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Robert El Bashir

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# A New Proof of Prüfer's Theorem

ROBERT EL BASHIR*)

Praha

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The aim of this paper is to give an easy and direct proof of the well-known fact [1; Prüfer's Theorem: A bounded Abelian group is a direct sum of cyclic groups; page 44].

Cílem je podat jednoduchý a přímý důkaz dobře známého faktu [1; Prüferova věta: Omezená Abelova grupa je direktním součtem cyklických grup; str. 44].

The aim of this paper is to give an easy and direct proof of the well-known fact [1; Prüfer's Theorem: A bounded Abelian group is a direct sum of cyclic groups; page 44).

By a bounded group we mean a group in which the orders of the elements remain under a fixed finite bound. We will prove this theorem only for a bounded Abelian $p$-group, because every torsion Abelian group is a direct sum of Abelian $p$-groups.

First, let $G$ be a $p$-group, i.e. (generally not Abelian) group in which the orders of the elements are powers of one and the same prime $p$.

On $G$ we define a relation $\sim$ by

$$g \sim h \Leftrightarrow (\langle g \rangle \cap \langle h \rangle \neq \langle 0 \rangle) .$$

**Lemma 1.**

(i) *The relation $\sim$ is a relation of equivalence on $G \setminus \{0\}$.*

(ii) *There is (as a subset with exception of $0$) only one subgroup of group $G$ of order $p$ in a class of equivalence $\sim$.*

We will prove only the transitivity of $\sim$. Let $g \sim h$ and $h \sim k$. Then there exists $n_1 \in \langle g \rangle \cap \langle h \rangle$ and $n_2 \in \langle h \rangle \cap \langle k \rangle$ of order $p$. In $\langle h \rangle$ there exists only one subgroup of order $p$. For it

$$\langle n_1 \rangle = \langle n_2 \rangle \leqq \langle g \rangle \cap \langle k \rangle \quad \text{and} \quad g \sim k .$$

*) Faculty of Mathematics and Physics, Charles University, 186 00 Praha 8, Sokolovská 83, Czechoslovakia

(ii) Follows from the definition of $\sim$.

In the following $G$ will be an Abelian group.

A *maximal collection* will be a complete system of representants of the classes of the equivalence $\sim$ where each of the representants is maximal with respect to the order in the corresponding class of equivalence.

Of course, a maximal collection exists if and only if every class of equivalence has bounded orders of the elements.

**Lemma 2.** *Let $G$ be an Abelian $p$-group $\neq \langle 0 \rangle$ with elements of finite heights (with exception of $0$). Then every class of equivalence has bounded orders of elements and every maximal collection is a generating system of $G$.*

**Proof.** At first we will prove that the classes of equivalence have bounded orders. From Lemma 1(ii) we see that in every class of equivalence there exists an element of order $p$ which is also an element of $\langle h \rangle$ for every $h$ from this class of equivalence — and for it this class of equivalence has bounded orders because that element of order $p$ has finite height (all elements of $G$ have finite heights).

Let $M$ be a maximal collection and let $g \in G$, $g \neq 0$. We will find $h_1, \ldots, h_s \in M$ such that $g \in \langle h_1 \rangle + \ldots + \langle h_s \rangle$.

In $M$ there is some maximal element $h_1$ of the same class of equivalence, which $g$ is an element of. Here $\dfrac{\operatorname{ord}(g)}{p}\, g$ is an element of order $p$, which is equivalent to $g$. We have $\operatorname{ord}(h_1) \geqq \operatorname{ord}(g)$ because $h_1$ is maximal.

From Lemma 1(ii) it follows: there exists $a_1 \in \mathbb{N}$ such that $\dfrac{\operatorname{ord}(g)}{p}\, g = \dfrac{\operatorname{ord}(g)}{p}\, a_1 h_1$. Since $0 = \dfrac{\operatorname{ord}(g)}{p}(a_1 h_1 - g)$, $\operatorname{ord}(a_1 h_1 - g) \leqq \dfrac{\operatorname{ord}(g)}{p}$.

Let $b_1 = a_1 h_1 - g$. If $\operatorname{ord}(b_1) = 1$, then $b_1 = a_1 h_1 - g = 0$ and $g \in \langle h_1 \rangle$. If $\operatorname{ord}(b_1) > 1$ we repeat the procedure (which we did with $g$) for $b_1$ and we will get $b_2 = a_2 h_2 - b_1 = a_2 h_2 - (a_1 h_1 - g)$ (for some $h_2 \in M$ and $a_2 \in \mathbb{N}$). Now,

$$\operatorname{ord}(b_2) \leqq \frac{\operatorname{ord}(b_1)}{p} = \frac{\operatorname{ord}(a_1 h_1 - g)}{p} \leqq \frac{\operatorname{ord}(g)}{p^2}.$$

The order of $g$ is a power of $p$. Hence there exists $s \in \mathbb{N}$ that $\operatorname{ord}(b_s) = 1$, i.e. for some $h_1, \ldots, h_s \in M$ is $b_s = a_s h_s - a_{s-1} h_{s-1} + \ldots + (-1)^{s-1} a_1 h_1 + (-1)^s g = 0$ and $g \in \langle h_1 \rangle + \ldots + \langle h_s \rangle$.

**The Proof of Prüfer's Theorem.** We may assume that $G$ is a non-zero bounded Abelian $p$-group.

We will lead the proof in this way: we will choose some maximal collection $M$, then we will form a direct sum and verify, that this direct sum contains some maximal collection as a subset. From Lemma 2 it will follow that such direct sum must be the whole group $G$ and the proof will be complete.

At first we line up the elements of a chosen maximal collection $M$ into an unincreasing (as to orders) ordinal sequence $h_1, h_2, \dots$. We can do this because $G$ is a bounded group.

Now we will form a suitable direct sum $\bigoplus_{\alpha < \gamma} \langle g_\alpha \rangle$.

Let us put $g_1 = h_1$ and in the following we will work with the sequence $h_2, \dots$. If we already have $g_\alpha$, $\alpha < \beta$, we take the smallest ordinal $\delta$, for which $(\sum_{\alpha < \beta} + \langle g_\alpha \rangle) \cap \langle h_\delta \rangle = \langle 0 \rangle$, and put $\gamma_\beta = \chi_\delta$.

In the following we will work with the sequence $h_{\delta+1}, \dots$.

We will finish this process as soon as there is no such $\delta$. From the sequence $h_1, h_2, \dots$ we have chosen a subsequence $(g_\alpha \mid \alpha < \gamma)$. From the construction of this subsequence it follows: $\underset{\alpha < \beta}{+} \langle g_\alpha \rangle \cap \langle g_\beta \rangle = \langle 0 \rangle$ for every $\beta < \gamma$. Thus the sum $\underset{\alpha < \gamma}{+} \langle g_\alpha \rangle$ is direct.

It remains to prove that $\bigoplus_{\alpha < \gamma} \langle g_\alpha \rangle$ contains some maximal collection. It will be verified if $\bigoplus_{\alpha < \gamma} \langle g_\alpha \rangle$ contains some maximal element from the same class of equivalence for every $h_\varepsilon$ from the original maximal collection $M$, $h_\varepsilon \notin (g_\alpha \mid \alpha < \gamma)$. Let us take such an $h_\varepsilon$. From the construction of the sequence $(g_\alpha \mid \alpha < \gamma)$ we can see that there exists an ordinal $\beta < \gamma$, for which $\bigoplus_{\alpha < \beta} \langle g_\alpha \rangle \cap \langle h_\varepsilon \rangle \neq \langle 0 \rangle$ and $\operatorname{ord}(g_\alpha) \geqq$ $\geqq \operatorname{ord}(h_\varepsilon)$ for every $\alpha < \beta$. (This is due to the fact that the original maximal collection $M$ was lined up into an unincreasing sequence with respect to the orders.) Thus there exists an element $k$ of order $p$ with $k \in \bigoplus_{\alpha < \beta} \langle g_\alpha \rangle \cap \langle h_\varepsilon \rangle$. We can write $k = a_1 g_1 + \dots + a_n g_n$ for some $g_1, \dots, g_n \in (g_\alpha \mid \alpha < \beta)$. New, $pa_1 g_1 + \dots$ $\dots + pa_n g_n = 0$, because $k$ is or order $p$. Then $pa_j g_j = 0$ for $j = 1, \dots, n$ (the sum $\bigoplus_{\alpha < \beta} \langle g_\alpha \rangle$ is direct) and $\operatorname{ord}(a_j g_j) \leqq p$ for $j = 1, \dots, n$.

Further, we make use of $\operatorname{ord}(g_j) \geqq \operatorname{ord}(h_\varepsilon)$.

Hence there exist $m_j \in \langle g_j \rangle$ with $\dfrac{\operatorname{ord}(h_\varepsilon)}{p} m_j = a_j g_j$, $j = 1, \dots, n$.

Let $H_\varepsilon = m_1 + \dots + m_n$. Clearly, $\dfrac{\operatorname{ord}(h_\varepsilon)}{p} H_\varepsilon = k$, since $k \in \langle H_\varepsilon \rangle \cap \langle h_\varepsilon \rangle$ and $H_\varepsilon \sim h_\varepsilon$. At the same time $\operatorname{ord}(H_\varepsilon) = \operatorname{ord}(h_\varepsilon)$, because $\dfrac{\operatorname{ord}(h_\varepsilon)}{p} H_\varepsilon$ is an element of order $p$. The element $H_\varepsilon$ is maximal in the same class of equivalence as $h_\varepsilon$ and $H_\varepsilon$ is also an element of $\bigoplus_{\alpha < \gamma} \langle g_\alpha \rangle$.

By the construction of the elements $H_\varepsilon$, the set

$$\{g_\alpha \mid \alpha < \gamma\} \cup \{H_\varepsilon \mid h_\varepsilon \in M \setminus \{g_\alpha \mid \alpha < \gamma\}\} \subseteqq \bigoplus_{\alpha < \gamma} \langle g_\alpha \rangle$$

is a maximal collection and from Lemma 2 it follows $\bigoplus_{\alpha < \gamma} \langle g_\alpha \rangle = G$.

### Reference

[1] FUCHS L.: Abelian groups, Akadémiai Kiadó, Budapest 1966.