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Trees in Commutative Nil-Semigroups of Index Two

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Praha

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Binary trees in commutative semigroups satisfying $2x = 3y$ are studied.
Studují se binární stromy v komutativních pologrupách splňujících $2x = 3y$.

1. Introduction

Throughout this short note, all semigroups are assumed to be commutative and their operations will usually be denoted additively.

1.1. A semigroup S will be called a *zp*-semigroup in the sequel if S is a nil-semigroup of index (at most) two. It means that S contains an absorbing element o $(= o_S)$ and $2a = o$ for every $a \in S$. In other words, S satisfies the equation $2a = 3b$ for all $a, b \in S$.

1.2 Lemma. *Let a zp-semigroup S be generated by a finite set with $m \geq 0$ elements. Then $|S| \leq 2^m$.*

Proof. Easy to see. ▲

1.3 Lemma. *Let S be a zp-semigroup. Define a relation $\preccurlyeq_S$ on S by $a \preccurlyeq_S b$ if and only if $a = b + u$ for some $u \in S \cup \{0\}$. Then:*

(i) The relation $\preccurlyeq_S$ is an ordering of S and it is compatible with respect to the addition.

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(ii) o is a smallest element.

(iii) If S is non-trivial, then $S \setminus (S + S)$ is the set of maximal elemens of $S(\leq)$.

(iv) The set $Ann(S) \setminus \{o\} = \{a \mid S + a = o \neq a\}$ is the set of minimal elements of $(S \setminus \{o\})(\preccurlyeq)$.

Proof. Easy to check. ▲

1.4. A *zp-semigroup* S will be called a *zs*-semigroup if $S = S + S$ (equivalent, either $S = \{o\}$ or $S(\preccurlyeq)$ has no maximal elements – see 1.3(iii)).

1.5. *Let S be a non-trivial zs-semigroup. Then S is infinite and not finitely generated.*

Proof. The ordered set $S(\preccurlyeq)$ has no maximal elements, and hence it is infinite. Consequently, it follows from 1.2 that S is not finitely generated. ▲

1.6. *Let A be a subset of a zp-semigroup such that $A \subseteq T + T$, T being the subsemigroup generated by A (eg., $A \subseteq A + A$). Then $T = T + T$ and T is a zs-semigroup.*

Proof. Use the fact that $T + T$ is a subsemigroup. ▲

2. Auxiliary concepts (A)

2.1. Define two relations α and β on the set $\mathbb{N}$ of positive integers as $\alpha = \{(i,2i), (i,2i+1) \mid i \in \mathbb{N}\}$ and $\beta = \{(i,2^k i + l) \mid i,k \in \mathbb{N}, 0 \leq l < 2^k\}$.

2.2 Lemma. *(i) α is irreflexive, antisymmetric and $\alpha \subseteq \beta$.*

(ii) $(i,j) \in \beta$ implies $i < j$.

(iii) $(1,i) \in \beta$ for every $i \in \mathbb{N}$, $i \neq 1$.

(iv) β is irreflexive, antisymmetric and transitive.

Proof. (i), (ii) and (iii) are easy. As concernes (iv), the properties of irreflexivity and antisymmetry are clear. Finally, if $i,r,s \in \mathbb{N}$, $0 \leq p < 2^r$, $0 \leq q < 2^s$, then $2^s(2^r i + p) + q = 2^{r+s} i + 2^s p + q$ and $2^s p + q < 2^s p + 2^s(p+1) \leq 2^s \cdot 2^r = 2^{s+r}$. The transitivity of β is now clear. ▲

2.3 Lemma. *The relation β is just the transitive closure of α. That is, $(i,j) \in \beta$ iff there are $m \geq 1$ and positive integers $i_0, \ldots, i_m$ such that $i_0 = i$, $i_m = j$ and $(i_k, i_{k+1}) \in \alpha$ (or $i_{k+1} \in \{2i_k, 2i_k + 1\}$) for every $k = 0, 1, \ldots, m-1$.*

Proof. Denote, for a short moment by τ the transitive closure of α (defined on $\mathbb{N}$). Since β is transitive and contains α by 2.2(i), (iv), we get $\tau \subseteq \beta$. To prove the converse inclusion, we will proceed by induction on k, where $(i,j) \in \beta$, $j = 2^k i + l$, $1 \leq k$, $0 \leq l < 2^k$.

If $k = 1$, then $j = 2i + l$, $0 \le l < 1$, and hence $(i,j) \in \alpha \subseteq \tau$. If $k \ge 2$ and $l < 2^{k-1}$, then $j = 2^{k-1}p + l$, $p = 2i$, $(i,p) \in \alpha$ and $(p,j) \in \beta$. By induction, $(p,j) \in \tau$, and hence $(i,j) \in \tau$. On the other hand, if $k \ge 2$ and $2^{k-1} \le l$, then $j = 2^{k-1}q + l_1$, $q = 2i + 1$, $l = 2^{k-1} + l_1$, $0 \le l_1 \le 2^{k-1}$, $(i,q) \in \alpha$ and $(q,j) \in \beta$. By induction, $(q,j) \in \tau$ and hence $(i,j) \in \tau$. ▲

2.4 Remark. According to 2.2 (iv), the relation β is a sharp ordering defined on $\mathbb{N}$, and hence $\gamma = \beta \cup id_{\mathbb{N}}$ is a (reflexive) ordering on $\mathbb{N}$.

2.5 Lemma. *Let $i,j \in \mathbb{N}$. Then $(i,j) \in \beta$, provided that at least one of the following is true:*
(1) $(2i, 2j) \in \beta$;
(2) $(2i, 2j + 1) \in \beta$;
(3) $(2i + 1, 2j) \in \beta$;
(4) $(2i + 1, 2j + 1) \in \beta$;
(5) $i \ne j$ *and* $(i, 2j) \in \beta$;
(6) $i \ne j$ *and* $(i, 2j + 1) \in \beta$.

Proof. (i) If $(2i, 2j) \in \beta$, then $2j = 2^{k+1}i + l$, $1 \le k$, $0 \le l < 2^k$. Clearly, l is even, $j = 2^k i + l/2$, $0 \le l/2 < 2^k$, and so $(i,j) \in \beta$.
(ii) If $(2i, 2j + 1) \in \beta$, then $2j + 1 = 2^{k+1}i + l$, $1 \le k$, $0 \le l < 2^k$. Clearly, l is odd, $j = 2^k i + (l - 1)/2$, $0 \le (l - 1)/2 < 2^k$, and so $(i,j) \in \beta$.
(iii) If $(2i + 1, 2j) \in \beta$, then $2j = 2^{k+1}i + 2^k + l$, $1 \le k$, $0 \le l < 2^k$. Clearly, l is even, $j = 2^k i + 2^{k-1} + l/2$, $0 \le l/2 < 2^{k-1}$, $2^{k-1} + l/2 < 2^k$, and so $(i,j) \in \beta$.
(iv) If $(2i + 1, 2j + 1) \in \beta$, then $2j + 1 = 2^{k+1}i + 2^k + l$, $1 \le k$, $0 \le l < 2^k$. Clearly, l is odd, $j = 2^k i + 2^{k-1} + (l - 1)/2$, $(l - 1)/2 < 2^{k-1}$, $2^{k-1} + (l - 1)/2 < 2^k$ and so $(i,j) \in \beta$.
(v) If $i \ne j$ and $(i, 2j) \in \beta$, then $2j = 2^k i + l$, $1 \le k$, $0 \le l < 2^k$. Clearly, l is even, $j = 2^{k-1} + l/2$, $0 \le l/2 < 2^{k-1}$. Since $i \ne j$, we have $k \ge 2$, and so $(i,j) \in \beta$.
(vi) If $i \ne j$ and $(i, 2j + 1) \in \beta$, then $2j + 1 = 2^k i + l$, $1 \le k$, $0 \le l < 2^k$. Clearly, l is odd, $j = 2^{k-1} + (l - 1)/2$, $0 \le (l - 1)/2^{k-1}$. Since $i \ne j$, we have $k \ge 2$, and so $(i,j) \in \beta$. ▲

2.6 Lemma. *Let $i,j \in \mathbb{N}$ be such that $(i,j) \in \beta$ and $2i \ne j \ne 2i + 1$. Then either $(2i,j) \in \beta$ or $(2i + 1,j) \in \beta$.*

Proof. We have $j = 2^k i + l$, $1 \le k$, $0 \le l < 2^k$. The inequalities $2i \ne j \ne 2i + 1$ imply $k \ge 2$. Now, if $l < 2^{k-1}$, then $j = 2^{k-1} \cdot 2i + l$ implies $(2i,j) \in \beta$. On the other hand, if $2^{k-1} \le l$, then $j = 2^{k-1}(2i + 1) + (l - 2^{k-1})$, $l - 2^{k-1} < 2^k - 2^{k-1} = 2^{k-1}$ and we have $(2i + 1,j) \in \beta$. ▲

2.7 Lemma. *Let $i,j \in \mathbb{N}$ be such that $(i,j) \in \beta$ and $2i \ne j \ne 2i + 1$. If j is even, then $j \ge 4$ and $(i,j/2) \in \beta$. If j is odd, then $j \ge 5$ and $(i,(j - 1)/2) \in \beta$.*

Proof. We have $j = 2^k i + l$, $1 \le k$, $0 \le l < 2^k$. Since $2i \ne j \ne 2i + 1$, we have in fact $k \ge 2$. Now, if j is even, then $j \ge 4$, l is even, $j/2 = 2^{k-1}i + l/2$, $0 \le l/2 < 2^{k-1}$, and hence $(i, j/2) \in \beta$. On the other hand, if j is odd, then $j \ge 5$, l is odd, $(j - 1)/2 = 2^{k-1}i + (l - 1)/2$, $0 \le (l - 1)/2 < 2^{k-1}$, and hence $(i, (j - 1)/2) \in \beta$. ▲

2.8 Lemma. *Let $i, j, k \in \mathbb{N}$ be such that $(i, k) \in \alpha$ and $(j, k) \in \alpha$. Then $i = j$.*

Proof. Obvious from the definition of α. ▲

2.9 Lemma. *Let $i, j, k \in \mathbb{N}$ be such that $(i, k) \in \beta$ and $(j, k) \in \beta$. Then just one of the following three cases takes place:*

(i) $i = j$:

(ii) $(i, j) \in \beta$;

(iii) $(j, i) \in \beta$.

Proof. We will proceed by induction on $2k - i - j$:

Firstly, if $(j, k) \in \alpha$, then $k \in \{2j, 2j + 1\}$ and, due to 2.5(5),(6), either $i = j$ or $(i, j) \in \beta$. Similarly, if $(i, k) \in \alpha$. Consequently, we can assume that $(i, k) \notin \alpha$ and $(j, k) \notin \alpha$. Then it follows from 2.3 that there are $p, q \in \mathbb{N}$ such that $(i, p) \in \beta$, $(p, k) \in \alpha$, $(j, q) \in \beta$, $(q, k) \in \alpha$. By 2.8, $p = q$ and, of course, $2p - i - j < 2k - i - j$. The rest follows by induction. ▲

2.10 Remark. If $(i, j) \in \beta$, then there exists just one α-chain between i and j (see 2.3, 2.8 and 2.9).

2.11 Remark. Let A be a non-empty subset of $\mathbb{N}$, $1 \notin A$ and put $B = \{i \mid (i, j) \in \beta \text{ for every } j \in A\}$. Then $1 \in B$ by 2.2(iii) and $i < j$ for all $i \in B$ and $j \in A$. Consequently $k = \max(B)$ exists and, if $l \in B$, then either $l = k$ or $(l, k) \in \beta$ (use 2.9).

2.12 Lemma. *Let $(i, j) \in \beta$.*

(i) If j is even, then $(i, j + 1) \in \beta$.

(ii) If j is odd, then $j \ge 3$ and $(i, j - 1) \in \beta$.

Proof. There is $k \in \mathbb{N}$ such that $(i, k) \in \gamma$ and $(k, j) \in \alpha$. Consequently, either $j = 2k$, $(k, j + 1) \in \alpha$ and $(i, j + 1) \in \beta$ or $j = 2k + 1$, $(k, j - 1) \in \alpha$ and $(i, j - 1) \in \beta$. ▲

2.13 Lemma. *The following conditions are equivalent for a permutation p of $\mathbb{N}$:*

(i) $(i, j) \in \beta$ *iff* $(p(i), p(j)) \in \beta$.

(ii) $(i, j) \in \alpha$ *iff* $(p(i), p(j)) \in \alpha$.

(iii) $(i, j) \in \alpha$ *implies* $(p(i), p(j)) \in \alpha$.

(iv) $(p(i), p(j)) \in \alpha$ *implies* $(i, j) \in \alpha$.

(v) $\{p(2i), p(2i + 1)\} = \{2p(i), 2p(i) + 1\}$ *for every* $i \ge 1$.

Proof. (i) implies (ii). Let $(i,j) \in \alpha$. Then $i < j$, $p(i) \neq p(j)$ and, by (i), $(p(i), p(j)) \in \beta$. Further, by 2.3, there are positive integers $m, k_0, \ldots, k_m$ such that $k_0 = p(i)$, $k_m = p(j)$ and $(k_0, k_1) \in \alpha, \ldots, (k_{m-1}, k_m) \in \alpha$. Using (i) again, we get $(i, p^{-1}(k_1)) \in \beta, \ldots, (p^{-1}(k_{m-1}), j) \in \beta$, and so $i < p^{-1}(k_1) < \ldots < p^{-1}(k_{m-1}) < j$ (use the fact that the numbers $i, k_1, \ldots, k_{m-1}, j$ are pair-wise different). Now, $(i,j) \in \alpha$ implies $m = 1$ and $(p(i), p(j)) \in \alpha$. Quite similarly, $(p^{-1}(i), p^{-1}(j)) \in \alpha$.

(iii) implies (ii). Let $(p(i), p(j)) \in \alpha$. By (iii), we have $(p(i), p(2i)) \in \alpha$ and $(p(i), p(2i+1)) \in \alpha$. Thus either $j = 2i$ or $j = 2i + 1$. In both cases, $(i,j) \in \alpha$.

The remaining implicatins are easy. ▲

2.14 Lemma. *If p is a permutation of* $\mathbb{N}$ *satisfying the equivalent conditions of* 2.12, *then* $p(1) = 1$ *and* $\{p(2), p(3)\} = \{2,3\}$.

Proof. Easy to check. ▲

2.14 Remark. Denote by $\mathscr{A}$ the set of permutations satisfying the equivalent conditions of 2.12. The $\mathscr{A}$ is a subgroup of the group $\mathbb{N}!$ of all permutations of $\mathbb{N}$. It is clear that permutations from $\mathscr{A}$ are just automorphism of the ordered set $\mathbb{N}(\gamma)$.

3. Auxiliary concepts (B)

3.1. In the sequel, $\mathscr{F}$ stands for the set of non-empty finite subsets of $\mathbb{N}$ and $\mathscr{F}_o = \mathscr{F} \cup \{\emptyset\}$.

For every $i \in \mathbb{N}$, let $T_i = \{2i, 2i+1\} \in \mathscr{F}$.

For $A \in \mathscr{F}_o$, let $\mu(A) = \{i \mid T_i \subseteq A\}$, $\varsigma(A) = \bigcup_{i \in \mu(A)} T_i = \{2i, 2i+1 \mid i \in \mu(A)\} \subseteq \subseteq A$, $\eta_1(A) = \mu(A) \cap (A \setminus \varsigma(A))$, $\eta_2(A) = \mu(A) \cap \varsigma(A)$ (so that $\eta_1(A) \cup \eta_2(A) = = \mu(A) \cap A$) and $\xi(A) = \mu(A) \cup (A \setminus \varsigma(A))$.

A set $A \in \mathscr{F}_o$ will be called reduced if $\mu(A) = \emptyset$.

3.2 Lemma. *Let* $A \in \mathscr{F}_o$. *Then:*

(i) $A \setminus \varsigma(A)$ *is reduced.*

(ii) $|\xi(A)| = |\mu(A)| + |A \setminus \varsigma(A)| - |\eta_1(A)| = |\varsigma(A)|/2 + |A \setminus \varsigma(A)| - |\eta_1(A)| = = |A| - |\varsigma(A)|/2 - |\eta_1(A) \leq |A|$.

(iii) $|\xi(A)| = |A|$ *iff A is reduced (and then* $\xi(A) = A$*).*

Proof. Easy to check. ▲

3.3 Lemma. *For every* $A \in \mathscr{F}_o$ *thhere exists* $m \geq 0$ *with* $\xi^{m+1}(A) = \xi^m(A)$.

Proof. By 3.2(ii), $|\xi(A) \leq |A|$ and the rest follows from 3.2(iii). ▲

3.4. Let $A \in \mathscr{F}_o$. Then we put $\bar{\xi}(A) = \xi^m(A)$ where $\xi^m(A) = \xi^{m+1}(A)$ (see 3.3).

3.5 Lemma. *For every* $A \in \mathscr{F}_o$, *the set* $\bar{\xi}(A)$ *is reduced and* $|\bar{\xi}(A)| \leq |A|$.

Proof. See 3.2 and 3.4. ▲

3.6 Lemma. *Let* $A, B \in \mathscr{F}_o$. *Then:*
(i) $\varsigma(A) \cup \varsigma(B) \subseteq \varsigma(A \cup B)$ *and* $(A \cup B) \setminus \varsigma(A \cup B) \subseteq (A \setminus \varsigma(A)) \cup (B \setminus \varsigma(B))$.
(ii) $\mu(A) \cup \mu(B) \subseteq \mu(A \cup B)$.

Proof. Easy to see. ▲

3.7 Lemma. *Let* $A, B \in \mathscr{F}_o$ *and* $i \in \mathbb{N}$. *Then* $i \in \xi(A) \cap \xi(B)$ *iff at least one of the following seven cases takes places:*
(1) i *is odd,* $i \in A \cap B$ *and* $i - 1 \notin A \cup B$;
(2) i *is even,* $i \in A \cap B$ *and* $i + 1 \notin A \cup B$;
(3) i *is odd,* $T_i \subseteq A$, $i \in B$ *and* $i - 1 \notin B$;
(4) i *is odd,* $T_i \subseteq B$, $i \in A$ *and* $i - 1 \notin A$;
(5) i *is even,* $T_i \subseteq A$, $i \in B$ *and* $i + 1 \notin B$;
(6) $T_i \subseteq A \cap B$.

Proof. Easy to see. ▲

3.8. Define a relation λ on $\mathscr{F}$ by $(B, A) \in \lambda$ iff $B = (A \setminus T_i) \cup \{i\}$ for some $i \in \mathbb{N}$ such that $T_i \subseteq A$. Moreover, put $\kappa = \lambda \cup id_{\mathscr{F}}$, denote by ϱ the transitive closure of λ defined on $\mathscr{F}$ and finally, put $\sigma = \varrho \cup id_{\mathscr{F}}$.

3.9 Lemma. *(i)* λ *is irreflexive and antisymmetric.*
(ii) If $(B, A) \in \lambda$, *then* $|B| < |A|$ *(more precisely,* $|A| - 2 \le |B| \le |A| - 1$*).*
(iii) κ *is reflexive and antisymmetric.*
(iv) If $(B, A) \in \kappa$, *then* $|B| \le |A|$.

Proof. Obvious from the definition of λ. ▲

3.10 Lemma. *(i)* ϱ *is irreflexive, antisymmetric and transitive (i.e.,* ϱ *is a sharp ordering of* $\mathscr{F}$*).*
(ii) $(B, A) \in \varrho$ *iff there are* $m \ge 1$ *and* $A_0, A_1, \dots, A_m \in \mathscr{F}$ *such that* $A_0 = B$, $A_m = A$ *and* $(A_i, A_{i+1}) \in \lambda$ *for* $i = 0, 1, \dots, m - 1$.
(iii) If $(B, A) \in \varrho$, *then* $|B| < |A|$.

Proof. Easy to see (use 3.9). ▲

3.11 Lemma. *(i)* σ *is reflexive, antisymmetric and transitive (i.e.,* σ *is a (reflexive) ordering of* $\mathscr{F}$*) and* σ *is the transitive closure of* κ.
(ii) $(B, A) \in \sigma$ *iff there are* $m \ge 1$ *and* $A_0, A_1, \dots, A_m \in \mathscr{F}$ *such that* $A_0 = B$, $A_m = A$ *and* $(A_i, A_{i+1}) \in \kappa$ *for* $i = 0, 1, \dots, m - 1$.
(iii) If $(B, A) \in \sigma$, *then* $|B| \le |A|$.

Proof. Easy to see (use 3.9 and 3.10). ▲

3.12 Lemma. *Let* $(B, A) \in \kappa$. *Then:*
(i) For every $i \in B$ *there is at least one* $j \in A$, *wih* $(i, j) \in \alpha \cup id_{\mathbb{N}}$.
(i) For every $k \in A$ *there is at least one* $l \in B$, *wih* $(l, k) \in \alpha \cup id_{\mathbb{N}}$.

Proof. Obvious from the definition of α, λ and κ. ▲

3.13 Lemma. *Let $(B, A) \in \sigma$. Then:*
(i) For every $i \in B$ there is at least one $j \in A$ with $(i,j) \in \gamma$.
(i) For every $k \in A$ there is at least one $l \in B$ with $(l, k) \in \gamma$.

Proof. Combine 2.3, 3.11(ii) and 3.12. ▲

3.14 Lemma. $(\xi(A), A) \in \sigma$ *for every* $A \in \mathscr{F}$.

Proof. If A is reduced, then $\xi(A) = A$ and there is nothing to show. Henceforth, let $\mu(A) = \{i_1, \dots, i_m, m \geq 1, i_1 < i_2 < \dots < i_m\}$. Now, put $A_m = A$ and $A_{j-1} = (A_j \setminus T_{i_j}) \cup \{i_j\}$ for $j = m, m-1, \dots, 1$. One checks easily by induction that $A_j = (A \setminus \bigcup_{k=j+1}^{m} T_{i_k}) \cup \{i_{j+1}, i_{j+2}, \dots, i_m\}$ for every $j = m-1, m-2, \dots, 0$. Clearly, $A_0 = \xi(A)$, $(A_{m-1}, A_m) \in \lambda$, $(A_{m-2}, A_{m-1}) \in \lambda, \dots, (A_0, A_1) \in \lambda$. Consequently, $(\xi(A), A) = (A_0, A_m) \in \varrho$. ▲

3.15 Corollary. *Let $A \in \mathscr{F}$. Then:*
(i) $(\xi^m(A), A) \in \sigma$ *for every* $m \geq 0$.
(ii) $(\bar{\xi}(A,) A) \in \sigma$.

3.16 Remark. One sees easily that minimal elements of the ordered set $\mathscr{F}(\sigma)$ are just reduced sets. Now, if $A \in \mathscr{F}$, then $\bar{\xi}(A)$ is reduced and $(\bar{\xi}(A), A) \in \sigma$. (3.15)

3.17 Example. (cf. 3.16) Put $A = \{2,3,4,5\}$ Then $\xi(A) = \{1,2\}$, $\{1,2\}$ is reduced, and so $\bar{\xi}(A) = \{1,2\}$. On the other hand, $(\{2,3\}, A) \in \lambda$ and $(\{1\},\{2,3\}) \in \lambda$. Thus $(\{1\}, A) \in \varrho$, $\{1\}$ is reduced and $\{1\} \neq \{1,2\}$.

3.18 Let S be a zp-semigroup and $f : \mathbb{N} \to S$ a mapping such that $f(2i) + f(2i+1) = f(i)$ for every $i \in \mathbb{N}$. Define a mapping $g : \mathscr{F}_o \to S$ by $g(\emptyset) = o_S$ and $g(A) = \sum_{i \in A} f(i)$ for every $A \in \mathscr{F}$.

3.18.1 Lemma. *If $(i,j) \in \beta$, then $f(i) \in S + f(j)$.*

Proof. The assertion is clear for $(i,j) \in \alpha$ and the general case follows by induction on the length of the corresponding α-chain. ▲

3.18.2 Lemma. *If $A \in \mathscr{F}$ such that $(i,j) \in \beta$ for some $(i,j) \in A$, then $g(A) = o$.*

Proof. By 3.18.1, $f(i) = f(j) + a$ for some $a \in S$. Then $f(i) + f(j) = 2f(j) + a = o$. ▲

3.18.3 Lemma. *Let $A \in \mathscr{F}$ be such that $\eta_1(A) = \emptyset$ (see 3.1). Then $g(A) = g(\xi(A))$.*

Proof. Easy to check directly. ▲

3.18.4 Lemma. $g(A \cup B) = g(A) + g(B)$ *for all* $A, B \in \mathscr{F}$, $A \cap B = \emptyset$.

Proof. Obvious. ▲

4. Auxiliary concepts (C)

4.1. A finite subset A of $\mathbb{N}$ will be called pre-pure if $(i,j) \notin \beta$ for all $i,j \in A$. The set A will be called pure if it is both pre-pure and reduced (see 3.1). We denote by $\mathcal{Q}$ ($\mathcal{P}$, resp.) the set of non-empty finite pre-pure (pure, resp.) subsets of $\mathbb{N}$ and we put $\mathcal{Q}_o = \mathcal{Q} \cup \{\emptyset\}(\mathcal{P}_o = \mathcal{P} \cup \{\emptyset\}$, resp.).

Notice that if A is pre-pure, then $\eta_1(A) = \emptyset = \eta_2(A)$ (see 3.1).

4.2 Lemma. *Let $(B,A) \in \lambda$ be such that $A \in \mathcal{Q}$. Then $B \in \mathcal{Q}$.*

Proof. We have $B = (A \setminus T_i) \cup \{i\}, i \in \mathbb{N}$, $T_i \subseteq A$. *Take* $j,k \in B$. If $j,k \in A$, then $(j,k) \notin \beta$, since $A \in \mathcal{Q}$. If $j \notin A$, $k \notin A$, then $j = i = k$ and $(j,k) \notin \beta$ again.

If $j \in A$ and $k \notin A$, then $k = i$, $2i \in A$, $(i,2i) \in \beta$, $(j,2i) \notin \beta$, and therefore $(j,k) = (j,i) \notin \beta$. Assume, finally, that $j \notin A$ and $k \in A$. Then $j = i$ and $2i \neq k \neq 2i + 1$. Further, since $A \in \mathcal{Q}$, we have $(2i,k) \notin \beta$ and $(2i + 1,k) \notin \beta$. Now, it follows from 2.6 that $(j,k) = (i,k) \notin \beta$. We have proved that $(j,k) \notin \beta$, so $B \in \mathcal{Q}$. ▲

4.3 Lemma. *Let $(B,A) \in \sigma$ be such that $A \in \mathcal{Q}$. Then $B \in \mathcal{Q}$.*

Proof. Combine 4.2 and 3.11(ii). ▲

4.4 Lemma. *Let $A,B,C \in \mathcal{Q}$ be such that $(B,A) \in \lambda$, $(C,A) \in \lambda$ and $B \neq C$. Then there is $D \in \mathcal{Q}$ such that $(D,B) \in \lambda$ and $(D,C) \in \lambda$.*

Proof. We have $B = (A \setminus T_i) \cup \{i\}$and $C = (C \setminus T_j) \cup \{j\}, i,j \in \mathbb{N}$, $T_i \cup T_j \subseteq A$. Since $B \neq C$, we have also $i \neq j$ and it follows that $T_j \subseteq B$ and $T_i \subseteq C$. If $i = 2j$ or $i = 2j + 1$, then $i \in A$, a contradiction with $(i,2i) \in \beta$. Thus $2j \neq i \neq 2j + 1$, $(B \setminus T_j) \cup \{j\} = D$ and $(D,B) \in \lambda$, where $D = (A \setminus (T_i \cup T_j)) \cup \{i,j\} \in \mathcal{Q}$ use (4.3). Quite similarly, $D = (C \setminus T_i) \cup \{i\}$and $(D,C) \in \lambda$. ▲

4.5 Lemma. *Let $A,B,C \in \mathcal{Q}$ be such that $(B,A) \in \sigma$ and $(C,A) \in \sigma$. Then there is $D \in \mathcal{Q}$ such that $(D,B) \in \sigma$ and $(D,C) \in \sigma$.*

Proof. There are $B_0, \ldots, B_m, C_0, \ldots, C_n \in \mathcal{Q}$, $m, n \in \mathbb{N}$, such that $B_0 = B$, $C_0 = C$, $B_m = A = C_n$ and all the pairs $(B_i, B_{i+1}), (C_j, C_{j+1})$, $i = 0, 1, \ldots, m - 1$, $j = 0, 1, \ldots, n - 1$ are in κ (use 4.3).

Firstly, assume that $m = 1$ and define sets $E_{n-1}, \ldots, E_0 \in \mathcal{Q}$ by induction in the following way: It follows from 4.4 that $(E_{n-1}, B) \in \kappa$ and $(E_{n-1}, C_{n-1}) \in \kappa$ for some $E_{n-1} \in \mathcal{Q}$. Now, if $1 \leq j < n$ and the sets $E_{n-1}, \ldots, E_j \in \mathcal{Q}$ are found such that $(E_{n-1}, C_{n-1}) \in \kappa$, $(E_{n-2}, C_{n-2}) \in \kappa, \ldots, (E_j, C_j) \in \kappa$, $(E_{n-1}, B) \in \kappa$, $(E_{n-2}, E_{n-1}) \in \kappa$, $\ldots$, $(E_j, E_{j+1}) \in \kappa$, then (by 4.4 again) there is $E_{j-1} \in \mathcal{Q}$ with $(E_{j-1}, C_{j-1}) \in \kappa$ and $(E_{j-1}, E_j) \in \kappa$. Consequently, $(E_0, B) \in \sigma$ and $(E_0, C) = (E_0, C_0) \in \kappa \subseteq \sigma$. We can put $D = E_0$ in this case.

In the general case, we proceed by induction on $m + n$. According to the preceding step of the proof, we can assume tat $m \geq 2$. Then, by induction, there is $F \in \mathcal{Q}$ with $(F, B_1) \in \sigma$, and $(F, C) \in \sigma$. Further, $(B, B_1) \in \kappa$ and, due to the first part of the proof, we find $D \in \mathcal{Q}$ such that $(D, B) \in \sigma$ and $(D, F) \in \sigma$. Then, of course, $(D, C) \in \sigma$. ▲

4.6 Remark. Let $A, B, C \in \mathcal{Q}$ be such that $(B, A) \in \varrho$ and $(C, A) \in \varrho$. By 4.5, $(D, B) \in \sigma$ and $(D, C) \in \sigma$ for some $D \in \mathcal{Q}$. If $D = B$, then $(B, C) \in \sigma$, and hence either $B = C$ or $(B, C) \in \varrho$. Similarly, if $D = C$, then either $B = C$ or $(C, B) \in \varrho$. Thus, if $B \neq C$, $(B, C) \notin \varrho$ and $(C, B) \notin \varrho$, then $(D, B) \in \varrho$ and $(D, C) \in \varrho$.

4.7 Lemma. *Let $A \in \mathcal{Q}$. Then:*

(i) $\xi^m(A)$ is pre-pure and $(\xi^m(A), A) \in \sigma$ for every $m \geq 0$
(ii) $\bar{\xi}(A)$ is pure and $(\bar{\xi}(A), A) \in \sigma$.

Proof. We have $(\xi^m(A), A) \in \sigma$ and $(\bar{\xi}(A), A) \in \sigma$ by 3.15. Consequently, both $\xi^m(A)$ and $\bar{\xi}(A)$ are pre-pure by 4.3. Finally $\bar{\xi}(A)$ is reduced, and hence pure. ▲

4.8 Remark. The ordering σ of $\mathcal{F}$ (see 3.11) induces an ordering of $\mathcal{Q}$ and we will denote it again by σ (but see also 4.3). By 4.5 the ordered set $\mathcal{Q}(\sigma)$ is downwards confluent and (see 3.16) minimal elements of $\mathcal{Q}(\sigma)$ are just pure sets. Of course, $\mathcal{Q}(\sigma)$ satisfies the minimum condition, and therefore for every $A \in \mathcal{Q}$ there exists a minimal element $M_A \in \mathcal{Q}$ with $(M_A, A) \in \sigma$. Because of the confluency, M_A is determined uniquely and it follows from 4.7(ii) that $M_A = \bar{\xi}(A)$ (cf. 3.17).

4.9. Lemma. *Let $A, B, C \in \mathcal{Q}$ be such that $A \cap B = \emptyset$, $A \cup B \in \mathcal{Q}$ and $(C, A) \in \kappa$. Then $C \cap B = \emptyset$ and $C \cup B \in \mathcal{Q}$.*

Proof. We can assume that $C \neq A$. Then $(C, A) \in \lambda$ and $C = (A \setminus T_i) \cup \{i\}$, $i \in \mathbb{N}$, $T_i \subseteq A$. Moreover, if $j \in C \cap B$, then $A \cap B = \emptyset$ implies $j = i$. But then $i, 2i \in A \cup B$ and $(i, 2i) \in \beta$ yields a contradiction with $A \cup B \in \mathcal{Q}$. Thus $C \cap B = \emptyset$ and it remains to show that $C \cup B \in \mathcal{Q}$. Let, on the contrary, $k, l \in C \cup B$ be such that $(k, l) \in \beta$. Since $(A \setminus T_i) \cup B \in \mathcal{Q}$ and $C \in \mathcal{Q}$, we have either $k = i$, $l \in B$ or $k \in B$, $l = i$.

If $k = i$ and $l \in B$, then $(i, l) \in \beta$ and $A \cap B = \emptyset$ implies $2i \neq l \neq 2i + 1$. Now, by 2.6, either $(2i, l) \in \beta$ or $(2i + 1, l) \in \beta$, a contradiction with $A \cup B \in \mathcal{Q}$.

If $k \in B$ and $l = i$, then $(k, i) \in \beta$, and hence $(i, 2i) \in \beta$ implies $(k, 2i) \in \beta$. But $k, 2i \in A \cup B$, a contradiction with $A \cup B \in \mathcal{Q}$. ▲

4.10 Lemma. *Let $A, B, C, D \in \mathcal{Q}$ be such that $A \cap B = \emptyset$, $A \cup B = \mathcal{Q}$, $(C, A) \in \kappa$ and $(D, B) \in \kappa$. Then $C \cap D = \emptyset$ and $C \cup D \in \mathcal{Q}$.*

Proof. By 4.9, $C \cap B = \emptyset$ and $C \cup B \in \mathcal{Q}$. Consequently, using 4.9 once more, we get $C \cap D = \emptyset$ and $C \cup D \in \mathcal{Q}$. ▲

4.11 Lemma. *Let $A, B, C, D \in \mathcal{Q}$ be such that $A \cap B = \emptyset$, $A \cup B \in \mathcal{Q}$, $(C, A) \in \sigma$ and $(D, B) \in \sigma$. Then $C \cap D = \emptyset$ and $C \cup D \in \mathcal{Q}$.*

Proof. There are $m \geq 1$ and $C_0, \ldots, C_m, D_0, \ldots, D_m \in \mathcal{Q}$ such that $C_0 = C$, $D_0 = D$, $C_m = A$, $D_m = B$ and (C_i, C_{i+1}), $(D_i, D_{i+1}) \in \kappa$ for every $i = 0, 1, \ldots, m-1$. Now, our result follows easily from 4.10 by induction on m. ▲

4.12 Lemma. *Let $A, B \in \mathcal{Q}$ be such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{Q}$. Then:*
(i) $\xi^m(A) \cap \xi^m(B) = \emptyset$ and $\xi^m(B) \in \mathcal{Q}$ for every $m \geq 1$.
(ii) $\bar{\xi}(A) \cap \bar{\xi}(B) = \emptyset$ and $\bar{\xi}(A) \cup \bar{\xi}(B) \in \mathcal{Q}$.

Proof. Combine 4.11 and 4.7. ▲

4.13 Lemma. *Let $A, B, C \in \mathcal{Q}$ be such that $A \cap B = \emptyset$, $A \cup B \in \mathcal{Q}$ and $(C, A) \in \kappa$. Then $(C \cup B, A \cup B) \in \kappa$.*

Proof. We can assume that $C \neq A$. Then $C = (A \setminus T_i) \cup \{i\}, i \in \mathbb{N}$, $T_i \subseteq A$, and we get $C \cup B = (A \setminus T_i) \cup B \cup \{i\} = ((A \cup B) \setminus T_i) \cup \{i\}$. Thus $(C \cup B, A \cup B) \in \in \lambda$. ▲

4.14 Lemma. *Let $A, B, C, D \in \mathcal{Q}$ be such that $A \cap B = \emptyset$, $A \cup B \in \mathcal{Q}$, $(C, A) \in \kappa$ and $(D, B) \in \kappa$. Then $(C \cup D, A \cup B) \in \sigma$.*

Proof. By 4.13, we have $(C \cup B, A \cup B) \in \kappa$. Further, by 4.9, $C \cap B = \emptyset$ and $C \cup B \in \mathcal{Q}$. Consequently, using 4.13 again, we get $(C \cup D, C \cup B) \in \kappa$. From this, $(C \cup D, A \cup B) \in \sigma$. ▲

4.15 Lemma. *Let $A, B, C, D \in \mathcal{Q}$ be such that $A \cap B = \emptyset$, $A \cup B \in \mathcal{Q}$, $(C, A) \in \sigma$ and $(D, B) \in \sigma$. Then $(C \cup D, A \cup B) \in \sigma$.*

Proof. Using 4.14, we can proceed similarly as in the proof of 4.11. ▲

4.16 Lemma. *Let $A, B \in \mathcal{Q}$ be such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{Q}$. Then:*
(i) $(\xi^m(A) \cup \xi^m(B), A \cup B) \in \sigma$ for every $m \geq 0$.
(ii) $(\bar{\xi}(A) \cup \bar{\xi}(B), A \cup B) \in \sigma$.

Proof. Combine 4.15 and 4.7. ▲

4.17 Lemma. *Let $A, B \in \mathcal{Q}$ be such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{Q}$. Then $\bar{\xi}(A \cup B) = \bar{\xi}(\bar{\xi}(A) \cup \bar{\xi}(B))$.*

Proof. It follows from 4.7 and 4.16(ii) that $(\bar{\xi}(A \cup B), A \cup B) \in \sigma$ and $(\bar{\xi}(\bar{\xi}(A) \cup \bar{\xi}(B)), A \cup B) \in \sigma$. However, both the sets $\bar{\xi}(A \cup B)$ and $\bar{\xi}(\bar{\xi}(A) \cup \bar{\xi}(B))$ are pure (see 4.7(ii)), and hence they coincide by 4.5 (see also 4.8). ▲

4.18 Lemma. *Let $A, B, C \in \mathcal{Q}$ be such that $A \cap C = \emptyset$, $A \cup C \in \mathcal{Q}$ and $(C, B) \in \sigma$. Then $A \cap B = \emptyset$.*

Proof. Let, on the contrary, $i \in A \cap B$. By 3.13(iii), $(j, i) \in \gamma$ for some $j \in C$. But $i, j \in A \cup C$ and $A \cup C \in \mathcal{Q}$. Henceforth, $i = j$ and $i \in A \cap C$, a contradiction. ▲

4.19 Lemma. *Let $A, B \in \mathcal{Q}$ be such that $A \cap \bar{\xi}(B) = \emptyset$ and $A \cup \bar{\xi}(B) \in \mathcal{Q}$. Then $A \cap B = \emptyset$.*

Proof. We have $(\bar{\xi}(B), B) \in \sigma$ by 3.15(ii) and we use 4.18. ▲

4.20 Lemma. *Let $A, B, C \in \mathcal{Q}$ be such that $A \cap C = \emptyset$, $A \cup C \in \mathcal{Q}$ and $(C, B) \in \sigma$. Then $A \cup B \in \mathcal{Q}$.*

Proof. Let on the contrary, $(i,j) \in \beta$ for some $i, j \in A \cup B$. Since $A, B \in \mathcal{Q}$, we have either $i \in A, j \in B$ or $i \in B, j \in A$.

Firstly, assume $i \in A$, $j \in B$. By 3.13(iii), $(k,j) \in \beta$ for some $k \in C$. Since $A \cap C = \emptyset$, we have $k \neq i$, and hence either $(i,k) \in \beta$ or $(k,i) \in \beta$ by 2.9, a contradiction with $A \cup C \in \mathcal{Q}$.

Next, let $i \in B$, $j \in A$. Again $(k,i) \in \beta$ for some $k \in C$, and therefore $(k,j) \in \beta$, a contradiction with $A \cup C \in \mathcal{Q}$. ▲

4.21 Lemma. *Let $A, B \in \mathcal{Q}$ be such that $A \cap \bar{\xi}(B) = \emptyset$ and $A \cup \bar{\xi}(B) \in \mathcal{Q}$. Then $A \cup B \in \mathcal{Q}$.*

Proof. Combine 3.15(ii) and 4.20. ▲

4.22 Lemma. *Let $A, B, C, D \in \mathcal{Q}$ be such that $(C, A) \in \sigma$, $(D, B) \in \sigma$, $C \cap D = \emptyset$ and $C \cup D \in \mathcal{Q}$. Then $A \cap B = \emptyset$ and $A \cup B \in \mathcal{Q}$.*

Proof. By 4.18 and 4.20, $A \cap D = \emptyset$ and $A \cup D \in \mathcal{Q}$. Using 4.18 and 4.20 once more, we get our result. ▲

4.23 Lemma. *The following conditions are equivalent for $A, B \in \mathcal{Q}$:*

(i) $A \cap B = \emptyset$ and $A \cup B \in \mathcal{Q}$.

(ii) There exists $m \geq 0$ such that $\xi^m(A) \cap \xi^m(B) = \emptyset$ and $\xi^m(A) \cup \xi^m(B) \in \mathcal{Q}$.

(iii) For every $m \geq 0$, $\xi^m(A) \cap \xi^m(B) = \emptyset$ and $\xi^m(A) \cup \xi^m(B) \in \mathcal{Q}$.

(iv) $\bar{\xi}(A) \cap \bar{\xi}(B) = \emptyset$ and $\bar{\xi}(A) \cup \bar{\xi}(B) \in \mathcal{Q}$.

Proof. Combine 4.7, 4.12 and 4.22. ▲

4.24 Lemma. *Let $A \in \mathcal{Q}$ be such that $k = max(A)$ is even. Then $k + 1 \notin A$ and $A \cup \{k + 1\} \in \mathcal{Q}$.*

Proof. Clearly, $k + 1 \notin A$ and $k = 2j, j \in N$. Now, assume that $A \cup \{k + 1\} \notin Q$. Since $A < k + 1$, there is $i \in A$ with $(i, k + 1) \in \beta$. If $k + 1 = 2i + 1$, then $i = j$ and $(i, k) = (i, 2i) \in \beta$, a contradiction with $A \in \mathcal{Q}$. Thus $k + 1 \neq 2i + 1$ and $(i,j) \in \beta$ by 2.7. On the other hand, $(j, 2j) \in \beta$, and hence $(i, k) = (i, 2j) \in \beta$, again a contradiction. ▲

4.25 Lemma. *Let $A \in \mathcal{P}$ be such that $A \neq \{1\}$ and $k = max(A)$ is odd. Then $k - 1 \notin A$ and $A \cup \{k - 1\} \in \mathcal{Q}$.*

Proof. We have $k = 2j + 1 \geq 3$ and, since A is reduced, we conclude that $k - 1 \notin A$. Now, assume that $A \cup \{k - 1\} \notin \mathcal{Q}$. Since *max* $(A \setminus \{k\}) < k - 1$, there is $i \in A$ with $(i, k-1) \in \beta$. If $k-1 = 2i$, then $i = j$ and $(i,k) = (i, 2i+1) \in \beta$ a contradiction with $A \in \mathcal{Q}$. Thus $k - 1 \neq 2i$ and $(i,j) \in \beta$ by 2.7. On the other hand, $(j, 2j+1) \in \beta$, and hence $(i,k) = (i, 2j+1) \in \beta$, again a contradiction. ▲

4.26 Corollary. Let $A \in \mathcal{P}$ be such that $A \neq \{1\}$. The there exists at least one $l \in \mathbb{N}$ such that $l \notin A$ and $A \cup \{l\} \in \mathcal{Q}$.

4.27 Lemma. *Let $A \in \mathcal{Q}$ and $i \in \mathbb{N}$ be such that $M = \{j \in A \mid (i,j) \in \beta\}$ is non-empty. Put $k = max(M)$.*

(i) If k is even, then $A \cup \{k+1\} \in \mathcal{Q}$.

(ii) If k is odd, then $k \geq 3$ and $A \cup \{k-1\} \in \mathcal{Q}$.

Proof. (i) If $l \in A$ is such that $(l, k+1) \in \beta$, then $(l,k) \in \beta$ by 2.12(ii), a contradiction with $A \in \mathcal{Q}$. Onn the other hand, if $l \in A$ is such that $(k+1, l) \in \beta$, then $(i, k+1) \in \beta$ (2.12(i)) implies $(i,l) \in \beta$ and $l \in M$, a contradiction with $k < l$. Thus $A \cup \{k+1\} \in \mathcal{Q}$.

(ii) If $l \in A$ is such that $(l, k-1) \in \beta$, then $(l,k) \in \beta$ by 2.12(i), a contradiction with $A \in \mathcal{Q}$. On the other hand, if $l \in A$ is such that $(k-1, l) \in \beta$, then $(i, k-1) \in \beta$ (2.12(ii)) implies $(i,l) \in \beta$ and $l \in M$, a contradiction with $k < l$. Thus $A \cup \{k-1\} \in \mathcal{Q}$. ▲

4.28. Let S be a zp-semigroup and $f : \mathbb{N} \to S$ a mapping such that $f(2i) + f(2i+1) = f(i)$ for every $i \in \mathbb{N}$. Define $g : \mathcal{Q}_o \to S$ by $g(\emptyset) = o_S$ and $g(A) = \sum_{i \in A} f(i)$ for every $A \in \mathcal{Q}$ (see 3.18).

4.28.1 Lemma. $g(\bar{\xi}(A)) = g(A)$ *for every* $A \in \mathcal{Q}$.

Proof. By 3.18.3, $g(\xi(A)) = g(A)$. Consequently, we get $g(\xi^m(A)) = g(A)$ by induction on $m \geq 0$. ▲

5. One particular zs-semigroup

5.1. Define a binary operation $\oplus$ on the set $\mathcal{P}_o$ of (finite) pure subsets of $\mathbb{N}$ (see 4.1) by $A \oplus B = \bar{\xi}(A \cup B)$ for all $A, B \in \mathcal{P}$ such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{Q}$ (see 4.7(ii), and $A \oplus B = \emptyset$ otherwise.

5.2 Lemma. *(i)* $A \oplus B = B \oplus A$.

(ii) $A \oplus \emptyset = \emptyset = \emptyset \oplus A$.

(iii) $A \oplus A = \emptyset$.

Proof. Obvious from the definition of the operation $\oplus$. ▲

5.3 Lemma. *Let $A, B, C \in \mathcal{P}_o$. Then $A \oplus (B \oplus C) \neq \emptyset$ iff the sets A, B, C are non-empty, pair-wise disjoint and $A \cup B \cup C \in \mathcal{Q}$. Then $A \oplus (B \oplus C) = \bar{\xi}(A \cup B \cup C)$.*

Proof. (i) Let $A \oplus (B \oplus C) \neq \emptyset$. Then the pure sets A, B, C are non-empty, $B \cap C = \emptyset$, $B \cup C \in \mathcal{Q}$, $B \oplus C = \xi(B \cup C)$, $A \cap \xi(B \cup C) \in \mathcal{Q}$ and $A \oplus (B \oplus C) = \xi(A \cup (\xi(B \cup C))$.

Using 4.19 and 4.21, we get $A \cap (B \cup C) = \emptyset$ and $A \cup (B \cup C) \in \mathcal{Q}$. Consequently, the sets A, B, C are pair-wise disjoint and $A \cup B \cup C \in \mathcal{Q}$. Finally, $A \oplus (B \oplus C) = \xi(A \cup \xi(B \cup C)) = \xi(\xi(A) \cup \xi(B u C)) = \xi(A \cup B \cup C)$ by 4.17.

(ii) Let the sets A, B, C be non-empty, pair wise disjoint and let $A \cup B \cup C \in \mathcal{Q}$. Then $B \cup C \in \mathcal{Q}$, so that $B \oplus C = \xi(B \cup C)$. Moreover, $A \cap \xi(B \cup C) = \emptyset$ and $A \cup \xi(B \cup C) \in \mathcal{Q}$ by 4.11. Thus $A \oplus (B \oplus C) = A \oplus \xi(B \cup C) = \xi(A \cup \xi(B \cup C)) \neq \emptyset$. ▲

5.4 Lemma. *Let $A, B, C \in \mathcal{P}_o$. Then $(A \oplus B) \oplus C \neq \emptyset$ iff the sets A, B, C are non-empty, pair-wise disjoint and $A \cup B \cup C \in \mathcal{Q}$. Then $(A \oplus B) \oplus C = \xi(A \cup B \cup C)$.*

Proof. Similar to that of 5.3. ▲

5.5 Lemma. *$\mathcal{P}_o(\oplus)$ is a commutative zp-semigroup and $\emptyset$ is the absorbing element of this semigroup.*

Proof. Combine 5.2, 5.3 and 5.4. ▲

5.6 Lemma. *For every $A \in \mathcal{P}$ there are $B, C \in \mathcal{P}$ such that $A = B \oplus C$.*

Proof. If $|A| = 1$, *then* $A = \{i\}$, $i \in \mathbb{N}$, and we put $B = \{2i\}$, $C = \{2i + 1\}$. Then $B \oplus C = A$. If $A = A_1 \cup A_2$, where $A_1 \cap A_2 = \emptyset$ and A_1, A_2 are non-empty, then $A_1, A_2 \in \mathcal{P}$ and $A = A_1 \oplus A_2$. ▲

5.7 Proposition. *$\mathcal{P}_o(\oplus)$ is a non-trivial commutative zs-semigroup.*

Proof. See 5.2, 5.5 and 5.6. ▲

5.8 Lemma. *Let $A_1, \ldots, A_m \in \mathcal{P}_o$, $m \geq 2$ Then $A_1 \oplus \ldots \oplus A_m \neq \emptyset$ iff the sets $A_1, \ldots, A_m$ are non-empty, pair-wise disjoint and $A_1 \cup \ldots \cup A_m \in \mathcal{Q}$. Then $A_1 \oplus \ldots \oplus A_m = \xi(A_1 \cup \ldots \cup A_m)$.*

Proof. We will proceed by induction on m. The case $m = 2$ is clear from the definition 5.1. If $m \geq 3$ and $B = A_1 \oplus \ldots \oplus A_{m-1}$ (see 5.7), then $A_1 \oplus \ldots \oplus A_m = B \oplus A_m$ and $B \oplus A \neq \emptyset$ iff $B \neq \emptyset \neq A_m$, $B \cap A_m = \emptyset$ and $B \cup A_m \in \mathcal{Q}$; then $B \oplus A_m = \xi(B \cup A_m)$. The rest is clear. ▲

5.9 Proposition. *(i) If $A = \{i_1, \ldots, i_m\}, m \geq 1$, is a pre-pure set, then $\{i_1\} \oplus \ldots \oplus \{i_m\} = \xi(A)$ (and so $A = \sum_{j=1}^{m} \oplus \{i_j\}$, provided that A is pure).*

(ii) The semigroup $\mathcal{P}_o$ is generated by the set $\{\{i\} \mid i \in \mathbb{N}\}$.

(iii) $\{2i\} \oplus \{2i + 1\} = \{i\}$ for every $i \in \mathbb{N}$.

Proof. Use 5.7 and 5.8. ▲

5.10 Lemma. *Let $A \in \mathscr{P}$ be such that $A \neq \{1\}$ and and let $k = max(A)$.*

(i) If k is even, then $k \geq 2, k + 1 \notin A$ and $A \cup \{k + 1\} \in \mathscr{Q}$ and $A \oplus \{k + 1\} = = \xi((A \setminus \{k\}) \cup \{k/2\})$.

(ii) If k is odd, then $k \geq 3$, $k - 1 \notin A$, $A \cup \{k - 1\} \in \mathscr{Q}$ and $A \oplus \{k - 1\} = = \xi((A \setminus \{k\}) \cup \{(k - 1)/2\})$.

Proof. See 4.24 and 4.25. ▲

5.11 Corollary. Let $A \in \mathscr{P}$ be such that $A \neq \{1\}$.Then $A \oplus \{l\} \neq \emptyset$ for at least one $l \in \mathbb{N}$.

5.12 Proposition. *$Ann(\mathscr{P}_o(\oplus)) = \{A \in \mathscr{P}_o \mid \mathscr{P}_o \oplus A = \emptyset\} = \{\emptyset, \{1\}\}$ (and hence $|Ann(\mathscr{P}_o(\oplus))| = 2$).*

Proof. Clearly, both the sets $\emptyset$ and $\{1\}$ belong to the annihilator. On the other hand, if $A \in \mathscr{P}$ is such that $A \neq \{1\}$, then it follows from 5.11 that A is not in the annihilator. ▲

5.13 Lemma. *Let $A \in \mathscr{P}$ and $i \in \mathbb{N}$ be such that $M = \{j \in A \mid (i,j) \in \beta\}$ is non-empty. Put $k = max(M)$.*

(i) If k is even, then $k \geq 2$, $k + 1 \notin A$, $A \cup \{k + 1\} \in \mathscr{Q}$ and $A \oplus \{k + 1\} = = \xi((A \setminus \{k\}) \cup \{k/2\})$.

(ii) If k is odd, then $k \geq 3$, $k - 1 \notin A$, $A \cup \{k - 1\} \in \mathscr{Q}$ and $A \oplus \{k - 1\} = = \xi((A \setminus \{k\}) \cup \{(k - 1)/2\})$.

Proof. See 4.27. ▲

5.14 Proposition. *Let $A, B \in \mathscr{P}_o$ be such that $A \neq B$ and $\{A,B\} \neq \neq \{\emptyset, \{1\}\} (= Ann(\mathscr{P}_o(\oplus)))$. Then there exists at least one $p \in \mathbb{N}$ such that either $A \oplus \{p\} = \emptyset \neq B \oplus \{p\}$ or $A \oplus \{p\} \neq \emptyset = B \oplus \{p\}$.*

Proof. It is divided into four parts:

(i) $A = \emptyset$ (or $B = \emptyset$), then $B \neq \{1\}$ (*or* $A \neq \{1\}$) and the assertion follows from 5.11.

(ii) Let $i \in A$ be such that $M = \{j \in B \mid (i,j) \in \beta\} \neq \emptyset$ and let $k = max(M)$. Clearly, $i \notin B$. If k is even, then $(i, k + 1) \in \beta$ by 2.12(i), and hence $A \oplus \{k + 1\} = \emptyset \neq B \oplus \{k + 1\}$ by 5.13(i).
If k is odd, then $k \geq 3$, $(i, k - 1) \in \beta$ by 2.12(ii), and hence $A \oplus \{k - 1\} = = \emptyset \neq B \oplus \{k - 1\}$ by 5.13(ii).

(iii) Let $j \in B$ such that $N = \{i \in A \mid (j, i) \in \beta) \neq \emptyset$. Now, we can proceed in the same way as in (ii).

(iv) In view of (i), (ii) and (iii), we can assume that $A, B \in \mathscr{P}$, $(i,j) \notin \beta$ and $(j, i) \notin \beta$ for all $i \in A$ and $j \in B$. Now, since $A \neq B$, we find $k \in A \setminus B$ (or $(l \in B \setminus A)$. Then $B \cup \{k\} \in \mathscr{Q} (A \cup \{l\} \in \mathscr{Q})$, and therefore $A \oplus \{k\} = \emptyset \neq \neq B \oplus \{k\} (A \oplus \{l\} \neq \emptyset = B \oplus \{l\})$. ▲

5.15 Proposition. *The semigroup $\mathscr{P}_o(\oplus)$ is subdirectly irreducible and the monolith of $\mathscr{P}_o$ (i.e., the smallest non-identical congruence) is just the congruence corresponding to the ideal $Ann(\mathscr{P}_o(\oplus))$. That is, $\mu_{\mathscr{P}_o} = \{(\emptyset,\{1\}),(\{1\},\emptyset)\} \cup id_{\mathscr{P}_o}$.*

Proof. Let $\varrho \neq id_{\mathscr{P}_o}$ be a congruence of $\mathscr{P}_o(\oplus)$ and let $\mathscr{K} = \{K \in \mathscr{P} \mid (K,\emptyset) \in \varrho\}$. There are $A, B \in \mathscr{P}_o$ such that $A \neq B$ and $(A,B) \in \varrho$. Now, it follows from 5.14 that $\mathscr{K} \neq \emptyset$ and we take $L \in \mathscr{K}$ such that $l = max(L)$ is smallest possible. If $l = 1$, then $L = \{1\}$ and $(\{1\},\emptyset) \in \varrho$. On the other hand, if $l \geq 2$, then, by 5.10, there is $q \in \mathbb{N}$ such that $L \oplus \{q\} \neq \emptyset$ and $max(L \oplus \{q\}) < l$. Of course, $(L \oplus \{q\},\emptyset) \in \varrho$ and this is a contradiction. ▲

5.16 Proposition. *Let S be a zp-semigroup and $f : \mathbb{N} \to S$ a mapping such that $f(2i) + f(2i+1) = f(i)$ for every $i \in \mathbb{N}$. Put $g(\emptyset) = o (= o_S)$ and $g(A) = \sum_{i \in A} = f(i)$ for every $A \in \mathscr{P}$. Then g is a homomorphism of the semigroup $\mathscr{P}_o(\oplus)$ into the semigroup S. Moreover, if $f(1) \neq o$, then g is injective.*

Proof. (i) First of all, let $A, B \in \mathscr{P}_o$ and $C = A \oplus B$. We have to show that $g(C) = g(A) + g(B)$.
If $A = \emptyset$ (or $B = \emptyset$), then $C = \emptyset$, $g(A) = o$ (or $g(B) = o$), $g(C) = o$, and hence $g(C) = o = g(A) + g(B)$.
If $i \in A \cap B$, then $C = \emptyset$, $g(C) = o$, $g(A) + g(B) = 2f(i) + u$ for some $u \in S \cup \{0\}$ and hence $g(C) = o = g(A) + g(B)$.
If $A \neq \emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B \notin \mathscr{Q}$, then $C = \emptyset$, $g(C) = o$ and $g(C) = o = \sum_{i \in A \cup B} f(i) = g(A) + g(B)$ by 3.18.2.
If $A \neq \emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B \in \mathscr{Q}$, then $C = \xi(A \cup B)$ and, by 4.28.1, $g(C) = g(A \cup B) = \sum_{i \in A \cup B} f(i) = \sum_{i \in A} f(i) + \sum_{i \in B} f(i) = g(A) + g(B)$.
(ii) Assume that $f(1) \neq o$ and put $\varrho = Ker(g)$. Then $(\{1\},\emptyset) \notin \varrho$, and hence the equality $\varrho = id_{\mathscr{P}_o}$ follows from 5.15. ▲

5.17 Proposition. *Let S be a zs-semigroup. Then for every $a \in S$, $a \notin S$, $a \neq o_S$, there exists an injective homomorphism g of $\mathscr{P}_o(\oplus)$ into S such that $g(\{1\}) = a$.*

Proof. By induction on $m \geq 0$, define a mapping $f_m : \{1,2,\dots,2m,2m+1\} \to S$ in the following way: Firstly, $f_0(1) = a$. Then if $m \geq 0$ and $f_0, \dots f_m$ are defined, then we put $f_{m+1} \mid \{1,2,\dots,2m+1\} = f_m$ and $f_{m+1}(2m+2) = x$ and $f_{m+1}(2m+3) = y$, where $x, y = S$ are chosen such that $x + y = f_m(m+1)$. Now, put $f = \cup f_m$, so that f is mapping of $\mathbb{N}$ into S such that $f(1) = a$ and $f(2i) + f(2i+1) = f(i)$ for every $i \in \mathbb{N}$. The rest follows from 5.16. ▲

5.18 Proposition. *Let S be a zs-semigroup. Then for every $a \in S$ there exists a homomorphism g of $\mathscr{P}_o(\oplus)$ into S such that $g(\{1\}) = a$.*

Proof. This is an immediate consequence of 5.17, the case $a = o$ being trivial. ▲

6. Trees in zp-semigroups

6.1. In this section, let S be a non-trivial zp-semigroup. An infinite sequence $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ of elements from S (i.e., a mapping from $\mathbb{N}$ into S) will be called an S-tree if $a_i = a_{2i} + a_{2i+1}$ for every $i \in \mathbb{N}$.

We denote by $\mathcal{T}\,(= \mathcal{T}(S))$ the set of trees.

6.1 Proposition. *$\mathcal{T}$ is a subsemigroup of the cartesian power S^{ω}.*

Proof. Clearly, the constant sequence $\mathbf{o} = (o)$ belongs to $\mathcal{T}$, and so $\mathcal{T}$ is non-empty. Furthermore, if $\mathbf{a}, \mathbf{b} \in \mathcal{T}$ then the sequence $\mathbf{a} + \mathbf{b} = (a_i + b_i)$ is a tree, too. ▲

6.2. If $\mathbf{a} = (a_1, a_2, a_3, \ldots)$, then we denote by $R(\mathbf{a})\,(= R(S, \mathbf{a}))$ the subsemigroup of S generated by the elements $a_1, a_2, a_3, \ldots$, i.e., $R(\mathbf{a}) = \langle a_i \mid i \in \mathbb{N} \rangle_S$.

6.3 Theorem. *Let $\mathbf{a} = (a_1, a_2 a_3, \ldots)$ be tree such that $a_1 \neq o$. Then there exists an isomorphism g of $\mathcal{P}_o(\oplus)$ onto $R(\mathbf{a})$ such that $g(\{i\}) = a_i$ for every $i \in \mathbb{N}$ (in particular, $g(\{1\}) = a_1$).*

Proof. Put $f(i) = a_i$ for every $i \in \mathbb{N}$, $g(\emptyset) = o_S$ and $g(A) = \sum_{i \in A} f(i)$ for every $A \in \mathcal{P}$. By 5.16, g is an injective homomorphism of the semigroup $\mathcal{P}_o(\oplus)$ into the semigroup S. Since $\mathcal{P}_o(\oplus)$ is generated by the set $\{\{i\} \mid i \in \mathbb{N}\}$ (5.9(ii)), the image $Im(g)$ is a subsemigroup of S generated by the set $g(\{\{i\} \mid i \in \mathbb{N}\}) = \bigcup_{i \in \mathbb{N}} f(i)$. Consequently, $Im(g) = R(\mathbf{a})$ and g is an isomorphism of $\mathcal{P}_o(\oplus)$ onto $R(\mathbf{a})$. ▲

6.4 Corollary. Let $\mathbf{a}, \mathbf{b} \in \mathcal{T}$ be trees such that $a_1 \neq o \neq b_1$. Then the zs-semigroups $R(\mathbf{a})$ and $R(\mathbf{b})$ are isomorphic.

6.5 Remark. According to 5.9(ii), the sequence $\mathbf{w} = (\{1\}, \{2\}, \{3\}, \ldots)$ of elements from $\mathcal{P}_o$ is a tree and $R(\mathbf{w}) = \mathcal{P}_o$.

6.6 Lemma. *Let $\mathbf{a}$ be a tree.*

(i) If $(i, j) \in \beta$, then $a_i = a_j + a$ for some $a \in R(\mathbf{a})$.

(ii) If $(i, j) \in \gamma$, then $a_i = a_j + u$ for some $u \in R(\mathbf{a}) \cup \{0\}$.

Proof. (i) The assertion is clear for $(i, j) \in \alpha$ and, in the general case, it follows by induction on the length of the corresponding α-chain.

(ii) This follows immediately from (i). ▲

6.7 Lemma. *Let $\mathbf{a}$ be a tree and let $i, j \in \mathbb{N}$ be not comparable in γ. Then $1 \neq i \neq j \neq 1$ and, if $k \in \mathbb{N}$ is maximal with respect to $(k, i), (k, j) \in \beta$ (see 2.11), then $a_k = a_i + a_j + u$ for some $u \in R(\mathbf{a}) \cup \{0\}$.*

Proof. There are $m, n, i_0, \ldots, i_m, j_0, \ldots, j_n \in \mathbb{N}$ such that $i_0 = k = j_0$, $i_m = i$, $j_n = j$ and all the pairs $(i_0, i_1), \ldots, (i_{m-1}, i_m), (j_0, j_1), \ldots, (j_{n-1}, j_n)$ are in α. Clearly, $(i_1, i) \in \gamma$, $(j_1, j) \in \gamma$, and hence $a_{i_1} = a_i + u_1$, $a_{j_1} = a_j + u_2$ for some $u_1, u_2 \in R(\mathbf{a}) \cup \{0\}$

(6.6 (ii)). If $i_1 \neq j_1$, then $(k, i_1) \in \alpha$, $(k, j_1) \in \alpha$ implies $a_{i_1} + a_{j_1} = a_k$, and therefore $a_k = a_i + a_j + u_1 + u_2 = a_i + a_j + u$, $u = u_1 + u_2 \in S \cup \{0\}$.

On the other hand, if $i_1 = i_j$, then using the maximality of k, we get either $(i_1, i) \notin \beta$ or $(j_1, j) \notin \beta$. But, if $(i_1, i) \notin \beta$, then $j_1 = i_1 = i$, and hence $(i, j) \in \gamma$, a contradiction. The other case is similar. ▲

6.8 Proposition. *Let* **a** *be a tree such that* $a_1 \neq o$ *and let* $i, j \in \mathbb{N}$. *The following conditions are equivalent:*

(i) $(i, j) \in \beta$

(ii) $a_i \in R(\mathbf{a}) + a_j$

(iii) $a_i \in S + a_j$.

Proof. (i) implies (ii) by 6.6(i) and (ii) implies (iii) trivially.

(iii) implies (i). Assume, on the contrary, that $a_i = a_j + a$, $a \in S$, and that $(i, j) \notin \beta$. If $(i, j) \in \gamma$ than $a_j = a_i + u$, $u \in S \cup \{0\}$, by 6.6(ii), and hence $a_i = = a_i + u + a = a_i + u + a + u + a = a_i + 2u + 2a = a_i + 2u + o = o$. But $(1, i) \in \gamma$ implies $a_1 = a_i + v$, so that $a_1 = o$, a contradiction. It follows that $(i, j) \notin \gamma$ and $(j, i) \notin \gamma$. Now, if k is an in 6.7, then $a_k = a_i + a_j + w$, $w \in S \cup \{o\}$. Again, we get $a_k = 2a_i + u + w = o$ and $a_1 = o$, a contradiction. ▲

6.9 Corollary. *Let* **a** *be a tree such that* $a_1 \neq o$ *and let* $i, j \in \mathbb{N}$. *The following conditions are equivalent:*

(i) $(i, j) \in \gamma$.

(ii) $a_i \prec_{R(\,)} a_j$.

(iii) $a_i \preccurlyeq_S a_j$.

6.10 Proposition. *Let* **a** *be a tree such that* $a_1 \neq o$. *Then the elements* $o, a_1, a_2, a_3, \ldots$ *are pair-wise different.*

Proof. If $a_i = a_j$, then $a_i \preccurlyeq a_j$ and $a_j \preccurlyeq a_j$ and $a_j \preccurlyeq a_i$ implies $(i, j) \in \gamma$ and $(j, i) \in \gamma$ (6.9). Thus $i = j$. (Notice that assertion follows immediately from 6.3). ▲

6.11 Proposition. *Let a be a tree such that* $a_1 \neq o$. *The following conditions are equivalent for permutation p of N:*

(i) The sequence $(a_{p(1)}, a_{p(2)}, a_{p(3)}, \ldots)$ *is a tree.*

(ii) p satisfies the equivalent conditions of 2.13.

Proof. (i) implies (ii). Put $b_i = a_{p(i)}$. Clearly, $b_1 \neq o$. Further, if $(i, j) \in \beta$, then $b_i \in S + b_j$, and so $(p(i), p(j)) \in \beta$ (use 6.8). Similarly, if $(p(i), p(j)) \in \beta$, then $(i, j) \in \beta$.

(ii) implies (i). We have $a_{p(2i)} + a_{p(2i+1)} = a_{p(i)} = a_{2p(i)} + a_{2p(i)+1} = a_p(i)$. ▲

6.12 Lemma. *Let* $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ *be a tree and* $m \in \mathbb{N}$. *Put* $b_{2^k+1} = a_{2^k m + l}$ *for all* $k \geq 0$ *and* $0 \leq l < 2^k$. *Then the sequence* $(b_1, b_2, b_3, \ldots)$ *is a tree (we have* $b_1 = a_m$*).*

Proof. Easy to check directly. ▲

7. Trees in zp-semigroups – continued

7.1. Let S be a non-trivial zp-semigroup. A finite sequence $(a_1, \ldots, a_m)$, $m \geq 1$ of elements from S will be called a partial tree if m is odd and $a_i = a_{2i} + a_{2i+1}$ for every $i = 1, 2, \ldots, (m-1)/2$.

7.2. Let $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ be partial trees. We say that $\mathbf{b}$ extends $\mathbf{a}$ if $m \geq n$ and $a_1 = b_1, \ldots, a_m = b_m$.

The relation of extension determines a (reflexive) ordering on the set $\mathscr{R}$ of partial trees. Maximal elements of this set are non-extendable partial trees.

If $\mathbf{c} = (c_1, c_2, c_3, \ldots)$ is a tree, then we say that $\mathbf{c}$ extends the partial tree $\mathbf{a}$ if $a_1 = c_1, \ldots, a_m = c_m$.

7.3. If $\mathbf{a} = (a_1, \ldots, a_m)$ is a partial tree, then $R(\mathbf{a}) (= R(S, \mathbf{a}))$ is the subsemigroup of S generated by the elements $a_1, \ldots, a_m$. According to 1.2, we have $|R(\mathbf{a})| \leq 2^m$.

7.4 Lemma. *Let $a = (a_1, \ldots, a_m)$, $m = 2k + 1$, $k \geq 0$, be a partial tree. Then $|R(\mathbf{a})| \leq 2^{k+1}$.*

Proof. $R(\mathbf{a})$ is generated by the set $\{a_i \mid k + 1 \leq i \leq m\}$ and 1.2 applies. ▲

7.5 Lemma. *Let S be a zs-semigroup and $\mathbf{a} = (a_1, \ldots, a_m)$, $m = 2k + 1$, $k \geq 0$, be a partial tree. Then there is a partial tree $\mathbf{b} = (b_1, \ldots, b_n)$ such that $n = m + 2 = 2k + 3$ and $\mathbf{b}$ extends $\mathbf{a}$ (i.e., $a_1 = b_1, \ldots, a_m = b_m$).*

Proof. We have $k + 1 \leq m$ and there are $b_{m+1}, b_{m+2} \in S$ with $a_{k+1} = b_{m+1} + b_{m+2}$. ▲

7.6 Lemma. *If S is a zs-semigroup, then every partial tree extends to a tree.*

Proof. Denote by m the length of a partial tree $\mathbf{a}$. By induction, put ${}_0\mathbf{a} = \mathbf{a}$ and, for $n \geq 0$, let ${}_{n+1}\mathbf{a}$ be a partial tree of length $m + 2n + 2$ such that ${}_{n+1}\mathbf{a}$ extends the partial tree ${}_n\mathbf{a}$ (see 7.5). One sees easily, that there exists just one tree $\mathbf{b} = \cup_n \mathbf{a}$ extending all the partial trees ${}_n\mathbf{a}$, $n \geq 0$. ▲

7.7 Corollary. *(cf. 5.17) If S is a zs-semigroup, then for every $a \in S$ there exists at least one tree $(a_1, a_2, a_3, \ldots)$ such that $a_1 = a$.*

7.8 Remark. *Let S be a zp-semigroup. Then S is a subsemigroup of a zs-semigroup T. Now, if $\mathbf{a}$ is a partial S-tree, then there exists a T-tree $\mathbf{b}$, such that $\mathbf{b}$ extends $\mathbf{a}$. Clearly, $R(\mathbf{a}) \subseteq R(\mathbf{b})$.*

8. A few remarks

8.1. Define an operation $\bigstar$ on the set $\mathscr{F}_o$ of finite subsets of $\mathbb{N}$ by $A \bigstar B = A \cup B$ if $A \neq \emptyset \neq B$, $A \cap B = \emptyset$, and $A \bigstar B = \emptyset$ otherwise.

8.1.1 Proposition. *$\mathscr{F}_o(\bigstar)$ is a free zp-semigroup over the set $N = \{\{i\} | i \in \mathbb{N}\}$ and $Ann(\mathscr{F}_o(\bigstar)) = \{\emptyset\}$.*

Proof. Easy to check. ▲

8.1.2. Denote by ν the congruence of $\mathscr{F}_o(\bigstar)$ generated by the ordered pairs $(\{i\}, \{2i, 2i + 1\})$, $i \in \mathbb{N}$, put $\mathscr{E}_0(\bigstar) = \mathscr{F}_o(\bigstar)/\nu$ and denote by π the natural projection of $\mathscr{F}_o$ onto $\mathscr{E}_o$ (so that $\nu = Ker(\pi)$).

8.1.3 Lemma. *$\mathscr{E}_o$ is a zs-semigroup.*

Proof. The semigroup $\mathscr{E}_o$ is generated by the set $\pi(N)$ and $\pi(N) \subseteq \subseteq \pi(N) \bigstar \pi(N)$. By 1.6, $\mathscr{E}_o$ is a zs-semigroup. ▲

8.1.4 Proposition. *There exists an isomorphism $\varrho : \mathscr{E}_o(\bigstar) \to \mathscr{P}_o(\oplus)$ such that $\varrho(\{i\}/\nu) = \varrho\pi(\{i\}) = \{i\}$ for every $i \in \mathbb{N}$.*

Proof. Since $\mathscr{F}_o(\bigstar)$ is free over N, there is a homomorphism $\alpha : \mathscr{F}_o(\bigstar) \to \mathscr{P}_o(\oplus)$ such that $\alpha \mid N = id_N$. Moreover, since $\mathscr{P}_o(\oplus)$ is generated by N (5.9 (ii)), the homomorphism α is projective and it follows from 5.9 (iii) that $\nu \subseteq ker(\alpha)$. Consequently, α induces a projective homomorphism $\varrho : \mathscr{E}_o(\bigstar) \to \mathscr{P}_o(\oplus)$ such that $\varrho(\{i\} \mid \nu) = \{i\}$ for every $i \in \mathbb{N}$. On the other hand, $\{2i\}/\nu \bigstar \{2i + 1\}/\nu = \{i\}/\nu$ and it follows from 5.16 that there exists a homomorphism $\sigma : \mathscr{P}_o(\oplus) \to \mathscr{E}_o(\bigstar)$ such that $\sigma(\{i\}) = \{i\}/\nu$ for every $i \in \mathbb{N}$. Now, $\sigma\varrho(\{i\}/\nu) = \{i\}/\nu$, i.e., $\sigma\varrho \mid \pi(N) = id_{\pi(N)}$, and hence $\sigma\varrho = id_{\mathscr{E}_o}$, since $\mathscr{E}_o$ is generated by $\pi(N)$. Thus ϱ is injective, ϱ is an isomorphism and $\sigma = \varrho^{-1}$. ▲

8.1.5 Lemma. *$\mathscr{G} = \mathscr{F}_o \setminus \mathscr{Q}$ is an ideal of the semigroup $\mathscr{F}_o(\bigstar)$.*

Proof. Clearly, $\emptyset \in \mathscr{G}$ and if $A \in \mathscr{F} \setminus \mathscr{Q}$ and $B \in \mathscr{F}_o$, thenn $A \cup B \notin \mathscr{Q}$. ▲

8.1.6 Lemma. *$\mathscr{G} = \pi^{-1}(o)$.*

Proof. We have $\pi(\emptyset) = \emptyset/\nu = o$ and, if $A \in \mathscr{F} \setminus \mathscr{Q}$, then $\varrho\pi(A) = \varrho(\sum_{i \in A} \bigstar \bigstar \{i\}/\nu) = \sum_{i \in A} \oplus \{i\} = o$, so that $\pi(A) = o$ and $A \in \pi^{-1}(o)$. Thus $\mathscr{G} \subseteq \subseteq \pi^{-1}(o)$. On the other hand, if $A \in \mathscr{Q}$, then $\varrho\pi(A) = \sum_{i \in A} \oplus \{i\} = \xi(A) \neq o$ (5.9(i)). ▲

8.1.7 Lemma. *If $A, B \in \mathscr{Q}$, then $\pi(A) = \pi(B)$ iff $\xi(A) = \xi(B)$.*

Proof. The assertion follows easily from 5.9(i). ▲

8.1.8 Proposition. $\nu = (\mathscr{G} \times \mathscr{G}) \cup \{(A,B) \mid A, B \in \mathscr{Q}, \xi(A) = \xi(B)\}$.

Proof. Combine 8.1.6 and 8.1.7. ▲

8.2 Remark. As it follows from 8.1.4, the zs-semigroup $\mathscr{P}_o(\oplus)$ is, as a semigroup, given by generators $a_1, a_2, a_3, \ldots$ and relations $a_i + a_j = a_j + a_i$, $2a_i = 3a_j$, $a_i = a_{2i} + a_{2i+1}$, $i, j \in \mathbb{N}$.

8.3. Let M be a set, $\mathcal{M}$ the set of all subsets of M and $\mathcal{N}$ a subset of $\mathcal{M}$. Further, let $\mathcal{S}$ be a subset of $\mathcal{M}$ such that $\emptyset \in \mathcal{S}$ and, if $A, B \in \mathcal{S} \setminus \{\emptyset\}$ are such that $A \cap B \in \mathcal{N}$, then $A \cup B \in \mathcal{S}$. Now, define an operation $\circledast$ *on* $\mathcal{S}$ by $A \circledast B = A \cup B$ if $A, B \in S \setminus \{\emptyset\}$, $A \cap B \in \mathcal{N}$ and $A \circledast B = \emptyset$ otherwise.

8.3.1 Lemma.

(i) $A \circledast B = B \circledast A$ *for all* $A, B \in S$.

(ii) $A \circledast \emptyset = \emptyset = \emptyset \circledast A$ *for all* $A \in \mathcal{S}$.

(iii) $A \circledast A = \emptyset$ *for every* $A \in \mathcal{S} \setminus \mathcal{N}$.

(iv) $A \circledast A = A$ *for every* $A \in \mathcal{S} \cap \mathcal{N}$.

(iv) $A \circledast A = \emptyset$ *for every* $A \in \mathcal{S}$ *iff either* $\mathcal{S} \cap \mathcal{N} = \emptyset$ *or* $\mathcal{S} \cap \mathcal{N} = \{\emptyset\}$.

Proof. Easy. ▲

8.3.2 Lemma. *Let* $A, B, C \in \mathcal{S}$. *Then:*

(i) $A \circledast (B \circledast C) \neq \emptyset$ *iff the sets* A, B, C *are non-empty,* $B \cap C \in \mathcal{N}$ *and* $(A \cap B) \cup (A \cap C) \in \mathcal{N}$ *(then* $A \circledast (B \circledast C) = A \cup B \cup C$*).*

(ii) $(A \circledast B) \circledast C \neq \emptyset$ *iff the sets* A, B, C *are non-empty,* $A \cap B \in \mathcal{N}$ *and* $(A \cap B) \cup (B \cap C) \in \mathcal{N}$ *(then* $(A \circledast B) \circledast C = A \cup B \cup C$*).*

Proof. Easy. ▲

8.3.3 Corollary. If $A, B, C \in \mathcal{S}$ *are such that* $A \circledast (B \circledast C) \neq \emptyset \neq (A \circledast B) \circledast C$, *then* $A \circledast (B \circledast C) = A \cup B \cup C = (A \circledast B) \circledast C$.

8.3.4 Lemma. *If* $\mathcal{N}$ *is an ideal of* $\mathcal{M}$, *then* $\mathcal{S}(\circledast)$ *is a (commutative) semigroup with absorbing element.*

Proof. Combine 8.3.1, 8.3.2 and 8.3.3. ▲

8.3.5 Lemma. *If* $\mathcal{N}$ *is an ideal of* $\mathcal{M}$ *such that* $\mathcal{S} \cap \mathcal{N} \subseteq \{\emptyset\}$ *(then* $\mathcal{S} \cap \mathcal{N} = \{\emptyset\}$*), then* $\mathcal{S}(\circledast)$ *is a zp-semigroup.*

Proof. Combine 8.3.4 and 8.3.1(v). ▲

8.3.6 Proposition. *Assume that* $\mathcal{N}$ *is an ideal of* $\mathcal{M}$ *such that* $\mathcal{S} \cap \mathcal{N} = \{\emptyset\}$ *and for every* $A \in \mathcal{S}$, $A \neq \emptyset$, *there exist* $B, C \in \mathcal{S}$, $B \neq \emptyset \neq C$, *with* $B \cap C \in \mathcal{N}$ *and* $B \cup C = A$. *Then* $\mathcal{S}(\circledast)$ *is a zs-semigroup.*

Proof. By 8.3.4, $\mathcal{S}(\circledast)$ is a zp-semigroup and the rest is clear. ▲

8.3.7 Example. Assume that M is infinite, $\mathcal{N}$ is an ideal of $\mathcal{M}$ and that every set from $\mathcal{N}$ is finite.

(i) Let $\mathcal{S}_1 = \mathcal{I}_c \cup \{\emptyset\}$, $\mathcal{I}_c$ being the set of countable infinite subsets of M. Then $\mathcal{S}_1(\circledast)$ is a non-trivial zs-semigroup. If M is countable, then $Ann(\mathcal{S}_1(\circledast)) = \mathcal{I}_f \cup \{\emptyset\}$, where $\mathcal{I}_f$ is the set of cofinite subsets of M. If M is uncountable, then $Ann(\mathcal{S}_1(\circledast)) = \{\emptyset\}$.

(ii) Let $\mathscr{S}_2 = \mathscr{I} \cup \{\emptyset\}, \mathscr{I}$ being the set of infinite subsets of $\mathscr{M}$. Ten $\mathscr{S}_2(\circledast)$ is a non-trivial zs-semigroup and $Ann(\mathscr{S}_2(\circledast) = \mathscr{I}_f \cup \{\emptyset\}$, where $\mathscr{I}_f$ is the set of cofinite subsets of M.

References

[1] FLAŠKA, V. AND KEPKA, T., *Comutative zeropotent semigroups*, Acta Univ. Carol. Math. Phys. **47/1** (2006), 3 – 14.