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AN UNBOUNDED BERGE'S MINIMUM THEOREM WITH APPLICATIONS TO DISCOUNTED MARKOV DECISION PROCESSES

Raúl Montes-de-Oca and Enrique Lemus-Rodríguez*

This paper deals with a certain class of unbounded optimization problems. The optimization problems taken into account depend on a parameter. Firstly, there are established conditions which permit to guarantee the continuity with respect to the parameter of the minimum of the optimization problems under consideration, and the upper semicontinuity of the multifunction which applies each parameter into its set of minimizers. Besides, with the additional condition of uniqueness of the minimizer, its continuity is given. Some examples of nonconvex optimization problems that satisfy the conditions of the article are supplied. Secondly, the theory developed is applied to discounted Markov decision processes with unbounded cost functions and with possibly noncompact actions sets in order to obtain continuous optimal policies. This part of the paper is illustrated with two examples of the controlled Lindley's random walk. One of these examples has nonconstant action sets.

Keywords: Berge's minimum theorem, moment function, discounted Markov decision process, uniqueness of the optimal policy, continuous optimal policy

Classification: 90C40, 93E20, 90A16

1. INTRODUCTION

Let X and A be nonempty Borel spaces. For each $x \in X$, let $\gamma(x)$ be a nonempty subset of A. Let $Gr(\gamma) := \{(x, a) : x \in X, a \in \gamma(x)\}$, and let $G : Gr(\gamma) \to \mathbb{R}$ be a nonnegative function. Consider the following minimization problem:

$$\inf_{a \in \gamma(x)} G(x, a), \quad x \in X. \tag{1}$$

Let $f^* : X \to A$, such that $f^*(x) \in \gamma(x)$, $x \in X$, be a minimizer of (1) assuming, of course, that such a minimizer exists, and let G^* be the corresponding optimal value function, i. e.

$$G^*(x) = G(x, f^*(x)) = \inf_{a \in \gamma(x)} G(x, a), \quad x \in X. \tag{2}$$

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The first part of this paper is concerned with establishing a version of the Berge's Minimum Theorem (see [5], p. 116) which permits to obtain the continuity of G^*, and the upper semicontinuity of the multifunction $x \to \gamma^*(x) := \{a \in \gamma(x) : G(x,a) = G^*(x)\}$.

The main condition that has been imposed in this part of the article is that G has to be a moment function i. e., that G grows without a bound on the complement of compact sets. Additionally, for the continuity of f^*, its uniqueness is required.

These conditions permit to deal with unbounded problems, i. e. with minimization problems with possibly unbounded function G and possibly noncompact restrictions sets $\gamma(x)$, $x \in X$, and they also work for minimization problems for which G and/or the restrictions sets $\gamma(x)$, $x \in X$, are nonconvex (see Section 3 below).

It is important to mention that the moment functions have been used in different classes of stochastic control problems (see,[10, 20, 21, 24] and [30]).

The first antecedent in the study of the continuity of G^* and the upper semicontinuity of $x \to \gamma^*(x)$ requiring the compactness of the restriction sets $\gamma(x)$, $x \in X$, is known as the Minimum Theorem (and related results) due to Berge (see [5] pp. 115–117). (In fact, Berge in [5] works with maximization problems and in his book he naturally referred to the result related to the continuity of G^* and the upper semicontinuity of $x \to \gamma^*(x)$ as the Maximum Theorem).

In Lemmas 6.11.8 and 6.11.9 of [27] the continuity of G^* and f^* has been analyzed under the assumption that $\gamma(x) = A$ for all $x \in X$, provided that A is a compact set.

Also in [11] a result concerning the continuity of G^* and f^* is presented, but the convexity of G is assumed. Nevertheless, it is important to consider the nonconvex case as well, regarding not only the economical application [15, 17] but also its importance when the action set is finite or disconnected, and hence, nonconvex [18].

The major bulk of the research on Berge's Theorem assumes boundedness on the reward (or cost) function and compactness of the $\gamma(x)$, $x \in X$. See,[4],[8],[9],[14],[16], [25],[32],[33],[34]. The importance of the unbounded, noncompact case is apparent in such a work as [15] or [17] in economics, and [10], where large bibliography related to Markov decision processes (MDPs) can be found on this subject. Hence, presenting a version of the Berge's Theorem for the unbounded noncompact case can, in our opinion, considerably extend its usefulness.

On the other hand, correspondences or multifunctions are basic tools in contemporary economic mathematical modelling in such problems as consumer theory or game-theoretical modelling of economic interactions, where they naturally appear from the start as basic building blocks of the corresponding mathematical model. Berge's Maximum Theorem provides extremely valuable information about the continuity of the optimal actions of the agents involved, for instance, under adequate conditions, it guarantees the continuity of the indirect utility function in consumer theory, and the upper semi-continuity of the best responses correspondence, in game theory. Further information on the application of Berge's Theorem to Dynamic Programming and to the problems mentioned above can be found in [25] and [31].

The second part of the paper deals with the application of the results of the first part to MDPs on Borel spaces, with (possibly) unbounded cost function, with (possibly) noncompact action sets, and with the expected total discounted cost as objective function (see [10]).

For this type of MDPs the existence of stationary optimal policies as minimizers of the Optimality Equation (OE) is assumed (see [10]). And for such a kind of discounted Markov decision process, denote by f^* its optimal policy, and by V^* its optimal value function [10].

In this part, the function G given in (1) is the right-hand side of the OE and the main conditions which ensure the continuity of f^* and V^* are the uniqueness of f^* (see [6] for conditions for the uniqueness of optimal policies of discounted MDPs) and the fact that the cost function c is a moment function.

The theory presented in this part of the article is applied to the very important models proposed by Lindley [22], that in the paper will be referred to as Lindley's random walk, useful in queueing and dam management theories; for the controlled case of the Lindley's random walk see, for instance, [35].

When dealing with MDPs in economic applications, the continuity of the optimal policy f^* greatly simplifies or clarifies the analysis of the corresponding stability of the model [16, 17, 23]. However, this is not always stressed in a more theoretical research, becoming then an important research area. This paper suggests a line of research on the uniqueness of the optimal policy based on the work started in [6].

It is interesting to note that research on discounted MDPs, in a non-Berge's Theorem approach, usually analyzes continuity of the value function, see for example [12].

The continuity of the optimal policy f^* is established in [13] for linear models, with finite horizon, constant multifunction $x \to \gamma(x)$, under convexity assumptions. The nonlinear case, with infinite horizon, a possibly nonconstant multifunction $x \to \gamma(x)$ and nonconvexity restrictions is, of course, of the great interest, precisely motivated by [13], and constitutes an important portion of this paper.

The paper is organized as follows. Section 2 presents the minimization problem. Section 3 gives the version of the Minimum Theorem and some nonconvex examples. Section 4 applies Section 3 to discounted MDPs, and Section 5 presents two examples to illustrate the theory developed in the previous section.The final (Conclusions) Section is followed by two appendixes which contain the complete details of the proofs of the examples in the article.

2. PRELIMINARIES

For short, throughout the paper, u.s.c will be used for upper semicontinuous, and l.s.c. for lower semicontinuous.

Let X and A be nonempty Borel spaces (i. e. measurable subsets of complete and separable metric spaces).

Now, some basics on multifunctions are supplied in this section, for more information see [1].

A *multifunction* γ from X to A is a function from X to A whose value $\gamma(x)$, for each $x \in X$, is a nonempty subset of A.

The *graph* of the multifunction γ is a subset of $X \times A$ defined as $Gr(\gamma) := \{(x, a) : x \in X,\ a \in \gamma(x)\}$.

Definition 2.1. A multifunction γ from X to A is said to be

(a) *Borel-measurable* if $\{x \in X : \gamma(x) \cap O \neq \emptyset\}$ is a Borel subset of X for every open set $O \subset A$;

(b) *upper semicontinuous* if $\{x \in X : \gamma(x) \cap F \neq \emptyset\}$ is closed in X for every closed $F \subset A$;

(c) *lower semicontinuous* if $\{x \in X : \gamma(x) \cap O \neq \emptyset\}$ is open for every open set $O \subset A$;

(d) *continuous* if it is both u.s.c. and l.s.c.

The terms correspondence (instead of multifunction) and hemicontinuous (instead of semicontinuous) are more convenient in general, but in order to stay close to Berge's original terminology (see [5]), they will not be adopted in this paper.

Remark 2.1. It is well-known that Definition 2.1(b), in the case of a compact-valued γ, is equivalent to: if $x_n \to x$ in X and $a_n \in \gamma(x_n)$, then there exists a subsequence $\{a_{n(k)}\}$ of $\{a_n\}$ and $a \in \gamma(x)$, such that $a_{n(k)} \to a$ (see Theorem 17.20 p. 565 in [1]). Besides, Definition 2.1(c) is also equivalent to: if $x_n \to x$ in X, then for each $a \in \gamma(x)$ there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ and $a_k \in \gamma(x_{n(k)})$ for each k such that $a_k \to a$ (see Theorem 17.21 p. 565 in [1])

Now, the minimization problem will be established.

Throughout the remainder of Sections 2 and 3, let X and A be fixed nonempty Borel spaces and γ denotes a given Borel-measurable multifunction from X to A. $\mathbb{F}$ denotes the set of measurable functions $f : X \to A$ such that $f(x) \in \gamma(x)$ for all $x \in X$ ($f \in \mathbb{F}$ is called a *selector* for the multifunction γ). Furthermore, $G : Gr(\gamma) \to \mathbb{R}$ is a given nonnegative (or bounded below) and measurable function, and

$$G^*(x) := \inf_{a \in \gamma(x)} G(x, a), \tag{3}$$

$x \in X$. If $G(x, \cdot)$ attains its minimum at some point in $\gamma(x)$, there will be written "min" instead of "inf" in (3).

Remark 2.2. In Rieder [28] it is proved that if $Gr(\gamma)$ is a Borel subset of $X \times A$, G is lower semicontinuous, bounded below, and inf-compact on $Gr(\gamma)$ (i. e. for every $x \in X$ and $r \in \mathbb{R}$, the set $\{a \in \gamma(x) : G(x, a) \leq r\}$ is compact), then there exists a selector $f^* \in \mathbb{F}$ such that for each $x \in X$, $G(x, \cdot)$ attains its minimum in $f^*(x)$.

Lemma 2.1. If γ is closed-valued (i. e. $\gamma(x)$ is closed for each $x \in X$) and u.s.c., then $Gr(\gamma)$ is closed in $X \times A$.

Proof. This is a consequence of Proposition 7, p. 110 [3]. □

Lemma 2.2. If γ is l.s.c. and G is u.s.c, then G^* is u.s.c.

Proof. See the proof of Lemma 17.29 in [1]. □

3. THE MINIMUM THEOREM

The Moment Condition (MC). There is a sequence $\{\mathbb{K}_n\}$ of compact sets such that $\mathbb{K}_n \uparrow Gr(\gamma)$, and

$$\lim_{n\to\infty}\Big(\inf_{(x,a)\notin\mathbb{K}_n} G(x,a)\Big) = +\infty. \tag{4}$$

Remark 3.1. (a) In case $x \to \gamma(x)$ is constant, that is $\gamma(x) = A$, for all $x \in X$, with A as a compact set, and X which is σ-compact (i. e. there exist an increasing sequence of compact sets $\{W_n\}$ such that $W_n \uparrow X$), G is trivially a moment, because, if $\mathbb{K}_n = W_n \times A$, $n = 1, 2, \dots$, then $\mathbb{K}_n \uparrow Gr(\gamma)$ and (4) holds because $Gr(\gamma)\backslash\mathbb{K}_n$ is empty, for each n, and the minimum over the empty set is equal to $+\infty$.

(b) A nonnegative measurable function H on a Borel space Y is said to be a *moment* on Y (see,[10, 20, 21],[24] and [30]) if there is an increasing sequence of compact sets $Y_n \uparrow Y$ such that

$$\lim_{n\to\infty}\Big(\inf_{y\notin Y_n} H(y)\Big) = +\infty.$$

Thus, the MC states that G is a moment on $Gr(\gamma)$.

In many important optimization problems coerciveness of the objective function is observed. It is remarkable that this simple and useful concept is sometimes ignored, being so natural in the context of the problems studied in this paper. See [26] for more information.

Define, for each $x \in X$, $\gamma^*(x) := \{a \in \gamma(x) : G(x,a) \le G^*(x)\} = \{a \in \gamma(x) : G(x,a) = G^*(x)\}$, and for each $\zeta \subset X$, where ζ is a nonempty compact set, $\Omega_\zeta := \{(x,a) \in Gr(\gamma) : x \in \zeta,\, a \in \gamma^*(x)\}$. Observe that for each $x \in X$, $\gamma^*(x)$ is nonempty and compact if G is lower semicontinuous and inf-compact on $Gr(\gamma)$ (see Remark 2.2).

Now the version of the Minimum Theorem will be presented.

Theorem 3.1. Suppose that the multifunction γ is closed-valued and continuous, G is continuous and inf-compact on $Gr(\gamma)$, and the MC holds. Then G^* is continuous and the multifunction $x \to \gamma^*(x)$ is u.s.c.

Proof. Fix $\zeta \subset X$, where ζ is a nonempty compact set. Let $(x_k, a_k) \in \Omega_\zeta$, $k = 1, 2, \dots$ and $(x,a) \in Gr(\gamma)$, such that $(x_k, a_k) \to (x,a)$. Note that $x \in \zeta$. Moreover, $G(x_k, a_k) \le G^*(x_k)$, for all k, and as from Lemma 2.2, G^* is u.s.c., it results that $G(x,a) \le G^*(x)$ (recall that G is continuous). Then $a \in \gamma^*(x)$, i. e. Ω_ζ is closed in $Gr(\gamma)$, and by Lemma 2.1, it is also closed in $X \times A$.

Now, suppose that for each $m = 1, 2, \dots$, there is $(w_m, a_m) \in \Omega_\zeta$ and $(w_m, a_m) \notin \mathbb{K}_m$ (here $\mathbb{K}_m$, $m = 1, 2, \dots$, are the compact sets in the MC). Since $w_m \in \zeta$, for all $m = 1, 2, \dots$, ζ is compact, and G^* is u.s.c., it follows that

$$G(w_m, a_m) \le G^*(w_m) \le \sup_{x\in\zeta} G^*(x) < +\infty,$$

for all m (recall that an u.s.c. function attains its maximum over a compact set, see Theorem 2.4.3 p. 44 [1]).

Consequently,

$$\limsup_{m\to\infty} G(w_m, a_m) < +\infty. \tag{5}$$

Now, as

$$\inf_{(x,a)\notin\mathbb{K}_m} G(x,a) \leq G(w_m, a_m),$$

$m = 1, 2, \dots,$ then

$$+\infty = \lim_{m\to\infty}\Big(\inf_{(x,a)\notin\mathbb{K}_m} G(x,a)\Big) \leq \limsup_{m\to\infty} G(w_m, a_m),$$

which is a contradiction to (5).Therefore, there exists a positive integer η such that $\Omega_\zeta \subset \mathbb{K}_\eta$. As Ω_ζ is closed, it follows that it is compact.

Since ζ is arbitrary, the conclusion is that Ω_ζ is compact for each nonempty compact $\zeta \subset X$.

Let $\{z_n\}$ be a sequence in X, and let z be an element in X such that $z_n \to z$. Take $\zeta^{'} = \{z_n\} \cup \{z\}$ (notice that $\zeta^{'}$ is a compact set and that G is bounded on $\Omega_{\zeta'}$). As $\Omega_{\zeta'}$ is a compact set and for each $x \in X$, $\gamma^*(x)$ is nonempty, so there exist $a_n \in \gamma^*(z_n)$, $n = 1, 2, \dots$, and $a^* \in \gamma^*(z)$, such that for a certain subsequence $\{(z_{n(k)}, a_{n(k)})\}$ of $\{(z_n, a_n)\}$, $(z_{n(k)}, a_{n(k)}) \to (z, a^*)$, $k \to \infty$. Now, suppose that for $r \geq 0$, $G^*(z_n) \leq r$ for all n. Hence, letting $n \to \infty$ in the last inequality, applying Theorem 1.17 p.19 in [31] (specifically the characterization of lim sup for a sequence given) , and the fact that $(z, a^*) \in \Omega_{\zeta'}$, it results that

$$r \geq \limsup_{n\to\infty} G^*(z_n) = \limsup_{n\to\infty} G(z_n, a_n) \geq G(z, a^*) = \ G^*(z),$$

i.e. for $r \geq 0$, $\{x \in X : G^*(x) \leq r\}$ is closed in X. On the other hand, obviously $\{x \in X : G^*(x) \leq r\}$ is empty for $r < 0$ (recall that G is nonnegative). Consequently, G^* is l.s.c. Therefore, as from Lemma 2.2, G^* is u.s.c., the continuity of G^* follows.

Besides, as the multifunction $x \to \gamma^*(x)$ is compact-valued and $a_{n(k)} \to a^*$, it follows from Remark 2.1 that this multifunction is u.s.c. □

Corollary 3.1. Under the Assumptions of Theorem 3.1, if there exists a unique $f^* \in \mathbb{F}$ such that

$$G^*(x) = G(x, f^*(x)), \tag{6}$$

$x \in X$, then f^* is continuous.

P r o o f. Note that for each $x \in X$, $\gamma^*(x) = \{f^*(x)\}$. Then for $W \subset A$, $\{x \in X : \gamma^*(x) \cap W \neq \emptyset\} = \{x \in X : f^*(x) \in W\}$. Now, as the multifunction $x \to \gamma^*(x)$ is u.s.c., it results that for each closed $F \subset A$, $\{x \in X : \gamma^*(x) \cap F \neq \emptyset\} = \{x \in X : f^*(x) \in F\}$ is closed in X. Consequently, f^* is continuous (see Theorem 8.3, p. 79 [7]). □

The following examples show that with the set of assumptions and conditions in this article, it is possible to consider minimization problems with nonconvex function G and/or with nonconvex restriction sets $\gamma(x)$, $x \in X$, as well.

As it has been mentioned in Section 1, the proofs related to the examples in this Section and in Section 5, will be given in Appendix A and Appendix B below.

Let k be a fixed positive integer, $k \geq 2$. Define

$$\varphi(x) := |x^{1/3} + 1| + 1, \tag{7}$$

$x \in \mathbb{R}$ (notice that $\varphi(x) > 0$, for all $x \in \mathbb{R}$), and

$$P(x, w) := (1/(2(k+1)))w^{2(k+1)} + \varphi(x)w^2 + \varphi(x)w + \varphi(x), \tag{8}$$

$x \in \mathbb{R}$ and $w \in \mathbb{R}$.

Lemma 3.1. For each $x \in \mathbb{R}$, there exists a unique $h^*(x) < 0$, such that

$$P(x, h^*(x)) = \min_{w \in \mathbb{R}} P(x, w).$$

Moreover, $P(x, h^*(x)) < P(x, 0) = \varphi(x), x \in \mathbb{R}$.

Remark 3.2. The polynomial $P(x, \cdot)$ (with respect to the second variable) defined above will have a degree greater or equal than 6 for $k \geq 2$, and hence, the minimizer can not, in general, be explicitly given, because its derivative will be of a degree ≥ 5, and so its roots will not be determined by radicals in most of the cases. For a clear general exposition on this subject, aimed at non-specialists, see [19].

Example 3.1. Take $X = A = \gamma(x) = \mathbb{R}$, $x \in \mathbb{R}$, and

$$G(x, a) = a^{1/3} + \varphi(x), \tag{9}$$

$a > 0$, $x \in \mathbb{R}$ and

$$G(x, a) = P(x, a), \tag{10}$$

$a \leq 0$, $x \in \mathbb{R}$.

Lemma 3.2. Example 3.1 satisfies the Assumptions in Corollary 3.1, and G is nonconvex.

Example 3.2. Take $X = \mathbb{R}, A = \gamma(x) = (-\infty, 0] \cup [1, +\infty)$, $x \in \mathbb{R}$, and

$$G(x, a) = a^{1/3} + \varphi(x), \tag{11}$$

$a \geq 1$, $x \in \mathbb{R}$, and

$$G(x, a) = P(x, a), \tag{12}$$

$a \leq 0$, $x \in \mathbb{R}$.

Lemma 3.3. Example 3.2 satisfies the Assumptions in Corollary 3.1, and both G and the restriction set $A = (-\infty, 0] \cup [1, +\infty)$ are nonconvex.

As mentioned before, the continuity of the value function has being studied by several authors and it should be stressed that, if the uniqueness of the minimizer is observed, Berge's Theorem immediately grants us the continuity of the optimal selector. As here an extension of the unbounded case of Berge's Theorem is presented, this continuity can be straightforwardly established in important discounted MDPs as the linear-quadratic one or in the greatly non-linear Lindley's random walks using the results of [6].

4. DISCOUNTED MARKOV DECISION PROCESSES

Decision Model. Let $(X, A, \{A(x) : x \in X\}, Q, c)$ be the usual discrete-time Markov decision model (see [10]), where both the state space X and the control space A are Borel spaces. For each $x \in X$, $A(x) \subset A$ is the measurable subset of admissible actions at a state x. The set $\mathbb{K} := \{(x, a) : x \in X,\, a \in A(x)\}$ is assumed to be a Borel subset of $X \times A$. Consider the transition probability law $Q(B|x, a)$, where $B \in \mathbb{B}(X)$ ($\mathbb{B}(X)$ denotes the Borel sigma-algebra of X) and $(x, a) \in \mathbb{K}$ is a stochastic kernel on X, given $\mathbb{K}$ (i.e. $Q(\cdot|x, a)$ is a probability measure on X for every $(x, a) \in \mathbb{K}$, and $Q(B|\cdot)$ is a measurable function on $\mathbb{K}$ for every $B \in \mathbb{B}(X)$). Finally, the cost per stage c is a nonnegative and measurable function on $\mathbb{K}$.

Remark 4.1. In many important cases the transition law Q is induced by a system equation of the form

$$x_{t+1} = F(x_t, a_t, \xi_t), \tag{13}$$

with $t = 0, 1, \ldots,$ where $\{\xi_t\}$ is a sequence of independent and identically distributed (i.i.d.) random elements with values in some Borel space S, and with a common density Δ. F is a measurable function from $\mathbb{K} \times S$ to X and the transition probability law Q is given by

$$Q(B|x, a) = \int I_B(F(x, a, s))\Delta(s)\,\mathrm{d}s, \tag{14}$$

$B \in \mathbb{B}(X)$ and $(x, a) \in \mathbb{K}$, where $I_{[\cdot]}$ denotes the indicator function of the subset $[\cdot]$.

Policies. A *control policy* π is a (measurable, possibly randomized) rule for choosing actions, and at each time $t\,(t = 0, 1, \ldots)$ the control prescribed by π may depend on the current state as well as on the history of the previous states and actions. The set of all policies will be denoted by Π. Given the initial state $x_0 = x$, any policy π defines a unique probability distribution of the state-action process $\{(x_t, a_t)\}$ (for details, see [10]). This distribution will be denoted by P_x^π, while E_x^π stands for the corresponding expectation operator, and the stochastic process $\{x_t\}$ will be called Markov decision process (MDP). $\mathbb{F}$ denotes the set of measurable functions $f : X \to A$ such that $f(x) \in A(x)$ for all $x \in X$. A policy $\pi \in \Pi$ is *stationary* if there exists $f \in \mathbb{F}$ such that, under π, the action $f(x_t)$ is applied at each time t. The class of stationary policies is naturally identified with $\mathbb{F}$.

Optimality Criterion. Given $\pi \in \Pi$ and initial state $x_0 = x \in X$, let

$$V(\pi, x) = E_x^\pi\Big[\sum_{t=0}^{\infty} \alpha^t\, c(x_t, a_t)\Big] \tag{15}$$

be the *total expected discounted cost* when using the policy π, given the initial state x. The number $\alpha \in (0,1)$ is called the *discount factor*.

A policy π^* is said to be *discounted optimal* if $V(\pi^*, x) = V^*(x)$ for all $x \in X$, where

$$V^*(x) = \inf_{\pi} V(\pi, x), \tag{16}$$

$x \in X$. V^* defined in (16) is called the *optimal value function*.

An MDP with the total expected discounted cost as the optimality criterion will be referred to as *discounted* MDP (and the plural of this term will be denoted by MDPs).

Assumption 4.1.

(a) The one-stage cost c is lower semicontinuous, and inf-compact on $\mathbb{K}$ (see Remark 2.2).

(b) The transition law Q is strongly continuous, i. e.

$$w(x,a) := \int u(y)\, Q(\mathrm{d}y|x,a)$$

is continuous and bounded on $\mathbb{K}$, for every measurable bounded function u on X.

(c) There is a policy π' such that $V(\pi', x) < \infty$, for each $x \in X$.

Lemma 4.1. (Hernández-Lerma and Lasserre [10] Theorem 4.2.3) Let Assumption 4.1 hold. Then the optimal value function V^* defined in (16) is the (pointwise) minimal solution of the *Optimality Equation* (OE), i. e. for all $x \in X$,

$$V^*(x) = \min_{a \in A(x)} [c(x,a) + \alpha \int V^*(y) Q(\mathrm{d}y|x,a)], \tag{17}$$

and, if u is another solution to the OE, then $u(\cdot) \geq V^*(\cdot)$.

There is also $f^* \in \mathbb{F}$ such that:

$$V^*(x) = c(x, f^*(x)) + \alpha \int V^*(y) Q(\mathrm{d}y|x, f^*(x)), \tag{18}$$

$x \in X$, and f^* is optimal.

The following assumption will be valid for discounted MDPs for which Assumption 4.1 holds. Take $\gamma(x) = A(x)$, $x \in X$.

Assumption 4.2.

(a) γ is closed-valued and continuous;

(b) f^* is unique;

(c) $c(\cdot,\cdot)$ is a continuous function, $\int V^*(y)Q(\mathrm{d}y|x,a)$ is finite for every $(x,a) \in \mathbb{K}$, and $\int V^*(y)Q(\mathrm{d}y|\cdot,\cdot)$ is a continuous function;

(d) c satisfies the MC.

Remark 4.2. (a) In [6] conditions which ensure the uniqueness of f^* in (18) are provided.

(b) In the next Section two examples for which Assumptions 4.1 and 4.2 hold are presented.

Theorem 4.1. Consider a discounted MDP for which Assumptions 4.1 and 4.2 hold. Then V^* and f^* are continuous functions.

Proof. Fix a discounted MDP for which Assumptions 4.1 and 4.2 hold. Let $(X, A, \{A(x) : x \in X\}, Q, c)$ be the Markov decision model for this MDP. Let f^* be the optimal policy whose existence is guaranteed in (18), let V^* be the optimal value function defined in (16), and take $\gamma(x) = A(x)$, $x \in X$ (notice that $\mathbb{K} = Gr(\gamma)$). Define

$$G(x,a) := c(x,a) + \alpha \int V^*(y)Q(\mathrm{d}y|x,a), \tag{19}$$

$(x,a) \in \mathbb{K}$. (Observe that the minimization problem is defined via the OE (17).) Now, Assumptions in Corollary 3.1 for these G and f^* will be verified. Firstly, note that from Assumptions 4.2(a), 4.2(b), and 4.2(c), the multifunction γ is closed-valued and continuous, G is continuous, and the uniqueness of f^* follows. Secondly, observe that

$$A_r(x) := \{a \in A(x) : G(x,a) \le r\} \subseteq \{a \in A(x) : c(x,a) \le r\},$$

$x \in X$ and $r \in \mathbb{R}$. Hence, Assumptions 4.1(a) and 4.2(c) imply that G is inf-compact on $Gr(\gamma)$.

Thirdly, let $\mathbb{K}_n$, $n = 1, 2, \dots$ be the compact sets in the MC for the cost function c (see Assumption 4.2(d)). Note that since c is nonnegative, then V^* and $\int V^*(y)Q(\mathrm{d}y|\cdot,\cdot)$ are also nonnegative. Hence, since

$$\begin{aligned} \inf_{(x,a)\notin\mathbb{K}_n} c(x,a) &\le \inf_{(x,a)\notin\mathbb{K}_n} [c(x,a) + \alpha \int V^*(y)Q(\mathrm{d}y|x,a)] \\ &= \inf_{(x,a)\notin\mathbb{K}_n} G(x,a), \end{aligned} \tag{20}$$

letting $n \to \infty$ in (20), it follows that G satisfies the MC. Therefore, V^* and f^* are continuous functions as a consequence of Theorem 3.1 and Corollary 3.1. □

5. EXAMPLES ON THE CONTROLLED LINDLEY'S RANDOM WALK

Example 5.1. Let $X = A = A(x) = [0, \infty)$, for all $x \in X$. The dynamic of the system is given by

$$x_{t+1} = [x_t + a_t - \xi_t]^+, \tag{21}$$

$t = 0, 1, \ldots$. Here $z^+ = \max\{0, z\}$, and ξ_0, $\xi_1, \ldots$ are i.i.d. random variables taking values in $S = [0, \infty)$ and with a common density Δ. Besides, it is assumed that Δ is a bounded continuous function. Let H be the distribution function of ξ, where ξ is a generic element of the sequence $\{\xi_t\}$ (note that H is a continuous function). The cost function is given by:

$$c(x, a) = x + (a - 1)^2, \tag{22}$$

x, $a \in [0, \infty)$.

Lemma 5.1. Assumptions 4.1 and 4.2 hold for Example 5.1.

With the results developed so far a theoretical but nevertheless interesting situation can arise:

Example 5.2. Let $X = A = [0, \infty)$, and $A(x) = [x, \infty)$, $x \in X$. The dynamics of the system is given by

$$x_{t+1} = [x_t + g(a_t) - \xi_t]^+, \tag{23}$$

$t = 0, 1, \ldots$. Here $\xi_0, \xi_1, \ldots$ are i.i.d. random variables taking values in $S = [0, \infty)$ and with a common density Δ. Besides, it is assumed that Δ is a bounded continuous function. The cost function is given by

$$c(x, a) = x^2 + a^2, \tag{24}$$

$(x, a) \in \mathbb{K}$.

Assumption 5.1. $g : [0, \infty) \to \mathbb{R}$ is positive, continuous, convex and decreasing.

Lemma 5.2. Under Assumption 5.1, Example 5.2 satisfies Assumptions 4.1 and 4.2.

6. CONCLUSIONS

It would seem that theorems like Berge's that are widely known to the mathematical economists, should be better known to all researchers interested in MDPs. And the fact that such a Theorem can still be a source of new findings when the optimal policy is unique, suggests that further research on this area is still promising.

APPENDIX A: PROOFS RELATED TO EXAMPLES 3.1 AND 3.2

Proof of Lemma 3.1. Let x be a fixed element of $\mathbb{R}$. Computing the first and the second derivatives of P with respect to w, denoted by P_w and P_{ww} respectively, it is obtained that

$$P_w(x, w) = w^{2k+1} + 2\varphi(x)w + \varphi(x), \tag{25}$$

and

$$P_{ww}(x, w) = (2k + 1)w^{2k} + 2\varphi(x). \tag{26}$$

$P_w(x, \cdot)$ has odd degree, hence it has at least one real root. As $P_{ww}(x, \cdot)$ is positive, by the well-known Rolle's Theorem, there only exists one real root for $P_w(x, \cdot)$ denoted by $h^*(x)$. Furthermore, again, the positiveness of $P_{ww}(x,\cdot)$ implies that $h^*(x)$ is the unique minimum for (8). Finally, $h^*(x)$ is negative because $P_w(x,w) > 0$ for $w \geq 0$, and, obviously, $P(x,h^*(x)) < P(x,0) = \varphi(x)$. Since x is arbitrary, Lemma 3.1 follows.□

P r o o f o f L e m m a 3.2. Fix $x \in \mathbb{R}$. If $a > 0$, then, trivially, $G(x,a) > 0$. Suppose that $a \leq 0$, then

$$\begin{aligned} G(x,a) &= \frac{1}{2(k+1)}a^{2(k+1)} + \varphi(x)a^2 + \varphi(x)a + \varphi(x) \\ &= \frac{1}{2(k+1)}a^{2(k+1)} + \varphi(x)(a^2+a+1) \\ &= \frac{1}{2(k+1)}a^{2(k+1)} + \varphi(x)[(a+1/2)^2 + 3/4] > 0. \end{aligned}$$

Consequently, as x is arbitrary, G is nonnegative (in fact G is positive).

Clearly, G is continuous (observe that, $G(x,0) = \lim_{a\to 0^+} G(x,a) = \lim_{a\to 0^+}(a^{1/3} + \varphi(x)) = \varphi(x)$, for each $x \in \mathbb{R}$).

Let $A_r(x) := \{a \in \mathbb{R} : G(x,a) \leq r\}$, $x \in \mathbb{R}$, $r \in \mathbb{R}$. Observe that for each $x \in \mathbb{R}$ and $r \in \mathbb{R}$, $A_r(x)$ is closed in $\mathbb{R}$ as a consequence of the continuity of G. Since G is positive, it follows that $A_r(x) = \emptyset$ (and hence $A_r(x)$ is compact) if $r \leq 0$. Note that, for $x \in \mathbb{R}$,

$$\lim_{a\to+\infty} G(x,a) = \lim_{a\to+\infty}(a^{1/3} + \varphi(x)) = +\infty, \tag{27}$$

and

$$\begin{aligned} \lim_{a\to-\infty} G(x,a) &= \lim_{a\to-\infty} P(x,a) \\ &= \lim_{a\to-\infty} a^{2(k+1)}\Big[\frac{1}{2(k+1)} + \varphi(x)\big(\frac{1}{a^{2k}} + \frac{1}{a^{2k+1}} + \frac{1}{a^{2(k+1)}}\big)\Big] \\ &= +\infty. \end{aligned} \tag{28}$$

Therefore, if for some $x \in \mathbb{R}$ and $r > 0$, $A_r(x)$ is unbounded, using (27) and (28) it is possible to choose $a' \in A_r(x)$ such that $G(x,a') > r$, which is a contradiction. In conclusion G is inf -compact on $Gr(\gamma)$, and there exists a selector $f^* \in \mathbb{F}$ such that for each $x \in \mathbb{R}$, $G(x,\cdot)$ attains its minimum in $f^*(x)$ (see Remark 2.2).

Now, note that from Lemma 3.1

$$G(x,h^*(x)) = P(x,h^*(x)) < G(x,a),$$

$x \in \mathbb{R}$, $a \leq 0$, $a \neq h^*(x)$; and also by Lemma 3.1, for $x \in \mathbb{R}$, $a > 0$,

$$\begin{aligned} G(x,h^*(x)) &= P(x,h^*(x)) \\ &< P(x,0) = \varphi(x) \\ &\leq a^{1/3} + \varphi(x) = G(x,a). \end{aligned}$$

Then, $h^*(x)$ is a minimum of $G(x,\cdot)$, for all $x \in \mathbb{R}$, and evidently $h^*(x) = f^*(x)$, for all $x \in \mathbb{R}$. Consequently, the uniqueness of f^* follows.

Now, it is direct to verify that G satisfies the MC. To prove this, take $\mathbb{K}_n = [-n, n] \times [-n, n]$, $n = 1, 2, \ldots$.

Obviously, for each n, $\mathbb{K}_n$ is compact and $\mathbb{K}_n \uparrow \mathbb{K} = Gr(\gamma) = \mathbb{R}^2$. Fix a positive integer n and take $(x, a) \notin \mathbb{K}_n$. Consider the following four cases: $-n \leq x \leq n$ and $a > n$; $-n \leq x \leq n$ and $a < -n$; $|x| > n$ and $a > 0$; or $|x| > n$ and $a \leq 0$.

In the first case, it follows from (9) that

$$G(x, a) = a^{1/3} + \varphi(x) \geq a^{1/3} \geq n^{1/3}. \tag{29}$$

If $-n \leq x \leq n$ and $a < -n$, then, from (10), it follows that

$$\begin{aligned} G(x, a) &= \frac{1}{2(k+1)} a^{2(k+1)} + \varphi(x) a^2 + \varphi(x) a + \varphi(x) \\ &= \frac{1}{2(k+1)} a^{2(k+1)} + \varphi(x)[(a + 1/2)^2 + 3/4] \\ &\geq \frac{1}{2(k+1)} a^{2(k+1)} \geq \frac{1}{2(k+1)} n^{2(k+1)}. \end{aligned} \tag{30}$$

Similarly, it is possible to obtain that, for $|x| > n$ and $a > 0$,

$$G(x, a) \geq n^{1/3}, \tag{31}$$

and that, for $|x| > n$ and $a \leq 0$,

$$G(x, a) \geq (3/4) n^{1/3}. \tag{32}$$

Hence, (30) – (32) imply that for every $(x, a) \notin \mathbb{K}_n$

$$G(x, a) \geq \min\{(3/4) n^{1/3}, (1/(2(k+1))) n^{2(k+1)}\}.$$

Since n is arbitrary, it results that

$$\inf_{(x,a) \notin \mathbb{K}_n} G(x, a) \geq \min\{(3/4) n^{1/3}, (1/(2(k+1))) n^{2(k+1)}\}, \tag{33}$$

for every $n = 1, 2, \ldots$. Then, letting $n \to +\infty$ in (33), it results that $G(\cdot, \cdot)$ satisfies the MC.

Finally, $G(\cdot, \cdot)$ is nonconvex as a consequence of the nonconvexity of $G(-1, a)$, $a \in \mathbb{R}$, which is given by $G(-1, a) = a^{1/3} + 1$, $a > 0$, and $G(-1, a) = 1 + a + a^2 + 1/(2(k+1)) a^{2(k+1)}$, $a \leq 0$ (observe that $G(x, 0) = \varphi(x)$, $x \in \mathbb{R}$ is also nonconvex). $\square$

Proof of Lemma 3.3. Similar to the proof of Lemma 3.2. $\square$

APPENDIX B: PROOFS RELATED TO EXAMPLES 5.1 AND 5.2

Proof of Lemma 5.1. The cost function c is nonnegative, continuous, and observe that for each $x \in [0, \infty)$, $A_r(x) = \emptyset$, if $r < x$, $A_r(x) = [1 - (r - x)^{1/2}, 1 + (r - x)^{1/2}]$ if $x \leq r \leq x + 1$, and $A_r(x) = [0, 1 + (r - x)^{1/2}]$, if $x + 1 < r$, then the inf-compactness of c follows, concluding that Assumption 4.1(a) holds.

The proof of the strong continuity of the transition law Q induced by (21) is as follows: if $u : X \to \mathbb{R}$ is a measurable and bounded function, then from (21) and the well-known Change of Variable Theorem, it is obtained that

$$\int u(y)Q(\mathrm{d}y|x,a) = u(0)[1 - H(x,a)] + \int I_{[0,x+a]}(z)u(z)\Delta(x+a-z)\,\mathrm{d}z,$$

$(x,a) \in \mathbb{K}$, where $I_{[\cdot]}$ denotes the indicator function of the set $[\cdot]$ and $H(x,a) = P[\xi \leq x+a]$. The continuity of H implies the continuity of $u(0)[1-H(x,a)]$ on $\mathbb{K}$. As u is bounded and Δ is a bounded continuous function, it results from the Dominated Convergence Theorem that

$$\int I_{[0,x+a]}(z)u(z)\Delta(x+a-z)\,\mathrm{d}z$$

is a continuous function on $\mathbb{K}$. In conclusion, Q is strongly continuous.

Let $f \in \mathbb{F}$, given by $f(x) = 1$, for all $x \in X$. Then for each $x \in X$,

$$E_x^f[c(x_0,a_0)] = c(x,f(x)) = x. \tag{34}$$

Now, for each $x \in X$,

$$\begin{aligned} E_x^f[c(x_1,a_1)] &= \int c(y,f(y))Q(dy|x,f) = \int yQ(\mathrm{d}y|x,f) \\ &= \int I_{[0,\infty)}(s)[x+1-s]^+\Delta(s)\,\mathrm{d}s \\ &= \int I_{[0,x+1]}(s)(x+1-s)\Delta(s)\,\mathrm{d}s \\ &= (x+1)P[\xi \leq x+1] - \int I_{[0,x+1]}(s)s\Delta(s)\,\mathrm{d}s \leq x+1, \end{aligned} \tag{35}$$

and by a direct induction argument, it follows that

$$E_x^f[c(x_t,a_t)] \leq x+t, \tag{36}$$

$t = 0, 1, \ldots$.

Now, for each $x \in X$, using (34) and (36),

$$V(f,x) = \sum_{t=0}^{\infty} \alpha^t E_x^f[c(x_t,a_t)] \leq x/(1-\alpha) + \alpha/(1-\alpha)^2. \tag{37}$$

Therefore, Assumption 4.1 holds for Example 5.1.

On the other hand, clearly $\gamma(x) = A(x) = [0,\infty)$, $x \in X$, is closed-valued and continuous (in fact γ is constant).

Observe that it is trivial to prove that $c(\cdot,\cdot)$ is strictly convex, $F(x,a,s) = [x+a-s]^+$, x, $a \in \mathbb{R}$ and $s \in S$, is convex in (x,a) for each $s \in S$ and increasing in x, for each $a \in \mathbb{R}$ and $s \in S$, and the multifunction $x \to A(x) = [0,\infty)$ is convex, that is, it is valid that $(1-\lambda)a+\lambda a' \in A((1-\lambda)x+\lambda x')$ for all x, x', a, $a' \in [0,\infty)$ and $\lambda \in [0,1]$, and A and $A(x)$ are convex for each $x \in X$. Moreover, X is convex as well. So this example satisfies Condition 1 in [6], and the uniqueness of the optimal policy follows.

Now the finiteness and the continuity of $\int V^*(y)Q(\mathrm{d}y|\cdot,\cdot)$ will be verified.

From (37),

$$0 \le V^*(x) \le x/(1-\alpha) + \alpha/(1-\alpha)^2, \tag{38}$$

$x \in X$.

Then, from (38) and a computation similar to the one in (35), it follows that for each $(x,a) \in \mathbb{K}$,

$$\begin{aligned}
\int V^*(y)Q(\mathrm{d}y|x,a) &= \int I_{[0,\infty)}(s)V^*([x+a-s]^+)\Delta(s)\,\mathrm{d}s \\
&\le [1/(1-\alpha)]\int I_{[0,\infty)}(s)[x+a-s]^+\Delta(s)\,\mathrm{d}s + \alpha/(1-\alpha)^2 \\
&\le \frac{x+a}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2} < +\infty.
\end{aligned}$$

In [6] it has been proved that, if condition C1 holds, then the optimal value function V^* is an increasing function on $X=[0,+\infty)$ (see Lemma 6.1 in [6]). Hence it is obtained that V^* is continuous almost everywhere (a.e.) in $[0,+\infty)$ (see Theorem 4.3.1 in [2] and the paragraph just next to the end of the proof of this theorem). Let $(x_k,a_k) \in \mathbb{K}$, $k = 1,2,\ldots$, such that $(x_k,a_k) \to (x,a) \in \mathbb{K}$. Let $T>0$ such that for each $k=1,2,\ldots$,

$$0 \le x_k \le T \quad \text{and} \quad 0 \le a_k \le T. \tag{39}$$

From (38), for each $k=1,2,\ldots$, and $s \in S$,

$$0 \le V^*([x_k+a_k-s]^+)\Delta(s) \le h_k(s),$$

where $h_k(s) = ([x_k+a_k-s]^+/(1-\alpha)+\alpha/(1-\alpha)^2)\Delta(s)$, $s \in S$. Observe that from (39) and a computation similar to the one in (35), for each $k=1,2,\ldots$,

$$\begin{aligned}
0 &\le \int h_k(s)\,\mathrm{d}s \\
&= \int I_{[0,\infty)}(s)\Big(\frac{[x_k+a_k-s]^+}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2}\Big)\Delta(s)\,\mathrm{d}s, \\
&\le \frac{x_k+a_k}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2} \le \frac{2T}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2} < +\infty.
\end{aligned} \tag{40}$$

Moreover, it is direct to verify that $\{h_k\}$ converges pointwisely to the function $h(s) = ([x+a-s]^+/(1-\alpha)+\alpha/(1-\alpha)^2)\Delta(s)$, $s \in S$, and that $|h_k(s)| \le (2T/(1-\alpha) + \alpha/(1-\alpha)^2)\Delta(s), s \in S, k=1,2,\ldots$. Now, using the standard Dominated Convergence Theorem it follows that $\int h_k(s)ds \to \int h(s)\,\mathrm{d}s$.

On the other hand, due to the continuity a.e. of V^*, it is obtained as well that

$$V^*([x_k+a_k-s]^+)\Delta(s) \to V^*([x+a-s]^+)\Delta(s),$$

when $k \to \infty$, s-a.e. So, applying Theorem 17, p.92 in [29], it results that

$$\begin{aligned}\lim_{k\to\infty}\int V^*(y)Q(\mathrm{d}y|x_k,a_k) &= \lim_{k\to\infty}\int I_{[0,\infty)}(s)V^*([x_k+a_k-s]^+)\Delta(s)\,\mathrm{d}s\\ &= \int I_{[0,\infty)}(s)V^*([x+a-s]^+)\Delta(s)\,\mathrm{d}s\\ &= \int V^*(y)Q(\mathrm{d}y|x,a),\end{aligned}$$

i. e. $\int V^*(y)Q(\mathrm{d}y|\cdot,\cdot)$ is a continuous function.

Therefore, Assumption 4.2(c) holds.

Finally, let $\mathbb{K}_n = [0,n]\times[0,n]$, $n = 1,2,\ldots$. Evidently, for each n, $\mathbb{K}_n$ is compact and $\mathbb{K}_n \uparrow \mathbb{K} = Gr(\gamma) = [0,\infty)\times[0,\infty)$. Now, fix a positive integer n and take $(x,a) \notin \mathbb{K}_n$. There are two cases: $0 \le x \le n$ and $a > n$, or $x > n$ and $a \ge 0$. In the first case, it follows from (22) that

$$c(x,a) \ge (n-1)^2. \tag{41}$$

If $x > n$ and $a \ge 0$, then again, from (22) it results that

$$c(x,a) \ge n. \tag{42}$$

(41) and (42) imply that for every $(x,a) \notin \mathbb{K}_n$,

$$c(x,a) \ge \min\{(n-1)^2, n\} \tag{43}$$

and since n is arbitrary, it follows that

$$\inf_{(x,a)\notin\mathbb{K}_n} c(x,a) \ge \min\{(n-1)^2, n\}, \tag{44}$$

for every $n = 1,2,\ldots$. Now, letting $n \to +\infty$ in (44) it results that $c(\cdot,\cdot)$ satisfies the MC. □

P r o o f o f L e m m a 5.2. Clearly, X and A are convex sets, and c is nonnegative, inf-compact, continuous, and strictly convex on $\mathbb{K}$; besides, $x \to A(x)$ is closed-valued.

It is direct to verify that $F(x,a,s) = [x+g(a)-s]^+$, $x,a \in \mathbb{R}$ and $s \in S$ is convex in (x,a) for each $s \in S$ and increasing in x, for each $a \in A(x)$ and $s \in S$, and the multifunction $x \to A(x)$ is convex, and $A(x)$ is convex, for each $x \in X$.

Similar to Example 5.1 (see the proof of Lemma 5.1), it is possible to prove that:

- Q induced by (23) is strongly continuous.
- For $f \in \mathbb{F}$, given by $f(x) = x$, $x \in X$,

  $$V(f,x) \le \eta x^2 + \beta x + \theta,\ x \in X, \tag{45}$$

  where $\eta = 2/(1-\alpha)$, $\beta = (4g(0)\alpha)/(1-\alpha)^2$, and $\theta = 2(g(0))^2[(\alpha(1+\alpha)/(1-\alpha)^3) + (\alpha(1+4\alpha+\alpha^2)/(1-\alpha)^4)]$.

Hence, Assumption 4.1 and 4.2(b) hold. In particular, Condition C1 in [6] holds.

Again, as in Example 5.1 (see the proof of Lemma 5.1), it is also possible to establish that:

- For each $(x, a) \in \mathbb{K}$,

$$\int V^*(y)Q(\mathrm{d}y|x, a) \leq \eta(x + g(0))^2 + \beta(x + g(0)) + \theta < +\infty. \tag{46}$$

  (To verify (46) it is necessary to use (45).)

- V^* is continuous almost everywhere in $[0, +\infty)$ (this follows from Condition C1 in [6]).

- $\int V^*(y)Q(\mathrm{d}y|\cdot, \cdot)$ is a continuous function.

Consequently, Assumption 4.2(c) holds.

Now the continuity of $x \to A(x)$ will be proved. Firstly, it will be proved that $x \to A(x)$ is l.s.c, and later that $x \to A(x)$ is u.s.c. Let $x_n \to x$ in X, and $a \in A(x) = [x, \infty)$. If $x = a$, then take $a_n = x_n \in A(x_n)$, $n = 1, 2, \dots$, and $a_n \to a$. If $a \neq x$, i.e. $x < a$, then take $a_n = x_n + (a - x)$, $n = 1, 2, \dots$, and observe that $a_n \in A(x_n)$, $n = 1, 2, \dots$, and $a_n \to a$; hence, Remark 2.1 implies that $x \to A(x)$ is l.s.c. Let $F \subset A$ be a closed set, and let $x_n \in \{x \in X : [x, \infty) \cap F \neq \emptyset\}$, $n = 1, 2, \dots$, and suppose that $x_n \to y \in X$. For each $n = 1, 2, \dots$, let $b_n \in [x_n, \infty) \cap F$. If there exists a positive integer m such that $b_m > y$, then $b_m \in [y, \infty) \cap F$, i.e. $y \in \{x \in X : [x, \infty) \cap F \neq \emptyset\}$. If $x_n \leq b_n \leq y$, for all $n = 1, 2, \dots$, then, since $x_n \to y$, it follows that $\lim_{n\to\infty} b_n = y$. As $b_n \in F$, for all n, and F is closed, it results that $y \in F$, i.e. $y \in \{x \in X : [x, \infty) \cap F \neq \emptyset\}$; hence, $\{x \in X : [x, \infty) \cap F \neq \emptyset\}$ is closed in X. Therefore, Definition 2.1(b) implies that $x \to A(x)$ is u.s.c.

Finally, for each $n = 1, 2, \dots$, let $\mathbb{K}_n = \{(x, a) : x \in [0, n], a \in [x, n]\}$. It is direct to verify that for each n, $\mathbb{K}_n$ is compact, and also that $\mathbb{K}_n \uparrow \mathbb{K}$. Let n be a fixed positive integer, and take $(x, a) \in \mathbb{K} \backslash \mathbb{K}_n$. Then $a > n$ which implies that $c(x, a) = x^2 + a^2 \geq a^2 > n^2$. So

$$\inf_{(x,a) \notin \mathbb{K}_n} c(x, a) > n^2. \tag{47}$$

Since n is arbitrary, it follows that (47) holds for each $n = 1, 2, \dots$. Hence, letting $n \to \infty$ in (47), it results that c satisfies the MC.

Therefore, Assumptions 4.2(a) and 4.2(d) hold. □

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