification $h\colon M \to L$ by $\uparrow a_L$. Recall from Baboolal [2] that for any compactification $h\colon M \to L$, L not necessarily regular continuous, we defined the remainder of L in $h\colon M \to L$ to be M/θ^* where θ^* is the pseudocomplement of θ in the congruence lattice of M and $\theta = \ker h$. The above definition of the remainder for a regular continuous L coincides with the general one since the congruence Δ_{a_L} corresponds to $\downarrow a_L$ and the pseudocomplement of Δ_{a_L}, which is ∇_{a_L}, corresponds to $\uparrow a_L$.

3. The frame analog of the two-point compactification

We construct a compactification for a class of regular continuous frames, the analog of the two point compactification for locally compact Hausdorff spaces.

To begin with let L be regular continuous. Suppose L has elements u and v with the property that $u \wedge v = 0$, $\uparrow u \vee v$ is compact, but neither $\uparrow u$ nor $\uparrow v$ is compact. Let $N_1 = \{x \in L\colon \uparrow x \vee u \text{ is compact}\}$ and $N_2 = \{x \in L\colon \uparrow x \vee v \text{ is compact}\}$.

Recall also that a filter F on a frame L is said to be *regular* if $x \in F$ implies there exists $y \in F$ such that $y \prec x$.

At several points in the proofs later on we make use of the following simple, but useful, observation. This is perhaps very well known or part of the folklore of the subject.

Proposition 3.1. *If L is a regular continuous frame then $z \ll e$ iff $\uparrow z^*$ is compact.*

Proof. Suppose $z \ll e$. Take $e = \bigvee S$ with $z^* \leqslant s$ for each $s \in S$. By the interpolation property we can find t such that $z \ll t \ll e$. Now $t \ll e$ implies there exists finite $S_0 \subseteq S$ such that $t \leqslant \bigvee S_0$. Now since $z^* \vee t = e$ we have $z^* \vee \bigvee S_0 = e$ and, because of the condition satisfied by z^*, we have then that $\bigvee S_0 = e$. This shows that $\uparrow z^*$ is compact.

Now suppose $\uparrow z^*$ is compact, and let $e = \bigvee S$. Then $e = \bigvee (z^* \vee s)(s \in S)$ so by the compactness of $\uparrow z^*$ we have $e = z^* \vee \bigvee S_0$ for some finite subset S_0 of S. Hence $z \leqslant \bigvee S_0$ and thus $z \ll e$. □

Lemma 3.2. *N_1 and N_2 are regular proper filters of L.*

Proof. Note that 0 is not in N_i $(i = 1, 2)$ since $\uparrow u$ and $\uparrow v$ are not compact. Using the fact that $\uparrow s, \uparrow t$ is compact if and only if $\uparrow s \wedge t$ is compact, and $\uparrow s$ compact, $s \leqslant t$ implies $\uparrow t$ is compact, it follows that N_1 and N_2 are proper filters of L. We now show regularity of N_1: Let $w \in N_1$. Then $\uparrow w \vee u$ is compact. Now $e = \bigvee z(z \ll e)$, so $\uparrow w \vee u$ compact implies there exists $z \ll e$ such that $w \vee u \vee z = e$. Now $z \ll e$ if and only if $\uparrow z^*$ is compact and thus $z^* \in N_1$.

Now $z \vee u \vee w = e \Longrightarrow (z \vee u)^{**} \vee w = e \Longrightarrow (z \vee u)^* \prec w \Longrightarrow z^* \wedge u^* \prec w$. Now $u \wedge v = 0$ implies $v \leqslant u^*$, and since $v \in N_1$ (as $\uparrow u \vee v$ is compact), we have $u^* \in N_1$. Thus $z^* \in N_1$, $u^* \in N_1$ and hence $z^* \wedge u^* \in N_1$. Thus N_1 is regular. The argument that N_2 is regular follows by symmetry and we are therefore done. □

Lemma 3.3. *$\uparrow w$ is compact if and only if $w \in N_1 \cap N_2$.*

Proof. Assume $\uparrow w$ is compact. Then $\uparrow w \vee u$ and $\uparrow w \vee v$ are compact. Thus $w \in N_1 \cap N_2$. Conversely if $w \in N_1 \cap N_2$, then $\uparrow w \vee u$ and $\uparrow w \vee v$ are compact. Hence $\uparrow (w \vee u) \wedge (w \vee v)$ is compact. Since $u \wedge v = 0$ this means $\uparrow w$ is compact. □

Now define $a \lhd b$ in L by: $a \lhd b \Leftrightarrow a \prec b$ and for each $i = 1, 2$ either $a^* \in N_i$ or $b \in N_i$.

Lemma 3.4. *$\lhd$ is a strong inclusion on L.*

Proof. (i) Assume $x \leqslant a \lhd b \leqslant y$. Now $a \prec b$, so clearly $x \prec y$. Take any N_i $(i = 1, 2)$. If $a^* \in N_i$, then $x^* \in N_i$ since $a^* \leqslant x^*$ and N_i is a filter. If $b \in N_i$, then $b \leqslant y$ implies $y \in N_i$. Thus $x \lhd y$.

(ii) $0 \lhd 0$ since $0 \prec 0$ and $0^* = e \in N_i$ for each i. Also $e \lhd e$ since $e \prec e$ and $e \in N_i$. Then suppose $x, y \lhd a$. Then $x \prec a$, $y \prec a$, so $x \vee y \prec a$. Fix i. If $a \in N_i$, then $x \vee y \lhd a$. If $x^* \in N_i$, $y^* \in N_i$, then $x^* \wedge y^* \in N_i$, i.e. $(x \vee y)^* \in N_i$. Thus $x \vee y \lhd a$. Now suppose $x \lhd a$, $x \lhd b$. Then $x \prec a$, $x \prec b$, so $x \prec a \wedge b$. Fix i. If $a \wedge b \in N_i$, then

$x \lhd a \wedge b$. If not, then either a lies outside N_i or b lies outside N_i. Thus $x^* \in N_i$ and hence $x \lhd a \wedge b$.

(iii) $x \lhd a$ implies $x \prec a$ follows from the definition.

(iv) Now suppose $x \lhd a$. Then $x \prec a$. If either $\uparrow x^*$ or $\uparrow a$ is compact, then $x \blacktriangleleft a$ in which case there exists $y \in L$ such that $x \blacktriangleleft y \blacktriangleleft a$. Since $\blacktriangleleft \subseteq \lhd$ this means $x \lhd y \lhd a$ so that interpolation holds. If $\uparrow x^*$ is not compact and $\uparrow a$ is not compact then both x^* and a lie outside $N_1 \cap N_2$ by the above lemma. There are two cases here: (a) $x^* \in N_1$ and $a \in N_2$, and (b) $x^* \in N_2$ and $a \in N_1$. Symmetry considerations make it sufficient to consider just one of these cases, say $x^* \in N_1$ and $a \in N_2$. We seek y such that $x \lhd y \lhd a$. By the fact that $\uparrow x^* \vee u$ is compact, $x \prec a$ and $a = \bigvee z(z \ll a)$, we can find $z \ll a$ such that $x^* \vee u \vee z = e$. Also from the fact that $\uparrow a \vee v$ is compact, $x \prec a$ and $x^* = \bigvee t(t \ll x^*)$, we can find $t \ll x^*$ such that $a \vee v \vee t = e$. Now we have $x \prec u \vee z$ and $x \prec t^*$, and thus $x \prec (u \vee z) \wedge t^* = (u \wedge t^*) \vee (z \wedge t^*)$. We also have $u \wedge t^* \prec a$, since the element $v \vee t$ is such that $a \vee v \vee t = e$ and $(u \wedge t^*) \wedge (v \vee t) = (u \wedge t^* \wedge v) \vee (u \wedge t^* \wedge t) = 0$. Also $z \wedge t^* \leqslant z \prec a$. Hence $(u \wedge t^*) \vee (z \wedge t^*) \prec a$. Thus $x \prec (u \wedge t^*) \vee (z \wedge t^*) = t^* \wedge (u \vee z) \prec a$. Put $y = t^* \wedge (u \vee z)$. We claim that $x \lhd y \lhd a$.

$x \lhd y$: Obviously $x \prec y$, $x^* \in N_1$. We show that $y \in N_2$. Now $t \ll e$ implies $\uparrow t^*$ is compact and hence $t^* \in N_1 \cap N_2$ by the above lemma. Also $u \vee z \in N_2$ since $u \in N_2$. Thus $t^* \wedge (u \vee z) \in N_2$, i.e. $y \in N_2$. Hence $x \lhd y$.

$y \lhd a$: Obviously $y \prec a$ and $a \in N_2$. Now $z \ll a$ implies $z \ll e$ and so $\uparrow z^*$ is compact. Hence $z^* \in N_1$. Also $v \in N_1, v \leqslant u^*$ implies $u^* \in N_1$. Furthermore, $z^* \wedge u^* \leqslant y^*$ since $z^* \wedge u^* \wedge y = z^* \wedge u^* \wedge (t^* \wedge (u \vee z)) = (z^* \wedge u^* \wedge t^* \wedge u) \vee (z^* \wedge u^* \wedge t^* \wedge z) = 0$. Hence $y^* \in N_1$ and thus $y \lhd a$. Thus $\lhd$ interpolates.

(v) $x \lhd a \Longrightarrow a* \lhd x^*$: $x \lhd a$ implies $x \prec a$ and hence $a^* \prec x^*$. Again, if either $\uparrow x^*$ or $\uparrow a$ is compact, then $x \blacktriangleleft a$ and hence $a^* \blacktriangleleft x^*$ from which $a^* \lhd x^*$. As in (iv) we need only consider the case $x^* \in N_1$, $a \in N_2$. In this case $a \leqslant a^{**}$ implies $a^{**} \in N_2$. Since also $x^* \in N_1$ we have then that $a^* \lhd x^*$ as required.

(vi) For each $a \in L$, $a = \bigvee x(x \blacktriangleleft a)$. But as remarked earlier, $x \blacktriangleleft a$ implies $x \lhd a$. Hence $a = \bigvee x(x \lhd a)$.

Thus $\lhd$ is a strong inclusion on L. □

Let $\vee\colon \alpha L \to L$ be the least compactification corresponding to the least strong inclusion $\blacktriangleleft$ on L and let $k\colon L \to \alpha L$ be its right adjoint. Let $J = \bigvee k(x)(x \ll e)$, so that by Section 2, $\uparrow J$ would be the remainder of L in αL. Note further that $J < L$. We then have:

Lemma 3.5. *If $w \in L$, then $\uparrow w$ is compact if and only if $k(w) \vee J = L$.*

P r o o f. Assume $\uparrow w$ is compact. Now $w \vee \bigvee x(x \ll e) = e$. Thus $\bigvee w \vee x(x \ll e) = e$, so by compactness of $\uparrow w$ we have $w \vee x = e$ for some $x \ll e$. Thus $w \vee x^{**} = e$, whence $x^* \prec w$. Since $\uparrow w$ is compact, this means $x^* \blacktriangleleft w$ and hence $x^* \in k(w)$. Now $\bigvee J = e$ implies $\bigvee x^* \vee y(y \in J) = e$, from which, since $\uparrow x^*$ is compact (as $x \ll e$), we have $x^* \vee y = e$ for some $y \in J$. Thus $k(w) \vee J = L$.

Conversely, assume $k(w) \vee J = L$. Then $e = x \vee y$ for some $x \blacktriangleleft w$, $y \in J$. Now either $\uparrow x^*$ is compact or $\uparrow w$ is compact. If $\uparrow x^*$ is compact, then $x \ll e$, so that $x \ll z \ll e$ for some $z \in L$. Hence $x \blacktriangleleft z \ll e$ so that $x \in k(z) \subseteq J$. Thus $e = x \vee y \in J$ which would imply $J = L$, a contradiction. Thus $\uparrow w$ is compact. □

Remark 3.6. Observe that the necessity of the above lemma that $\uparrow w$ compact implies $k(w) \vee J = L$ is in fact true for any strong inclusion $\lhd$ on L, whether least or not, where as before k is the right adjoint of the join map and $J = \bigvee k(x)(x \ll e)$.

Theorem 3.7. *Let L be regular continuous. Suppose L has elements u and v with the property that $u \wedge v = 0$, $\uparrow u \vee v$ is compact, but neither $\uparrow u$ nor $\uparrow v$ is compact. Let $N_1 = \{x \in L\colon \uparrow x \vee u \text{ is compact}\}$ and $N_2 = \{x \in L\colon \uparrow x \vee v \text{ is compact}\}$. The compactification $\vee\colon \gamma L \to L$ arising from the strong inclusion $\lhd$ given by: $a \lhd b \Leftrightarrow a \prec b$ and for each $i = 1, 2$ either $a^* \in N_i$ or $b \in N_i$ is such that the remainder of L in it is disconnected.*

P r o o f. Let $J = \bigvee k(x)(x \ll e)$ where $k\colon L \to \gamma L$ is the right adjoint of the join map. We show that the remainder $\uparrow J$ in γL is disconnected. We claim that $k(u \vee v) = k(u) \vee k(v)$: For this, obviously $k(u) \vee k(v) \subseteq k(u \vee v)$. For the reverse, take $s \in k(u \vee v)$. Then $s \lhd u \vee v$ and hence $s \prec u \vee v$. Since $u \wedge v = 0$ we have that $s \wedge u \prec u$ and $s \wedge v \prec v$. We have $(s \wedge u)^* \vee u = e$, so $\uparrow (s \wedge u)^* \vee u = \uparrow e$ is compact. Thus $(s \wedge u)^* \in N_1$. Also, since $\uparrow u \vee v$ is compact, we have $u \in N_2$. Thus for each i we have either $(s \wedge u)^* \in N_i$ or $u \in N_i$, i.e. $s \wedge u \lhd u$. Similarly $s \wedge v \lhd v$. Thus $s = (s \wedge u) \vee (s \wedge v) \in k(u) \vee k(v)$, proving the claim. Since $\uparrow u \vee v$ is compact we have by the above remark that $k(u \vee v) \vee J = L$ and hence that $k(u) \vee k(v) \vee J = L$. Thus $(k(u) \vee J) \vee (k(v) \vee J) = L$, $(k(u) \vee J) \wedge (k(v) \vee J) = J$ since $k(u) \wedge k(v) = k(0) = 0$. Furthermore, $k(u) \vee J \neq J$ for otherwise $k(u) \subseteq J$ and hence $k(v) \vee J = L$. Since $J = \bigvee k(x)(x \ll e)$ we have by the compactness of γL that $k(v) \vee k(x) = L$ for some $x \ll e$. Taking joins we then have $v \vee x = e$. Hence $\uparrow v \vee x = \uparrow e$ is compact so that $x \in N_2$. Now since $x \ll e$ we have $\uparrow x^*$ is compact and therefore $x^* \in N_2$. Thus $0 = x \wedge x^* \in N_2$ implying that $\uparrow v$ is compact, a contradiction. Hence $k(u) \vee J \neq J$ and similarly $k(v) \vee J \neq J$. Thus $\uparrow J$ is disconnected in γL. □

4. The characterization

We recall from [2] that a compactification $h\colon M \to L$ is called a perfect compactification if the right adjoint r of h satisfies the condition: $r(u \vee u^*) = r(u) \vee r(u^*)$ for all $u \in L$. This is the frame analog of the topological definition due to Sklyarenko [6] described earlier. The Stone-Čech compactification of a completely regular frame is an example of such a compactification as was shown in [2].

As a first step towards our characterization we now prove the following:

Theorem 4.1. *Let L be regular continuous. Then every compactification $h\colon M \to L$ has a remainder which is compact and connected if and only if whenever $\uparrow u \vee v$ is compact and $u \wedge v = 0$ in L, then either $\uparrow u$ is compact or $\uparrow v$ is compact.*

Proof. We prove the sufficiency first. Let $h\colon M \to L$ be a compactification of L. To avoid unnecessary symbols let us simply denote the unique $a_L \in M$ determining the remainder of L in M described earlier by $a \in M$. Now $\uparrow a$ is compact, being a closed sublocale of compact M. Assume $\uparrow a$ is not connected. Then there exists $c, d \in \uparrow a$, $c, d \neq a$ such that $c \vee d = e$ and $c \wedge d = a$. Since M, being compact regular, is normal there exists $f, g \in M$ such that $c \vee f = e$, $d \vee g = e$, and $f \wedge g = 0$. Now $(c \vee f) \wedge (d \vee g) = e \Longrightarrow ((c \vee f) \wedge d) \vee ((c \vee f) \wedge g) = e \Longrightarrow (c \wedge d) \vee (f \wedge d) \vee (c \wedge g) \vee (f \wedge g) = e \Longrightarrow a \vee f \vee g = e$. Consider the frame $\downarrow a$. We claim that in this frame $\uparrow^{\downarrow a}$ $(f \wedge a) \vee (g \wedge a)$ is compact. For this consider the map $\varphi\colon \uparrow f \vee g \to \uparrow^{\downarrow a} a \wedge (f \vee g))$ given by $\varphi(x) = x \wedge a$. We have $\varphi(f \vee g) = a \wedge (f \vee g)$, $\varphi(e) = e \wedge a = a$ so φ preserves top and bottom. It is then clearly a frame map. Furthermore, $\varphi(x) = \varphi(y) \Longrightarrow x \wedge a = y \wedge a \Longrightarrow x = x \wedge (a \vee f \vee g) = (x \wedge a) \vee (x \wedge (f \vee g)) = (y \wedge a) \vee (f \vee g) \leqslant y \vee y = y$, so that $x = y$, by symmetry. Thus φ is one to one. Furthermore, φ is also onto. Indeed, take $y \in M$, $a \wedge (f \vee g) \leqslant y$ and $y \leqslant a$. Then $\varphi(y \vee (f \vee g)) = (y \vee (f \vee g)) \wedge a = (y \wedge a) \vee ((f \vee g) \wedge a) = y \vee ((f \vee g) \wedge a) = y$. Thus $\uparrow f \vee g \cong \uparrow^{\downarrow a} a \wedge (f \vee g)$, and since $\uparrow f \vee g$ is compact, being a closed sublocale of M, $\uparrow^{\downarrow a} a \wedge (f \vee g)$ must also be compact. Thus $\uparrow^{\downarrow a} (f \wedge a) \vee (g \wedge a)$ is compact. Since $h\colon \downarrow a \to L$ is an isomorphism and $\uparrow^{\downarrow a} (f \wedge a) \vee (g \wedge a)$ is compact, we must have $\uparrow h(f) \vee h(g)$ compact in L. Since $h(f) \wedge h(g) = 0$, we must have either $\uparrow h(f)$ compact or $\uparrow h(g)$ compact, say $\uparrow h(f)$ compact.

Now take any $0 \neq z \prec f$. Then $z^* \vee f = e$ and hence $h(z^*) \vee h(f) = e$. Now $h(z^*) = \bigvee w (w \ll h(z^*))$, and hence $h(f) \vee \bigvee w (w \ll h(z^*)) = e$, i.e. $\bigvee (h(f) \vee w)(w \ll h(z^*)) = e$. Due to compactness of $\uparrow h(f)$ we can therefore find $w \ll h(z^*)$ such that $h(f) \vee w = e$. Now $w \ll h(z^*) \Longrightarrow w \ll e$ and hence $r(w) \leqslant a$ by the definition of a. Now $h(f^*) \leqslant w$ since $h(f) \vee w = e$ and hence $f^* \leqslant r(w) \leqslant a$. Thus since $g \wedge f = 0$, we have $g \leqslant f^* \leqslant a$, and therefore $e = d \vee g \leqslant d \vee a = d$ since $a \leqslant d$.

Hence $c = c \wedge e = c \wedge d = a$, a contradiction since $c \neq a$. Thus the remainder $\uparrow a$ is connected.

For the necessity suppose every compactification $h\colon M \to L$ has a remainder which is compact and connected. Assume the condition on L is not satisfied. Then there exists $u, v \in L$, $u \wedge v = 0$, $\uparrow u \vee v$ compact but neither $\uparrow u$ nor $\uparrow v$ is compact. It follows that $u \neq 0$ and $v \neq 0$. From Section 3 we can construct a compactification of L such that the remainder of L in it is disconnected. Thus the condition on L must be satisfied. □

We can now prove our main result:

Theorem 4.2. *The following conditions are equivalent for a non-compact regular continuous frame L.*

(1) *The least compactification of L is perfect.*

(2) *Whenever $\uparrow u \vee v$ is compact, $u, v \in L$, $u \wedge v = 0$ then either $k(u) \vee J = L$ or $k(v) \vee J = L$ where $J \subsetneqq L$ is the unique element in αL such that $\downarrow J \xrightarrow{\vee} L$ is an isomorphism, and $k\colon L \to \alpha L$ is the right adjoint of $\vee$.*

(3) *Whenever $\uparrow u \vee v$ is compact, $u, v \in L$, $u \wedge v = 0$, then either $\uparrow u$ is compact or $\uparrow v$ is compact.*

(4) *For every compactification $h\colon M \to L$ the remainder of L in it is compact and connected.*

Proof. (1) ⇒ (2) Assume $\vee\colon \alpha L \to L$ is perfect. Take $u, v \in L$, $u \wedge v = 0$ with $\uparrow u \vee v$ compact. By Lemma 3.5, $k(u \vee v) \vee J = L$. Since $(\alpha L, \vee)$ is perfect, we then have $k(u) \vee k(v) \vee J = L$. Now, we cannot have both $k(u) \subseteq J$ and $k(v) \subseteq J$, otherwise $J = L$ which is not possible. Thus either $k(u) \nsubseteq J$ or $k(v) \nsubseteq J$. Hence, by the remarks at the end of Section 1, we have either $k(u) \vee J = L$ or $k(v) \vee J = L$.

(2) ⇒ (3) Suppose $\uparrow u \vee v$ is compact, $u \wedge v = 0$. Then either $k(u) \vee J = L$ or $k(v) \vee J = L$, and hence by Lemma 3.5, either $\uparrow u$ is compact or $\uparrow v$ is compact.

(3) $\Longrightarrow$ (1) We recall from Baboolal [2] that if $h\colon M \to L$ is a compactification of L with $r\colon L \to M$ the right adjoint of h, then $h\colon M \to L$ is perfect if and only if the following condition is satisfied: $x \lhd u \vee u^*$, $x \leqslant u$ implies $x \lhd u$ for all $x, u \in L$, where $\lhd$ is the associated strong inclusion arising from $h\colon M \to L$. In the present case of $\vee\colon \alpha L \to L$ with right adjoint $k\colon L \to \alpha L$, we have to show $x \blacktriangleleft u \vee u^*$, $x \leqslant u$ implies $x \blacktriangleleft u$.

Consider first the case when $\uparrow u \vee u^*$ is compact. Then either $\uparrow u$ is compact or $\uparrow u^*$ is compact. Now $x \blacktriangleleft u \vee u^*$ implies $x \prec u \vee u^*$, and $x \leqslant u$ implies $x \prec u$: for, there exists v such that $x \wedge v = 0$, $v \vee u \vee u^* = e$. Thus $x \wedge (v \vee u^*) = (x \wedge v) \vee (x \wedge u^*) = 0$ and $v \vee u^* \vee u = e$ with a separating element $v \vee u^*$. If $\uparrow u$ is compact then $x \blacktriangleleft u$. If, on the other hand, $\uparrow u^*$ is compact, then $x \prec u$ implies $u^* \prec x^*$ from which it follows that $\uparrow x^*$ is compact. This implies $x \blacktriangleleft u$ as well.

Now consider the case where $\uparrow u \vee u^*$ is not compact. Take $x \blacktriangleleft u \vee u^*$, $x \leqslant u$. As before, $x \prec u$. Also either $\uparrow x^*$ is compact or $\uparrow u \vee u^*$ is compact. Since the latter is not possible, we have $\uparrow x^*$ is compact. Hence $x \blacktriangleleft u$ and thus $(\alpha L, \vee)$ is a perfect compactification. □

References

[1] *J. M. Aarts, P. Van Emde Boas*: Continua as remainders in compact extensions. Nieuw Arch. Wisk., III. Ser. *15* (1967), 34–37.

[2] *D. Baboolal*: Perfect compactifications of frames. Czech. Math. J. *61(136)* (2011), 845–861.

[3] *B. Banaschewski*: Compactification of frames. Math. Nachr. *149* (1990), 105–115.

[4] *P. T. Johnstone*: Stone Spaces. Cambridge University Press, Cambridge, 1982.

[5] *C. F. K. Jung*: Locally compact spaces whose Alexandroff one-point compactifications are perfect. Colloq. Math. *27* (1973), 247–249.

[6] *E. G. Sklyarenko*: Some questions in the theory of bicompactifications. Am. Math. Soc. Transl., II. Ser. *58* (1966), 216–244.

Author's address: Dharmanand Baboolal, School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa, e-mail: baboolald@ukzn.ac.za.