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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 37 (1996), No. 1, 17--29

About the Heart of a Hypergroup

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Received 1. October 1995

This paper presents some types of hypergroups, associated to an arbitrary hypergroup, studying the properties and the heart of them, establishes results concerning the width and gives some other results about the sequence of hearts, which can be associated to a hypergroup, in connection with the subhypergroups generated by a non-empty set, by an union of subhypergroups or by the intersection of subhypergroups, if it is not empty.

Introduction

The notion of the heart ω_H of a hypergroup H, introduced by two among the founders of the Hypergroup Theory, Dresher and Ore [9], has been studied by many mathematicians.

Just this subject is involved by most of the results of this paper, which throws light also on the algebraic structure of the set $\mathcal{P}^*(H)$ of non-empty subsets of H, of the set of the hyperproducts of elements of H, and on some topics of join spaces and of subhypergroup theory.

Let $\langle H, \circ \rangle$ be a hypergroup, $P = \langle \mathcal{P}^*(H); \otimes \rangle$ be the set of non-empty subsets of H endowed with the hyperoperation $\langle \otimes \rangle$ defined: $\forall (A, B) \in \mathcal{P}^*(H)^2$, $A \otimes B = \{C \in \mathcal{P}^*(H) \mid C \subset A \circ B\}$. $\forall a \in H$, let $I_{pl}(a) = \{e \in H \mid a \in e \circ a\}$, $I_{pr}(a) = \{f \in H \mid a \in a \circ f\}$, $I_p(a) = I_{pr}(a) \cup I_{pl}(a)$. Let $\Delta = \{D \subset I_p(H) \mid \forall h \in H, |D \cup I_{pl}(h)| = 1 = |D \cap I_{pr}(h)|\}$, $I_p = \bigcup_{a \in H} I_p(a)$. Moreover, $\forall e \in I_p$, $\forall q \in H$, let

$$u_r(q, e) = \{y \in H \mid e \in q \circ y\}$$

$$u_l(q, e) = \{z \in H \mid e \in z \circ q\}$$

1. Theorem.

1. *If $Q \in \mathcal{P}^*(H)$, Q is an identity in P iff there exists $D \in \Delta$ such that $Q \supset D$.*

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2. *If $Q, Q' \in \mathscr{P}^*(H)$, then Q' is an inverse of Q iff there is $D \in \Delta$ such that*

$$\forall e \in D, \ \bigcup_{q \in Q} u_l(q, e) \cap Q' \neq \emptyset ,$$

$$\forall e \in D, \ \bigcup_{q \in Q} u_r(q, e) \cap Q' \neq \emptyset ,$$

3. *If ω_P is the heart of P, we have $\omega_P = P$.*

Proof.

1. Let Q be an identity. Then $\forall a \in H$ one has $\{a\} \in Q \otimes \{a\}, \{a\} \in \{a\} \otimes Q$, whence $a \in Q \circ a$, $a \in a \circ Q$ from which there exists $(e_1, e_2) \in I_{pl}(a) \times I_{pr}(a)$ such that $\{e_1, e_2\} \subset Q$ hence $\forall a \in H$, $I_{pl}(a) \cap Q \neq \emptyset \neq Q \cap I_{pr}(a)$. On the converse if this condition is satisfied by a subset Q of H, we have: $\forall S \in \mathscr{P}^*(H)$, $\forall s \in S$, $\exists e_s \in I_{pr}(s) \cap Q$, whence $S \in S \otimes Q \circ Q \supset \bigcup_{s \in S} s \circ e_s \supset S$ from which Q is a right identity. Similarly on the left.

2. It's enough to remark that Q' is an inverse of Q iff there is $D \in \Delta$ such that $Q' \circ Q \supset D \subset Q \circ Q'$ and this condition is satisfied iff for all $h \in H$, we have $Q' \cap (Q \setminus I_{pl}(h)) \neq \emptyset \neq Q' \cap (Q \setminus I_{pr}(h))$ and $Q' \cap (I_{pl}(h)/Q) \neq \emptyset \neq Q' \cap (I_{pr}/Q)$.

3. It's enough to remark that H is an identity for P and it is inverse for any element of H. It follows, by Th. 129 [5], that $\omega_P = H \otimes H = P$.

2. Definition. Let A be a non-empty subset of a hypergroup H. Let's set

$$T(A) = \{(x_1, x_2, \ldots, x_n) \in H^n \mid \prod_{i=1}^{n} x_i = A\}$$

$$\lambda_H(A) = \min \{n \in \mathbb{N}^* \mid T(A) \neq \emptyset\}.$$

Clearly, $\lambda_H(\omega_H) = w(H)$ where $w(H)$ is the width of H (see [7]).

3. Definition. If $\langle H; \circ \rangle$ is a semi-hypergroup, let's denote $\prod_{\mathscr{C}}(H)$ the set of the hyperproducts P of elements of H, such that $\mathscr{C}(P) = P$.

4. Theorem. *Let $\langle H; \circ \rangle$ be a hypergroup, let $(x_1, x_2, \ldots, x_n) \in H^n$ be such that $\prod_{i=1}^{n} x_i \in \prod_{\mathscr{C}}(H)$, then $(x'_1, x'_2, \ldots, x'_n) \in H^n$ exists such that $x_1 \circ x_2 \circ \ldots \circ x_n \circ x'_n \circ \ldots \ldots \circ x'_1 = \omega_H$.*

Proof. $\forall k \in I_n = \{1, 2, \ldots, n\}$, let u_k be an element of ω_H, and let $x'_k \in H$ be such that $u_k \in x_k \circ x'_k$, then, since ω_H is a complete part, we have $\omega_H \supset x_k \circ x'_k$. It follows $x_1 \circ x_2 \circ \ldots \circ x_n \circ x'_n = \omega_H \circ x_1 \circ \ldots \circ x_n \circ x'_n = x_1 \circ \ldots \circ x_{n-1} \circ \omega_H \circ x_n \circ x'_n = x_1 \circ \ldots \circ x_{n-1} \circ \omega_H = \omega_H \circ \prod_{i=1}^{n-1} x_i$.

Hence $x_1 \circ x_2 \circ \ldots \circ x_n \circ x'_n \circ x'_{n-1} = \omega_H \circ \prod_{i=1}^{n-2} x_i \circ x_{n-1} \circ x'_{n-1} = \omega_H \circ \prod_{i=1}^{n-2} x_i$.
Going on in the same way, one arrives to $x_1 \circ x_2 \circ \ldots \circ x_n \circ x'_n \circ \ldots \circ x'_2 = \omega_H \circ x_1$ whence finally $x_1 \circ x_2 \circ \ldots \circ x_n \circ x'_n \circ \ldots \circ x'_2 \circ x'_1 = \omega_H \circ x_1 \circ x'_1 = \omega_H$.

5. Corollary. *If $\langle H; \circ \rangle$ is a hypergroup, then $w(H) < \aleph_0$ iff $n \in \mathbb{N}^*$ and $(x_1, \ldots, x_n) \in H^n$ exist such that $\prod_{i=1}^{n} x_1 \in \prod_{\mathscr{C}}(H)$.*

6. Lemma. *Let $\langle H; \circ \rangle$ be a hypergroup, then $H - \omega_H$ is a complete part.*

7. Proposition. *Let $\langle H; \circ \rangle$ be a hypergroup. If $H - \omega_H$ is a hyperproduct, then ω_H also is a hyperproduct.*

If follows straight from Th. 4 and from the former lemma.

8. Remark. Let H be a hypergroup endowed with a complete hyperproduct. The following implication is satisfied for $\forall A \in \mathscr{P}^*(H): A \cap \Pi x_i = \emptyset \Rightarrow \mathscr{C}(A) \cap \prod_{i=1}^{n} x_i = \emptyset$.

Let's suppose $z \in \mathscr{C}(A) \cap \prod_{i=1}^{n} x_i$, then $a \in A$ exists such that $z \in \mathscr{C}(a)$, hence $\mathscr{C}(a) = \mathscr{C}(z)$. The hypothesis $\prod_{i=1}^{n} x_i = \mathscr{C}(\prod_{i=1}^{n} x_i)$ implies

$$\mathscr{C}(z) \subset \bigcup_{y \in \prod_{i=1}^{n} x_i} \mathscr{C}(y) = \mathscr{C}(\prod_{i=1}^{n} x_i) = \prod_{i=1}^{n} x_i .$$

Therefore $a \in A$, $a \in \mathscr{C}(z) \subset \prod_{i=1}^{n} x_i$, whence $\prod_{i=1}^{n} x_i \cap A \neq \emptyset$ which is absurd.

9. Theorem. *Let $\langle H; \circ \rangle$ be a hypergroup, such that $w(H) < \aleph_0$. Let's denote*

$$\lambda_m(H) = \min \{k \mid \exists Q \in \Pi_{\mathscr{C}}(H): k = \lambda(Q)\}$$
$$\lambda_M(H) = \max \{h \mid \exists Q \in \Pi_{\mathscr{C}}(H): h = \lambda(Q)\}$$

Then we have:

$$\text{I.} \quad \lambda_m(H) \in \{w(H), w(H) - 1\}$$
$$\text{II.} \quad \lambda_M(H) \in \{w(H), w(H) + 1\}$$

Proof.

I. By the Corollary 5., $\Pi_{\mathscr{C}}(H) \neq \emptyset$. Let $Q = \prod_{i=1}^{n} y_i \in \prod_{\mathscr{C}}(H)$. By Th. 63 [5], we have $Q = \mathscr{C}(Q) = \bigcup_{x \in Q} \mathscr{C}(x)$, we have clearly: $\forall x \in Q$, $\mathscr{C}(x) \cap Q \neq \emptyset$ it follows $\mathscr{C}(x) \supset Q = \mathscr{C}(Q) \supset \mathscr{C}(x)$ whence $\mathscr{C}(x) = Q$. Therefore by Th. 67 [5], $Q = \omega_H \circ x$, from which if $y \in H$ is such that $x \circ y \subset \omega_H$ (it exists since ω_H is conjugable, Th. 75, 110 [5]), it follows $Q \circ y = \omega_H$ whence $w(H) \leq \lambda(Q) + 1$.

If we suppose $\lambda_m(H) \neq w(H)$, we have clearly $\lambda_m(H) < w(H)$, then if Q is such that $\lambda(Q) = \lambda_m(H)$, one obtains $\lambda(Q) < w(H) \leq \lambda(Q) + 1$. Therefore $w(H) = \lambda(Q) + 1$, whence $\lambda_m(H) = w(H) - 1$.

II. It's immediate.

10. Remark. Hypergroups H exist such that

a) $\lambda_m(H) = w(H) - 1$ and others such that b) $\lambda_M(H) = w(H) + 1$.

a) See for instance the Example H1, 220 [5]. We have $\omega(H_1) = 3$ since $\omega_{H_1} = \{0, a_1, a_2, a_3\}$ and $\omega_{H_1} = (x_1 \circ x_2) \circ x_3$. But we have also $\lambda(\{x_3, x_5\}) = \lambda(x_1 \circ x_2) = 2$, whence $\lambda_m(H_1) = 2$.

b) See the Examples I, II, 267 [5], in both of them $\lambda_M(H) = w(H) + 1$.

11. Remark. If $\langle H; \circ \rangle$ is n-complete, we have by 112 [5], $\lambda_m(H) = w(H) \leq n$.

12. Remark. Let $\langle H; \circ \rangle$ be a strongly canonical hypergroup. If it is finite, then $\lambda_m(H) = 2 = w(H)$ (see Th. 211 [5]). If it is not finite it can happen $w(H) \nless \aleph_0$, see for instance the following example: let $\langle K; \leq \rangle$ be an infinite, totally ordered set, endowed with a minimum element 0. Let $\langle \circ \rangle$ be the hyperoperation defined in K (see [13]) $0 \circ 0 = 0$, $\forall x : x \neq 0$, $x \circ x = \{y \mid y < x\}$, $\forall (x, y) \in K;\ x \neq y$, $x \circ y = \max\{x, y\}$. Clearly, $\langle K; \circ \rangle$ is strongly canonical, $\omega_K = K$ and $w(K) \nless \aleph_0$.

Let $\langle H; \circ \rangle$ be a semi-hypergroup, let $\Pi(H)$ the set of hyperproducts of elements of H. In $\Pi(H)$ let's define the hyperoperation $\langle \odot \rangle$, $A \odot B = \{C \in \Pi(H) \mid C \subset A \circ B\}$.

13. Theorem. *If $\langle H; \circ \rangle$ is a hypergroup, then $\langle \Pi(H); \odot \rangle$ is a hypergroup.*

Proof. It's clear that $\langle \odot \rangle$ is associative. Let's prove now the reproducibility. Let $A = \prod_{i=1}^{p} a_i$, $B = \prod_{i=1}^{p} b_j$ elements of $\Pi(H)$. By the reproducibility of $\langle \circ \rangle$, there exists $y_1 \in H$ such that $a_p \in y_1 \circ b_q$. Similarly, there is y_2 such that $y_1 \in y_2 \circ b_{q-1}$, whence $a_p \in y_2 \circ b_{q-1} \circ b_q$. Going up in the same way, one obtains y_q such that $y_{q-1} \in y_q \circ b_1$. Hence $a_p \in y_q \circ b_1 \circ b_2 \circ \ldots \circ b_q$. Therefore if we let $X = \prod_{i=1}^{p-1} a_i \circ y_q$, we have $A \in X \odot B$.

Similarly, we can find $z_1, z_2, \ldots, z_q$ such that $a_1 \in b_1 \circ z_1$, $z_1 \in b_2 \circ z_2 \ldots z_{q-1} \in b_q \circ z_q$ whence $A = a_1 \circ a_2 \circ \ldots \circ a_p \subset b_1 \circ b_2 \circ \ldots \circ b_q \circ z_q \circ a_2 \circ a_2 \circ \ldots \circ a_p$.

14. Theorem. *If K is a subhypergroup of a hypergroup $\langle H; \circ \rangle$ and K belongs to $\Pi(H)$, then K is contained in ω_H.*

Proof. If A is an element of $\Pi(H)$ and $A \cap \omega_H \neq \emptyset$, then $A \subset \omega_H$ since ω_H is a complete part. Then it's enough to remark that $K \cap \omega_H$ contains the set $I_p(K)$ of partial identities of K, to obtain the Theorem.

15. Remark. Not all subhypergroups of a hypergroup H are in $\Pi(H)$. For instance:

	0	1	2
0	0	1	2
1	1	0	2
2	2	2	H

Let $\langle H; \circ \rangle$ be the hypergroup.
It's clear that $K = \{0, 1\}$ is a subhypergroup, and that $K \notin \Pi(H)$.
Moreover, $\omega_H = H \in \Pi(H)$.

Now a natural question arises: *do non-conjugable subhypergroups exist, which can be written as hyperproducts?*

For instance, *does a hypergroup H exists, which is endowed with an ultraclosed subhypergroup A such that $\lambda_H(A) = n$?*

The following example proves it exists.

Let $\langle A; \circ \rangle$ be a hypergroup such that $\omega_A = A$ and $w(A) = n$.

It could be this one (see [5], §2):

$$A = \bigcup_{i=1}^{n} A_i \text{ where } i \neq j \Rightarrow A_i \cap A_j \neq \emptyset,\ \forall (x, y) \in A_i \times A_j,\ x \circ y = A_i \cup A_j .$$

Now, let's set $H = A \cup T$ where $A \cap T = \emptyset$, $|T| \geq 3$ and the hyperoperation $\otimes$ in H is defined (see 112, [5]). $\forall (a, b) \in A^2$, $a \otimes b = a \circ b$, $\forall (a, t) \in A \times T$, $a \otimes t = t \otimes a = t$, $\forall (t, s) \in T^2$, $s \otimes t = A \cup (T - \{s, t\})$. We have clearly that $\langle A; \circ \rangle$ is an ultraclosed (non conjugable) subhypergroup of $\langle H; \otimes \rangle$ and $\lambda_H(A) = n$.

We have clearly $\omega_H = H$ and $w(H) = 2$.

Indeed since T contains $\{s_1, s_2\}$, $s_1 \neq s_2$; then

$$(s_1 \circ s_1) \circ s_2 = (A \cup (T - \{s_1\})) \circ s_2 \supset \{s_2\} \cup (T - \{s_2\}) \cup A = H .$$

Let $\langle H; \circ \rangle$ be a hypergroup. Let's consider the sequence

$$(*)\ H \supset \omega(H) = \omega_1 \supset \omega(\omega(H)) = \omega_2 \supset \ldots \supset \omega_k \supset \omega_{k+1} \supset \ldots \supset \omega_n \supset \ldots$$

16. Theorem. *The following conditions are equivalent:*

1. *the sequence $(*)$ is finite;*
2. *there is $(n, k) \in \mathbb{N}^2$, where $n > k + 1$, such that ω_n is a complete part of ω_k;*
3. *there is $(n, k) \in \mathbb{N}^2$ where $n > k + 1$, such that for any $(x, y) \in (\omega_k - \omega_n) \times (\omega_k - \omega_n)$; $x \circ y \cap (\omega_k - \omega_n) \neq \emptyset$ implies $x \circ y \subset \omega_k - \omega_n$;*
4. *there is $(n, k) \in \mathbb{N}^2$ where $n > k + 1$, such that for any ω_n is ω_k-conjugable.*

Proof. 1. $\Rightarrow$ 2. If the sequence $(*)$ is finite, then there is $n \in \mathbb{N}$ such that $\omega_n = \omega_{n-1}$, hence ω_{n-2} is a complete part of ω_n.

2. $\Rightarrow$ 3. If ω_n is a complete part of ω_k, then $\omega_k - \omega_n$ is a complete part of ω_k.

3. $\Rightarrow$ 4. One proves easily that for any $s \in \mathbb{N}^*$, ω_s is a closed subhypergroup of H. Moreover, for all a, b in ω_k, if $\{a, b\} \subset \omega_k - \omega_n$, we have $a \circ b \subset \omega_n$, if $a \neq b$ and $|\{a, b\} \cap \omega_n| = 1$, we have $a \circ b \subset \omega_k - \omega_n$. Then, by Th. 104, 3″) [5], we obtain that ω_n is ω_k-conjugable.

4. $\Rightarrow$ 1. By the Th., ω_n is a complete part subhypergroup of ω_k. Hence $\omega_{k+1} = \omega(\omega_k) \subset \omega_n \subset \omega_{k+1}$ from which $\omega_n = \omega_{k+1}$. So, we have: $\omega_{n+1} = \omega(\omega_n) = \omega(\omega_{k+1}) = \omega_{k+2} \supset \omega_n = \omega_{k+1} \supset \omega_{k+2}$. Therefore, $\omega_n = \omega_{k+2} = \omega_{n+1}$. Let $\omega_{n+s} = \omega_{k+1}$. It follows $\omega_{n+s+1} = \omega(\omega_{n+s}) = \omega(\omega_{k+1}) = \omega_{k+2} = \omega_{k+1}$. Then, for any m such that $m \geq n$, we have $\omega_m = \omega_n$.

17. Theorem. *Let $\langle H; \circ \rangle$ be a hypergroup such that the sequence $(*)$ is finite, and let K be a complete part subhypergroup of H. Then there is $p \in \mathbb{N}$ such that $\omega_{p+1}(K) = \omega_{p+1}(H)$.*

Proof. Let's remark that $\omega(K)$ is a subhypergroup of $\omega(H)$. Indeed, for any $a \in \omega(K)$, there is $e \in K$ such that $a \in a \circ e$; it's clear that $a \in \beta_k(e) \subset \beta_H(e) = \omega(H)$. Moreover, since K is a complete part subhypergroup of H, we have $\omega(H) \subset K$. Then $\omega_1(K) \subset \omega_1(H) \subset K$. For any $s \geq 1$, from $\omega_s(K) \subset \omega_s(H) \subset \omega_{s-1}(K)$, one obtains $\omega_{s+1}(K) \subset \omega_{s+1}(H) \subset \omega_s(K)$, hence a sequence $K \supset \omega_1(H) \supset \omega_1(K) \supset \omega_2(H) \supset \omega_2(K) \supset \ldots$.

By Th. 16, there is $(n, p) \in \mathbb{N} \times \mathbb{N}$, where $n > p + 1$, such that $\omega_n(H) = \omega_{p+1}(H)$, therefore $\omega_{p+1}(H) = \omega_{p+1}(K)$.

18. Remark. If $K_1, K_2 \leq H$, then

$$\omega(K_1 \cap K_2) \leq \omega(K_1) \cap \omega(K_2).$$

Generally, we have not equality.

19. Examples. I. Let h be a hypergroup, for which $\omega(h) \neq h$ and let be x, y arbitrary in H. Let's define on $H = h \cup \{b, c, d\}$ $(\{b, c, d\} \cap h = \emptyset)$ the following hyperoperations:

1.

$\otimes$	x	b	c	d
y	$y \circ x$	b	c	d
b	b	h	d	c
c	c	d	h	b
d	d	c	b	h

We can easily verify the associativity and the reproducibility, so $(H, \otimes)$ is a hypergroup. We consider $K_1 = h \cup \{b\}$, $K_2 = h \cup \{c\}$, $K_3 = h \cup \{d\}$, $\omega(K_1) = \omega(K_2) = \omega(K_3) = h$, $\omega(K_1 \cap K_2 \cap K_3) \neq h$

2.

$\square$	x	b	c	d
y	$y \circ x$	b	$h \cup \{c\}$	$\{b,d\}$
b	b	h	$\{b,d\}$	$h \cup \{c\}$
c	$h \cup \{c\}$	$\{b,d\}$	$h \cup \{c\}$	$\{b,d\}$
d	$\{b,d\}$	$h \cup \{c\}$	$\{b,d\}$	$h \cup \{c\}$

$(H, \square)$ is a hypergroup. We consider $K_1 = h \cup \{b\}$, $K_2 = h \cup \{c)$, $\omega(K_1) = h$; $\omega(K_2) = h \cup \{c\}$, $\omega(K_1 \cap K_2) = \omega(h) \neq h = \omega(K_1) \cap \omega(K_2)$

II. Let h and k be two hypergroups with $\omega_h \neq h$ and let be x, y arbitrary in h and t, f arbitrary in k. Let's define on $H = h \cup k \cup \{a, c\}$ $(\{a, c\} \cap h \cup k = \emptyset)$ the following hyperoperation

$\odot$	x	a	t	c
y	$y \circ x$	a	$h \cup k$	$\{a, c\}$
a	a	h	$\{a, c\}$	$h \cup k$
f	$h \cup k$	c	$f \circ t$	$\{a, c\}$
c	$\{a, c\}$	$h \cup k$	$\{a, c\}$	$h \cup k$

$(H, \odot)$ is a hypergroup. We consider $K_1 = h \cup \{a\}$, $K_2 = h \cup k$, $\omega(K_1) = h$; $\omega(K_2) = h \cup k$, $\omega(K_1 \cap K_2) = \omega(h) \neq h = \omega(K_1) \cap \omega(K_2)$

But, for H a hypergroup, whose sequence $(*)$ is finite, between $\omega(K_1 \cap K_2)$ and $\omega(K_1)$, $\omega(K_2)$ we can find the following.

20. Proposition. *If* $K_1, K_2 \leq H$, *where* H *has a finite sequence* $(*)$, *then* $\exists p \in \mathbb{N}^*$, $\omega_{p+1}(K_1 \cap K_2) = \omega_{p+1}(\omega(K_1) \cap \omega(K_2))$.

Proof. Let's consider $\bar{H} = K_1 \cap K_2$ and $\bar{K} = \omega(K_1) \cap \omega(K_2)$. $\bar{K}$ is a subhypergroup, complete part of $\bar{H}$. (We can verify this using the definition of a complete part of a hypergroup.) Then we use the proof of Th. 17.

Also, we can give a relation for n-subhypergroups of H: $\exists p \in \mathbb{N}^*$, $\omega_{p+1}(K \cap K_2 \cap \ldots \cap K_n) = \omega_{p+1}(\omega(K_1) \cap \omega(K_2) \cap \ldots \cap \omega(K_n))$.

21. Remark. If $K_1, K_2 \leq H$, then $\omega(K_1) \subset K_1 \cap \omega(\langle K_1 \cup K_2 \rangle)$. Generally, we haven't equality.

22. Example. Let h and k be two hypergroups and let be x_1, x_2 arbitrary in h and y_1, y_2 arbitrary in k. Let's define on $H = h \cup k \cup \{a\}$ $(a \notin h \cup k)$ the following hyperoperation

$\square$	x_1	a	y_1
x_2	$x_2 \circ x_1$	a	H
a	a	h	H
y_2	H	H	$y_2 y_1$

$(H, \square)$ is a hypergroup. Let's consider $K_1 = h \cup \{a\}$, $K_2 = k$, $K_1 \cup K_2 = H$, $\langle K_1 \cup K_2 \rangle = H \Rightarrow \omega(\langle K_1 \cup K_2 \rangle) = H$. So

$$\omega(K_1) = h \subsetneq K_1 \cap \omega(\langle K_1 \cup K_2 \rangle) = K_1 = h \cup \{a\}.$$

But, also in this case, for H, whose sequence $(*)$ is finite, we can find: $\exists p \in \mathbb{N}^*$, $\omega_{p+1}(\omega(K_1) = \omega_{p+1}(K_1 \cap \omega(\langle K_1 \cup K_2 \rangle))$.

Indeed, we have $\omega(K_1) \subset K_1 \cap \omega(\langle K_1 \cup K_2 \rangle) \subset K_1$ so $\omega(K_1)$ is a subhypergroup, complete part of $K_1 \cap \omega(\langle K_1 \cup K_2 \rangle)$, whence using the Th. 17, we obtain this equality.

23. Remark. If $K_1, K_2 \leq H$, then $\langle \omega(K_1) \cup \omega(K_2) \rangle \cup \omega(\langle K_1 \cup K_2 \rangle)$. Generally, we haven't equality.

24. Example. Let h be a hypergroup, which has an identity, i; and let be y, y' arbitrary in $h \setminus \{i\}$. Let's define on $H = h \cup \{a, c\}$ $(\{a, c\} \cap h = \emptyset)$ the following hyperoperation

$\odot$	i	a	y	c
i	i	a	y	H
a	a	i	y	H
y'	y'	y'	$y' \circ y$	H
c	H	H	H	H

$(H, \odot)$ is a hypergroup. Let's consider: $K_1 = \{i, a\}$. (In fact, K_1 is group.) $K_2 = h$

$$K_1 \cup K_2 = \{a\} \cup h \Rightarrow \langle K_1 \cup K_2 \rangle = H \Rightarrow \omega(\langle K_1 \cup K_2 \rangle) = H$$

$$\omega(K_1) = \{i\}, \omega(K_2) = \eta(h) \Rightarrow \langle \omega(K_1) \cup \omega(K_2) \rangle = \langle \{i\} \cup \omega(h) \rangle = \omega(h) \subset h \neq H$$

In general, $\langle \omega(K_1) \cup \omega(K_2) \rangle$ is not a complete part of $\omega(K_1) \cup \omega(K_2)$. In the case of the example given, $c^2 \cap (h) \neq \emptyset$, but $c^2 \not\subset \omega(h)$.

25. Remark. If A is a subset of a hypergroup H, then

$$\langle \omega(\langle A \rangle) \cap A \rangle \subset \omega(\langle A \rangle).$$

Indeed, $\omega(\langle A \rangle) \cap A \subset \omega(\langle A \rangle) \cap \langle A \rangle = \omega(\langle A \rangle)$ so that $\langle \omega(\langle A \rangle) \cap A \rangle \subset \omega(\langle A \rangle)$. Generally, we haven't equality.

26. Example. Let's define on $H = \{e, x, y, z\}$ the hyperoperation

$\circ$	e	x	y	z
e	e	x	$\{e, x, y\}$	z
x	x	e	$\{e, x, y\}$	z
y	$\{e, x, y\}$	$\{e, x, y\}$	$\{e, x, y\}$	z
z	z	z	z	$\{e, x, y\}$

Let's consider $A = \{e, x, z\}$.
$\langle A \rangle = H \Rightarrow \omega(\langle A \rangle) = \{e, x, z\}$.
So, $\langle \omega(\langle A \rangle) \cap A \rangle = \{e, x\} \subsetneq \omega(\langle A \rangle)$.

We notice $\langle \omega(\langle A \rangle) \cap A \rangle$ isn't a complete part of $\omega(\langle A \rangle)$. For the preceding example, $y^2 \cap \langle \omega(\langle A \rangle) \cap A \rangle \neq \emptyset$, but $y^2 \not\subset \langle \omega(\langle A \rangle) \cap A \rangle$.

27. Proposition. *Let H be a commutative hypergroup and K_1, K_2 be subhypergroups of H. If for any $a \in \langle K_1 \cup K_2 \rangle - (K_1 \cup K_2)$, there exists $(k_1, k_2) \in K_1 \times K_2$, such that $a \in k_1 k_2$ and if $\langle \omega(K_1) \cup \omega(K_2) \rangle$ is a closed subhypergroup of $\omega(\langle K_1 \cup K_2 \rangle)$ then*

$$\langle \omega(K_1) \cup \omega(K_2) \rangle = \omega(\langle K_1 \cup K_2 \rangle).$$

Proof. We shall prove that $\langle \omega(K_1) \cup \omega(K_2) \rangle$ is conjugable in $\langle K_1 \cup K_2 \rangle$. $\langle \omega(K_1) \cup \omega(K_2) \rangle$ is closed in $\langle K_1 \cup K_2 \rangle$ because, from $a \in bx$, where $(a, b) \in \langle \omega(K_1) \cup \omega(K_2) \rangle^2$ and $x \in \langle K_1 \cup K_2 \rangle$, it results $(a, b) \in (\omega^2 \langle K_1 \cup K_2 \rangle)$ and so $x \in \omega(\langle K_1 \cup K_2 \rangle)$. Using now the condition given in the proposition, $x \in \langle \omega(K_1) \cup \omega(K_2) \rangle$.

As regards an arbitrary element $a \in \langle K_1 \cup K_2 \rangle$, we have three situations:

$$a \in K_1 \Rightarrow \exists a' \in K_1,\ aa' \subset \omega_{K_1} \subset \langle \omega(K_1) \cup \omega(K_2) \rangle;$$

$$a \in K_2 \Rightarrow \exists a' \in K_2,\ aa' \subset \omega_{K_2} \subset \langle \omega(K_1) \cup \omega(K_2) \rangle;$$

$$a \in \langle K_1 \cup K_2 \rangle - (K_1 \cup K_2) \Rightarrow \exists k_1 \in K_1,\ \exists k_2 \in K_2,\ a \in k_1 k_2.$$

For k_i there exists $k_i' \in K_i$, such that $k_i k_i' \in \omega_{K_i}$, $i = 1, 2$.

So, $ak_1'k_2' \subset (k_1'k_2')(k_2k_2') \subset \omega(K_1) \circ \omega(K_2) \subset \langle \omega(K_1) \cup \omega(K_2) \rangle$, whence for every $t \in k_1'k_2'$, $at \subset \langle \omega(K_1) \cup \omega(K_2) \rangle$.

28. Remark. If H is a hypergroup, such that ω_H can be written as a hyperproduct and if h is a subhypergroup of H, then, generally, $\omega(h)$ can't be written as a hyperproduct.

We can consider h a hypergroup, for which $\omega_h \neq h$ and ω_h can't be written as a hyperproduct.

Let's define on $H = h \cup \{a\} (a \notin h)$ the following hyperoperation:

$$\begin{cases} x \circ y = xy \\ a \circ a = h \\ a \circ x = x \circ a = a, \ \forall x \in h \end{cases}$$

$\langle H; \circ \rangle$ is a hypergroup, for which $\omega_H = h = a \circ a$, but $\omega(h) = \omega(\omega(H))$ is not a hyperproduct.

29. Theorem. *Let $H_1, H_2, \ldots, H_m$ be hypergroups, such that for any $i = 1, 2, \ldots$ $\ldots, m$, ω_{H_i} can be written as a hyperproduct, with $w(H_i) = n_i$. Let $H = \mathop{\times}\limits_{i=1}^{m} H_i$. Then ω_H is a hyperproduct and $w(H) = \max \{n_i \mid i = \overline{1, m}\}$.*

Proof. Let $x_{i_1}, \ldots, x_{i_{n_i}} \in H_i$, such that $\omega(H_i) = \prod_{j=1}^{n_1} x_{ij}$ and let $q = \max \{n_i \mid i = \overline{1, m}\}$.

If $q > n_i$, then for any $k = n_i + 1, \ldots, k = q$ we define x_{i_k} in this manner: $x_{i_{n_i}+1} = e$, where e is a partial identity on the right of $x_{i_{n_i}}$; for $k \geq n_i + 2$, x_{i_k} is a partial identity on the right of $x_{i_{k-1}}$.

We obtain $\omega(H_i) = \prod_{j=1}^{n_i} x_{ij} \subset \prod_{j=1}^{q} x_{ij} = \left(\prod_{j=1}^{n_i+1} x_{ij} \right) . \prod_{n_i}^{q} x_{ij} \subset \omega(H_i)$ whence $\omega(H) = \prod_{j=1}^{q} x_{ij}$ and $\omega(H) = \prod_{k=1}^{q} (x_{ik}, \ldots, x_{mk})$.

For $p < q$, $P \neq \omega(H)$, for any hyperproduct P of p elements of H. So, $w(H) = q$.

Let H be a hypergroup and let's denote by $A \| B = \{a/b \mid a \in A, b \in B\}$, where $\{A, B\} \subset \mathscr{P}^*(H)$.

Let's define on $H \| H$ the hyperoperation: $(a/b) \square (c/d) = (ac) \| (bd)$.

Generally, $\square$ is not well defined.

30. Example.

1. Let's consider the following join space: $\langle \mathbf{Z}, \circ \rangle$, where $x \circ y = \{x + y, x + y + 1, \ldots, x + y + n\}$. Then $x/y = \{x - y, x - y - 1, \ldots, x - y - n\}$ and $x/y \square z/w = \{(x + z)/(y + w), \ldots, (x + z)/(y + w + n), (x + z + 1)/(y + w), \ldots, (x + z + 1)/(y + w + n), \ldots, (x + z + n)/(y + w), \ldots, (x + z + n)/(y + w + n)\} =$

$\{\{x-y+z-w, \dots, x-y+z-w-n\}, \dots, \{x-y+z-w-n, \dots, x-y+z-w+2n\},$ $\{x-y+z-w+1, \dots, x-y+z-w-n+1\}, \dots, \{x-y+z-w+1-n, \dots,$ $x-y+z-w+1-2n\}, \dots, \{x-y+z-w+n, \dots, x-y+z-w\}, \dots,$ $\{x-y+z-w, \dots, x-y+z-w-n\}\}.$

Let's remark that $x/y = x'/y'$ if and only if $x - y = x' - y'$.

Therefore, "□" is well defined.

2. Let $\langle H, \circ \rangle$ be the hypergroup:

$\circ$	x	y	z
x	x	H	H
y	H	y	z
z	H	z	y

H is not a join space. In fact, $y/z \cap z/z \in x$, but $y \circ z \cap z \circ z = \emptyset$.

We shall prove that "□" is well defined.

One has $x/x = H$; $y/x = \{y,z\} = \{x,y\} = z/x$; $x/y = \{x\} = x/z$; $y/z = \{x,z\} = z/y$ and $y/y = \{x,y\} = z/z$.

Whence, for any $\{a, b, c, d\} \subset \{y, z\}$, we have:

$$\begin{aligned}
x/x \,\square\, b/x &= a/x \,\square\, x/b = x/x \,\square\, a/b = H \| H ;\\
x/x \,\square\, x/a &= \{x\} \| H = \{H; \{x\}\};\\
x/x \,\square\, a/x &= H \| \{x\} = \{H; \{y,z\}\};\\
x/x \,\square\, x/x &= x/x = H ;\\
x/a \,\square\, x/b &= \{x\} = x/y = x/z ;\\
a/x \,\square\, b/x &= \{y,z\} = y/x = z/x ;\\
x/a \,\square\, b/c &= H \| \{y\} = H \| \{z\} = \{\{x\}, \{x,y\}, \{x,z\}\};\\
a/x \,\square\, b/c &= \{y\} \| H = \{z\} \| H = \{\{y,z\}, \{x,y\}, \{x,z\}\};\\
a/b \,\square\, c/d &\in \{y/y = z/z, z/y = y/z\} = \{\{x,y\}, \{x,z\}\};
\end{aligned}$$

So, "□" is well defined. If we denote by $\alpha = x/x$; $\beta = y/x$; $\gamma = x/y$; $\mu = y/z$ and $\Psi = y/y$, then $\langle H \| H; \square \rangle$ is the following hypergroup:

□	α	β	γ	μ	Ψ
α	α	α, β	α, γ	$H \| H$	$H \| H$
β	α, β	β	$H \| H$	β, μ, Ψ	β, μ, Ψ
γ	α, γ	$H \| H$	γ	γ, μ, Ψ	γ, μ, Ψ
μ	$H \| H$	β, μ, Ψ	γ, μ, Ψ	Ψ	μ
Ψ	$H \| H$	β, μ, Ψ	γ, μ, Ψ	μ	Ψ

Let's remark that $\langle H \| H, \square \rangle$ is not a join space. (We have $\mu/\mu \cap \mu/\Psi \quad \alpha$, $\mu \circ \Psi \cap \mu \circ \mu = \emptyset$.)

3. Let's consider the hypergroup $\langle H, \circ \rangle$, where $x \circ y = \{x, y\}$, for any $(x, y) \in H^2$. Then, "□" is not well defined. Indeed, for x, y, z, w four different

elements of H, we have $z/z = H$, $x/y = x/z = \{x\}$,so $x/y \,\square\, z/w = x/z \,\square\, z/w$. But, on the other hand, $x/y \,\square\, z/w = (x \circ z) \| (y \circ w) = \{x/y, z/y, x/w, z/w\} = \{\{x\},\{z\}\}$ and $x/z \,\square\, z/w = \{\{x\},\{z\},H\}$.

4. Let $L = \langle L; \wedge, \vee \rangle$ be a lattice, without inferior and superior limits. Let $\langle L, \circ \rangle$ be the hypergroup (join space) defined:

$$x \circ y = \{u \mid x \wedge y \le u \le y \vee y\}.$$

Also, in this case, "$\square$" is not well defined. Indeed, we have

$$x/y = \begin{cases} \{t \mid t \le x\}, & \text{if } x < y \\ \{t \mid x \le t\}, & \text{if } x > y \\ L, & \text{if } x = y \end{cases}$$

and $x_1/y_1 = x_2/y_2$ if and only if $(x_1, x_2) = (y_1, y_2)$ or $(x_1 = x_2 = x$ and $\{y_1, y_2\} \subset \{t \mid t < x\})$ or $(x_1 = x_2 = x$ and $\{y_1, y_2\} \subset \{t \mid x < t\})$.

On the other hand,

$$(x/y) \,\square\, (u/v) = \{z \mid x \wedge u \le z \le x \vee u\} \| \{z \mid y \wedge v \le z \le y \vee v\}.$$

Choose, x, y, y', u, v in L such that $y < u < v < y' < x$ (it is possible, because L is infinite).

So $x/y = x/y'$ and we have $x/y = (x \wedge u)/(y \wedge v) \in (x/y) \,\square\, (u/v)$, but $u/y \neq u/s = (x \wedge u)/s$, for any s, such that $y' \wedge v = v \le s \le y' Vv = y'$. Moreover, $u/y \neq z/t$, for any z, t such that $x \wedge u < z \le x \vee u$ and $y \wedge v \le t \le y \vee v$. Therefore, $u/y \notin (x/y') \,\square\, (u/v($, whence "$\square$" is not well defined.

31. Proposition. *Let H be a hypergroup, for which "$\square$" is well defined. Then $\langle H \| H, \square \rangle$ is a hypergroup. Moreover,*

1. *If H is regular, $H \| H$ is regular, too;*
2. *If H is join space, $H \| H$ is join space, too;*
3. $(H \| H)/\beta_{H\|H} = \{\beta_{H\|H}(a/b) \mid (\beta_H(a), \beta_H(b)) \in H/\beta_H \times H/\beta_H\}$.

Proof. 1. If $e \in E(H)$, then $e/e \in E(H \| H)$.

For any $a/b \in H \| H$, $a'/b' \in i_{H\|H}(a/b)$, where $a' \in i_H(a)$ and $b' \in i_H(b)$. ($E(H)$ is the set of identities of H and $i(x)$ is the set of inverses of x, for any $x \in H$.)

2. Let $(x_1/x_2) \underset{H\|H}{/} (y_1/y_2) \cap (z_1/z_2) \underset{H\|H}{/} (w_1/w_1) \neq \emptyset$, that is α_1/α_2 exists, such that $x_1/x_2 \in (y_1 \circ \alpha_1) \| (y_2 \circ \alpha_2)$ and $z_1/z_2 \in (w_1 \circ \alpha_1) \| (\omega_2 \circ \alpha_2)$. Then, there exist $(x_1', x_2') \in H^2$ and $(z_1', z_2') \in H^2$, such that $x_1/x_2 = x_1'/x_2'$ and such that $z_1/z_2 = z_1'/z_2'$, where $x_1' \in y_1 \circ \alpha_1$ and $x_2' \in y_2 \circ \alpha_2$, respectively, $z_1' \in w_1 \circ \alpha_1$ and $z_2' \in w_2 \circ \alpha_2$. Hence $x_1'/y_1 \cap z_1'/w_1 \neq \emptyset$ and $x_2'/y_2 \cap z_2'/w_2 \neq \emptyset$.

So, there exist $a \in x_1' \circ w_1 \cap y_1 \circ z_1'$ and $b \in x_2' \circ w_2 \cap y_2 \circ z_2'$.

We have $a/b \in (x_1' \circ w_1) \| (x_2' \circ w_2) \cap (z_2' \circ y_1) \| (z_2' \circ y_2) = (x_1'/x_2' \,\square\, w_1/w_2) \cap (z_1'/z_2' \,\square\, y_1/y_2) = (x_1/x_2 \,\square\, w_1/w_2) \cap (z_1/z_2 \,\square\, y_1/y_2)$, so $\langle H \| H, \square \rangle$ is a join space, too.

3. We shall prove that for $(\beta_H(a_1), \beta_H(b_1)) = (\beta_H(a_2), \beta_H(b_2))$ we have $\beta_{H\|H}(a_1/b_1) = \beta_{H\|H}(a_2/b_2)$.

There exist $\{m, n\} \subset \mathbb{N}^*$, $\{c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_m\} \subset H$ such that $\prod_{i=1}^{n} c_i \supset \{a_1, a_2\}$ and $\prod_{i=1}^{m} d_i \supset \{b_1, b_2\}$.

If $n > m$, one considers $e_{m+1} \in I_r(d_m)$, $e_{m+2} \in I_r(e_{m+1}), \ldots, e_n \in I_r(e_{n-1})$ and one has $\{b_1, b_2\} \subset \prod_{i=1}^{m} d_i \subset \prod_{i=1}^{m} d_i e_{m+1} \cdot \ldots \cdot e_n$, that is b_1 and b_2 belong to a hyperproduct of n elements.

Similarly, for $n < m$. Therefore, we can consider $n = m$, and we obtain $\{a_1/b_1, a_2/b_2\} \subset \left(\prod_{i=1}^{n} c_i\right) \| \left(\prod_{i=1}^{n} d_i\right) = \prod_{i=1}^{n} \Box\, (c_i/d_i)$.

32. Proposition. *Let H be a commutative hypergroup. If $H \| H$ is a join space, then H satisfies the condition $\forall (a, b, c, d) \in H^4$, such that $a/b \cap c/d \neq \emptyset \Rightarrow (a \circ d) \| \omega_H \cap (b \circ c) \| \omega_H \neq \emptyset$.*

Proof. Let $y \in a/b \cap c/d$, that is $a \in y \circ$ and $c \in y \circ d$. Let's consider c' a partial inverse of c (that is $c \circ c' \cap I_p \neq \emptyset$, where I_p is the set of partial identities of H).

There exists $z \in H$, such that $c' \in z \circ a$ and let $t \in z \circ c$. One has $a/c' \in (y \circ b) \| (z \circ a) = (y/z) \Box (b/a)$ and $c/t \in (y \circ d) \| (z \circ c) = (y/z) \Box (d/c)$. So, $y/z \in (a - c') \underset{H\|H}{/} (b/a) \cap (c/t) \underset{H\|H}{/} (d/c)$. Because $H \| H$ is a join space, it result $(a/c') \Box (d/c) \cap (b/a) \Box (c/t) \neq \emptyset$, that is $(a \circ d) \| (c \circ c) \cap (b \circ c) \| (a \circ t) \neq \emptyset$.

We have $c' \circ c \subset \omega_H$, and $a \circ t \subset a \circ z \circ c \supset c' \circ c$, so $a \circ z \circ c \subset \omega_H$, whence $(a \circ d) \| \omega_H \cap (b \circ c) \| \omega_H \neq \emptyset$.

References

[1] Corsini P. *Hypergroupes réguliers et hypermodules.* Ann. Univ. Ferrara, Sc. Mat. Ser. VII, v. XXI (1975).

[2] Corsini P. *Contributo alla teoria degli ipergruppi.* Atti Soc. Pelor. Sc. Mat. fis. nat., XXVI (1980).

[3] Corsini P. *Feebly canonical and 1-hypergroups.* Acta Universitatis Carolinae Math. et Phys., v. 24.

[4] Corsini P. *Sugli impergruppi canonici finiti con identità parziali scalari.* Rend. del Circ. Mat. di Palermo, serie II, t. XXXVI (1987).

[5] Corsini P. *Prolegomena of hyopergroup theory.* Aviani Editore (1993).

[6] Corsini P. and Freni D. *On the heart of hypergroups.* Mathematica Montisnigri, vol. II (1993).

[7] Corsini P. and Mittas J. *New topics in hypergroup theory.* Accepted by "Multiple valued logic and Applications" (1995).

[8] Corsini P. and Sureau Y. *Sur les sous-hypergroupes d'un hypergroupe.* Riv. Mat. Pura e Appl. n. 1 (1976).

[9] Dresher M. and Ore O. *Theory of multigroups.* Amer. J. of Math., 62 (1940).

[10] LEOREANU V. *On the width of a hypergroup*. Proceedings of "Algebraic Hyperstructures and Applications", M. Ştefănescu Editor, Hadronic Press (1994).
[11] LEOREANU V. *New results on subhypergroup theory and on the nucleus structure*. Accepted by P.U.M.A. (1995).
[12] LEOREANU V. *On the heart of join spaces and of regular hypergroups*. Accepted by Riv. Mat. Pura e Appl. (1995).
[13] MITTAS J. *Hypergroupes fortement canoniques et supérieurement canoniques*. Proceeding II of the Int. Symp. on Applications of Math. in Syst. Sci., Braşov, Romania (1978).
[14] PRENOWITZ W. and JANTOSCIAK J. *Geometries and Join Spaces*. J. Reine und Angewändte Math. 257 (1972).