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Multiplication Groups of Quasigroups and Loops I

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Transversals A, B to a subgroup H in a group G are called H-connected if $[A, B]$ is contained in H. In the sequel, the connected transversals will turn out to be important tools for the study of the multiplication groups of quasigroups and loops.

Transversály A, B podgrupy H grupy G se nazývají H-spojené, jestliže $[A, B]$ je obsaženo v H. Spojené transversály se později ukáží jako důležité nástroje pro studium multiplikačních grup kvazigrup a lup.

1. Preliminaries

1.1 Let G be a group. For all $a, b \in G$, $a^b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. If A, B are subsets of G, then $A^{-1} = \{a^{-1};\ a \in A\}$, $A^x = x^{-1}Ax$ for every $x \in G$, $AB = \{ab;\ a \in A,\ b \in B\}$ and $[A, B] = \{[a, b];\ a \in A,\ b \in B\}$ (then $[B, A] = [A, B]^{-1}$). Moreover, $\langle A \rangle$ denotes the subgroup generated by A.

We put $G' = \langle [G, G] \rangle$, $G'' = (G')'$, etc.

If H is a subgroup of G, then $\langle [H, G] \rangle$ is a normal subgroup of G (indeed, for $x, y \in G$ and $u \in H$, $y^{-1}u^{-1}x^{-1}uxy = y^{-1}u^{-1}yuu^{-1}y^{-1}x^{-1}uxy \in \langle H, G] \rangle$).

1.2 Let G be a group. For a non-empty subset A of G, $\mathbf{C}_G(A) = \{x \in G;\ xa = ax \text{ for each } a \in A\}$ and $\mathbf{N}_G(A) = \{x \in G;\ xA = Ax\}$. Then ${}_G(A)$ and $\mathbf{N}_G(A)$ are subgroups of G and $\mathbf{C}_G(A)$ is normal in $\mathbf{N}_G(A)$. Moreover, $A \subseteqq \mathbf{C}_G(\mathbf{C}_G(A))$, $\mathbf{N}_G(A) = \{x \in G;\ x^{-1}Ax = A\} = \{x \in G;\ xAx^{-1} = A\}$ and $\mathbf{C}_G(A) = \mathbf{C}_G(\langle A \rangle)$, $\mathbf{N}_G(A) = \mathbf{N}_G(\langle A \rangle)$.

If H is a subgroup of G, then $H \subseteqq \mathbf{N}_G(H)$ and $\mathbf{N}_G(H)$ is the greatest subgroup of G containing H as a normal subgroup. Further, $H \subseteqq \mathbf{C}_G(H)$ if H is abelian,; in that case, $H \subseteqq \mathbf{Z}(\mathbf{C}_G(H))$.

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$\mathbf{Z}(G) = \mathbf{C}_G(H)$ denotes the centre of G.

If H is a normal subgroup of G and K is a characteristic subgroup of H (i.e. K is invariant under all automorphisms of H), then K is normal in G (in particular, for H normal, $\mathbf{Z}(H)$ and H' are also normal in G).

1.3 Let H be a subgroup of a group G. Put $\mathbf{L}_G(H) = \bigcap_{x \in G} H^x$. Then $\mathbf{L}_G(H)$ (the so called *core* of H) is the greatest subgroup of H which is normal in G. Clearly, $\mathbf{L}_G(H) = \{u \in H;\ x^{-1}ux \in H \text{ for each } x \in G\}$.

For every $x \in G$, $\mathbf{L}_G(H) = \mathbf{L}_G(H^x)$. Moreover, $\mathbf{L}_{G/N}(H/N) = 1$, where $N = \mathbf{L}_G(H)$.

1.4 Lemma. *Let H, K be subgroups of a group G.*

(i) *If K, H are normal in G, then $[K, H] \subseteq K \cap H$. If, moreover, $K \cap H = 1$, then $[K, H] = 1$, $K \subseteq \mathbf{C}_G(H)$ and $H \subseteq \mathbf{C}_G(K)$.*

(ii) *If $K \cap H = 1$, K is normal in G and $\mathbf{C}_G(H) \subseteq K$, then $\mathbf{L}_G(H) = 1$.*

(iii) *If $KH \subseteq HK$, then $KH = HK$ is a subgroup of G.*

(iv) *If $G = KH$, then $H \cap \mathbf{C}_G(K) \subseteq \mathbf{L}_G(H)$ and $K \cap \mathbf{C}_G(H) \subseteq \mathbf{L}_G(K)$.*

(v) *If H is normal in G and $H \cap G' = 1$, then $H \subseteq \mathbf{Z}(G)$.*

1.5 Let G be a group. For every subset A of G, let $\text{cn}(A) = \text{cn}_G(A) = \bigcup_{x \in G} A^x$.

(i) $A \subseteq \text{cn}(A)$ and $\text{cn}(\text{cn}(A)) = \text{cn}(A)$.

(ii) $1 \in \text{cn}(A)$ iff $1 \in A$.

(iii) $\text{cn}(A) \cap \mathbf{Z}(G) = A \cap \mathbf{Z}(G)$.

(iv) $(\text{cn}(A))^{-1} = \text{cn}(A^{-1})$.

(v) If $A \subseteq \mathbf{Z}(G)$, then $\text{cn}(A) = A$.

(vi) If H is a subgroup of G, then $\text{cn}(H) = H$ iff H is normal in G.

(vii) If $x, y \in G$, then $xy \in \text{cn}(A)$ if $yx \in \text{cn}(A)$.

(viii) If A_i, $i \in I$, is a non-empty system of subsets of G, then $\text{cn}(\bigcup A_i) = \bigcup \text{cn}(A_i)$.

2. Stable transversals

2.1 Let H be a subgroup of a group G. A subgroup A of G is said to be a *left (right) partial transversal to G in H* if $a^{-1}b \notin H$ ($ab^{-1} \notin H$) for all $a, b \in A$, $a \neq b$.

2.2 Lemma. *Let A be a left (right) partial transversal to a subgroup H of a group G. Then, for every $x \in G$:*

(i) *xA (Ax) is a left (right) partial transversal to H.*

(ii) *Ax (xA) is a left (right) partial transversal to H^x ($H^{x^{-1}}$).*

(iii) *A^x is a left (right) partial transversal to H^x.*

(iv) *A^{-1} is a right (left) partial transversal to H.*

(v) *A is a left (right) partial transversal to $H \cap \langle A \rangle$.*

2.3 Lemma. *Let H be a subgroup of a group G and A a subset of G. The following conditions are equivalent:*

(i) $a^{-1}b \notin \mathrm{cn}(H)$ $(ab^{-1} \notin \mathrm{cn}(H))$ *for all* $a.\ b \in A$, $a \neq b$.

(ii) Ax (xA) *is a left (right) partial transversal to H for every* $x \in G$.

(iii) *For every $x \in G$, the sets Ax and xA are both left and right partial transversals to H.*

(iv) *A is a left (right) partial transversal to H^x for every $x \in G$.*

(v) *A^x is a left (right) partial transversal to H for every $x \in G$.*

(vi) *For all $x, y \in G$, A^x is both left and right partial transversal to H^y.*

(vii) *For all $x, y, z \in G$, the sets $(Ax)^y$, $(xA)^y$, $(A^y)\,x$, $x(A^y)$ are both left and right partial transversals to H^z.*

2.4 Let H be a subgroup of a group G. A subset A of G satisfying the equivalent conditions of 2.3 is called a *stable partial transversal* to H in G.

2.5 Lemma. *Let A be a stable partial transversal to a subgroup H of G. Then, for all $x, y, z \in G$, the sets $(Ax)^y$, $(xA)^y$, $(A^y)\,x$, $x(A^y)$ are stable partial transversals to H^z.*

2.6 Let H be a subgroup of a group G. A subset A of G is said to be a *left (right) pseudotransversal* to H in G if $G = AH$ $(G = HA)$.

2.7 Lemma. *Let A be a left (right) pseudotransversal to a subgroup H of a group G. Then, for every $x \in G$:*

(i) *xA (Ax) is a left (right) pseudotransversal to H.*

(ii) *Ax (xA) is a left (right) pseudotransversal to H^x $(H^{x^{-1}})$.*

(iii) *A^x is a left (right) pseudotransversal to H^x.*

(iv) *A^{-1} is a right (left) pseudotransversal to H.*

(v) *A is a left (right) pseudotransversal to $H \cap \langle A\rangle$ in $\langle A\rangle$.*

2.8 Lemma. *Let H be a subgroup of a group G and A a subset of G. The following conditions are equivalent:*

(i) *Ax (xA) is a left (right) pseudotransversal to H for every $x \in G$.*

(ii) *For every $x \in G$, the sets Ax and xA are both left and right pseudotransversals to H.*

(iii) *A is a left (right) pseudotransversal to H^x for every $x \in G$.*

(iv) *A^x is a left (right) pseudotransversal to H for every $x \in G$.*

(v) *For all $x, y \in G$, A^x is both left and right pseudotransversal to H^y.*

(vi) *For all $x, y, z \in G$, the sets $(Ax)^y$, $(xA)^y$, $(A^y)\,x$, $x(A^y)$ are both left and right pseudotransversals to H^z.*

Proof. (i) implies (ii). For every $x \in G$ there are $a \in A$ and $u \in H$ with $ax^{-1}u = 1$. Then $x = ua$ and we have shown that A is a right pseudotransversal.

2.9 Let H be a subgroup of a group G. A subset A of G satisfying the equivalent conditions of 2.8 is called a *stable pseudotransversal* to H in G.

2.10 Lemma. *Let A be a stable pseudotransversal to a subgroup H of G. Then,* *for all $x, y, z \in G$, the sets $(Ax)^y, (xA)^y, (A^y)\,x, x(A^y)$ are stable pseudotransversals to H^z in G.*

2.11 Let H be a subgroup of a group G. A subset A of G is said to be a *left (right) transversal* to H in G if it is both a left (right) partial transversal and a left (right) pseudotransversal to H in G.

2.12 Lemma. *Let A be a left (right) transversal to a subgroup H of a group G. Then, for every $x \in G$:*

(i) *xA (Ax) is a left (right) transversal to H.*

(ii) *Ax (xA) is a left (right) transversal to H^x $(H^{x^{-1}})$.*

(iii) *A^x is a left (right) transversal to H^x.*

(iv) *A^{-1} is a right (left) transversal to H.*

(v) *A is a left (right) transversal to $H \cap \langle A\rangle$ in $\langle A\rangle$.*

2.13 Lemma. *Let H be a subgroup of a group G and A a subset of G. The following conditions are equivalent:*

(i) *Ax (xA) is a left (right) transversal to H for every $x \in G$.*

(ii) *For every $x \in G$, the sets Ax and xA are both left and right transversals to$H$.*

(iii) *A is a left (right) transversal to H^x for every $x \in G$.*

(iv) *A^x is a left (right) transversal to H for every $x \in G$.*

(v) *For all $x, y \in G$, A^x is both left and right transversal to H^y.*

(vi) *For all $x, y, z \in G$, the sets $(Ax)^y, (xA)^y, (A^y)\,x, x(A^y)$ are both left and right transversals to H^z.*

2.14 Let H be a subgroup of a group G. A subset A of G satisfying the equivalent conditions of 2.13 is called a *stable transversal* to H in G.

2.15 Lemma. *Let A be a stable transversal to a subgroup H of G. Then, for all $x, y, z \in G$, the sets $(Ax)^y, (xA)^y, (A^y)\,x, x(A^y)$ are stable transversals to H^z in G.*

2.16 Lemma. *Let H be a subgroup of a group G and A a subset of G. Put $G_1 = \langle A\rangle$ and $H_1 = H \cap G_1$. If A is a stable (partial, pseudo) transversal to H in G, then A is a stable (partial, pseudo) transversal to H_1 in G_1.*

2.17 Lemma. *Let H be a subgroup of a group G.*

(i) *If K is a subgroup of H and if A is a (left, right, stable) partial transversal to H in G, then A is also a (left, right, stable) partial transversal to K in G.*

(ii) *If K is a subgroup of G with $A \subseteq K$, $L = K \cap H$ and if A is a (left, right, stable) (pseudo)transversal to H in G, then A is also a (left, right, stable) (pseudo)transversal to L in K.*

(iii) *If K is a subgroup of G with $H \subseteq K$ and if A is a (left, right) (pseudo)transversal to H in G, then $A \cap K$ is a (left, right) (pseudo)transversal to H in K. Moreover, if A is stable, then $A \cap K$ is stable (in K).*

2.18 Lemma. *Let H be a subgroup of a group G and φ a homomorphism of G onto a group K, $\mathrm{Ker}(\varphi) = N$.*

(i) *If $N \subseteq H$ and A is a (left, right, stable) partial transversal to H, then $\varphi|A$ is injective and $\varphi(A)$ is a (left, right, stable) partial transversal to $\varphi(H)$.*

(ii) *If A is a (left, right, stable) pseudotransversal to H in G, then $\varphi(A)$ is a (left, right, stable) pseudotransversal to $\varphi(H)$ in K.*

2.19 Lemma. *Let H and K be subgroup of a group G.*

(i) *If $H \cap K = 1$, then K (H) is both a left and right partial transversal to H (K).*

(ii) *If H is normal in G and $H \cap K = 1$, then K (H) is a stable partial transversal to H (K).*

(iii) *If $HK = G$, then K (H) is a stable pseudotransversal to H (K).*

2.20 Lemma. *Let H, K be subgroups of a group G such that $H \cap K = 1$ and $HK = G$. Then K (H) is a stable transversal to H (K).*

Proof. Fisrt, let $a, b \in H$, $x \in G$, $x = uv$, $u \in H$, $v \in K$. Then $w = (a^x)^{-1}(b^x) = v^{-1}u^{-1}a^{-1}buv$. If $w \in K$, then $(a^{-1}b)^u \in H \cap K = 1$ and hence $a^{-1}b = 1$, $a = b$. We have shown that H^x is a left partial transversal to K. By 2.3(v), H is a stable partial transversal to K.

Now, let $x = uv$, $u \in H$, $v \in K$. Then $H^xK = v^{-1}H^uvK = v^{-1}HK = v^{-1}G = G$. By 2.8(iv), H is a stable pseudotransversal to K in G.

2.22 Lemma. *Let H be a subgroup of a finite index in a group G and let A be a (left, right) transversal to H in G. The following conditions are equivalent:*

(i) *A is a stable partial transversal.*

(ii) *A is a stable pseudotransversal.*

(iii) *A is a stable transversal.*

Proof. Let $n = [G : H]$, $x \in G$, $B = x^{-1}Ax$. Then $\mathrm{card}(A) = \mathrm{card}(B) = n$.
(i) implies (iii). B is a left partial transversal and there is an injective mapping f: $B \to A$ with $bH = f(b)\,H$ for each $b \in B$. Now, f is a bijection, $f(B) = A$ and $BH = AH = G$. Thus B is a transversal.
(ii) implies (iii). B is a left pseudotransversal and there is an injective mapping g: $A \to B$ with $aH = g(a)\,H$ for each $a \in A$. Again, g is a bijection. If $b, c \in B$, $b \neq c$, then $bH = g^{-1}(b)\,H \neq g^{-1}(c)\,H = cH$ and we see that B is a transversal.

2.23 Lemma. *Let H be a subgroup of a group G and A a left (right) pseudotransversal to H in G.*

(i) *There exists a left (right) transversal B to H in G such that $B \subseteq A$.*

(ii) *If $\langle A \rangle \cap H = 1$, then A is a subgroup of G and a transversal to H in G.*

(iii) *If K is a subgroup of G with $H \subseteq K$, then $\mathbf{L}_G(K) = \{u \in K;\ u^a \in K$ for each $a \in A\}$ ($\mathbf{L}_G(K) = \{u \in K;\ u^{a^{-1}} \in K$ for each $a \in A\}$).*

(iv) *If K is a subgroup of G with $A \subseteq K$, then $\mathbf{L}_K(K \cap H) \subseteq \mathbf{L}_G(H)$.*

(v) *If K is a normal subgroup of H and if $K^{a^{-1}} \subseteq H$ ($K^a \subseteq H$) for each $a \in A$, then $K \subseteq \mathbf{L}_G(H)$.*

2.24 Lemma. *Let A be a stable (partial, pseudo) transversal to a subgroup H in a group G. Then A^{-1} is also a stable (partial, pseudo) transversal to H in G.*

3.1 Let H be a subgroup of a group G and A, B subsets of G. We shall say that A, B are *H-connected* if $[A, B] \subseteq H$.

3.2 Lemma. *Let A be a left pseudotransversal to a subgroup H of a group G and let B be a subset of G such that A, B are H-connected.*

(i) *If B is a left partial transversal to H, then B is a stable partial transversal.*

(ii) *If B is a left pseudotransversal to H in G, then both A and B are stable pseudotransversals to H in G.*

Proof. (i) Let $x \in G$ and $b, c \in B$ be such that $x^{-1}b^{-1}cx \in H$. There are $a \in A$ and $u \in H$ with $x = au$. Then $u^{-1}a^{-1}b^{-1}cau \in H$, hence $a^{-1}b^{-1}ca \in H$ and we have $b^{-1}c = b^{-1}a^{-1}ba \cdot a^{-1}b^{-1}ca \cdot a^{-1}c^{-1}ac \in H$. However, then $b = c$.
(ii) Let $x, y \in G$, $x = au$, $a \in A$, $u \in H$. Then there are $b \in B$ and $v \in H$ with $uy = bv$. Of course, $w = b^{-1}a^{-1}ba \in H$ and $b = a^{-1}baw^{-1}$. Now, $y = u^{-1}bv = u^{-1}a^{-1}baw^{-1}v = u^{-1}a^{-1}bauz = x^{-1}bxz$, where $z = u^{-1}w^{-1}v \in H$. We have shown that B is a stable pseudotransversal. By the reason of symmetry, A is also stable.

3.3 Corollary. *Let A, B be H-connected left transversals to a subgroup H of a group G. Then A, B are stable transversals.*

3.4 In the sequel, by *H-connected (pseudo)transversals* we will always mean H-connected left (pseudo)transversals (which are then both left and right (pseudo)transversals).

3.5 Lemma. *Let A, B be H-connected (pseudo)tranversals to a subgroup H in a group G and let K be a subgroup of G such that $H \subseteq K$. Then $A \cap K$, $B \cap K$ are H-connected (pseudo)transversals to H in K.*

Proof. By 2.17(iii), both $A \cap K$ and $B \cap K$ are left (pseudo)transversals to H in K. Clearly, they are H-connected in K.

3.6 Lemma. *Let A, B be H-connected (pseudo)transversals to a subgroup H in G. Put $G_1 = \langle A, B\rangle$ and $H_1 = G_1 \cap H$. Then A, B are H_1-connected (pseudo)-transversals to H_1 in G_1.*

3.7 Lemma. *Let A, B be H-connected pseudotransversals to a subgroup H in a group G. Then there exist H-connected transversals C, D to H in G such that $C \subseteq A$ and $D \subseteq B$.*

3.8 Lemma. *Let H be a subgroup of a group G, φ a homomorphism of G onto a group K and $N = \mathrm{Ker}(\varphi)$.*

(i) *If A, B are H-connected pseudotransversals to H in G, then $\varphi(A)$, $\varphi(B)$ are $\varphi(H)$-connected pseudotransversals to $\varphi(H)$ in K.*

(ii) *If $N \subseteq H$ and A, B are H-connected transversals to H in G, then $\varphi(A)$, $\varphi(B)$ are $\varphi(H)$-connected transversals to $\varphi(H)$ in K and $\varphi|A$, $\varphi|B$ are injective mappings.*

(iii) *If* $\mathbf{L}_G(NH) = N$, *A is a left pseudotransversal to H in G and if B is a subset of G such that A, B are H-connected, then* $\varphi(B)$ *is a left partial transversal to* $\varphi(H)$ *in K.*

Proof. (i) and (ii). Easy.
(iii) Let $b, c \in B$ be such that $\varphi(b^{-1}c) \in \varphi(H)$, i.e. $b^{-1}c \in NH$. For every $a \in A$, $a^{-1}b^{-1}ca = a^{-1}b^{-1}ab \cdot b^{-1}a^{-1}ac \cdot c^{-1}a^{-1}ca \in NH$, and so $b^{-1}c \in \mathbf{L}_G(NH) = N$ by 2.23(iii). We have shown that $\varphi(b) = \varphi(c)$.

3.9 Lemma. *Let H be a subgroup of a group G such that* $\mathbf{L}_G(H) = 1$. *If A, B are H-connected pseudotransversals to H in G, then A, B are transversals to H in G.*

Proof. This follows from 3.8(iii) for $\varphi = \mathrm{id}_G$, $K = G$ and $N = 1$.

3.10 Lemma. *Let A, B be H-connected pseudotransversals to a subgroup H in a group G and let* $\varphi\colon G \to K$ *be a homomorphism of G onto a group K such that* $\mathrm{Ker}(\varphi) = \mathbf{L}_G(NH)$ *for a normal subgroup N of G. Then* $\varphi(A)$, $\varphi(B)$ *are* $\varphi(H)$*-connected transversals to* $\varphi(H)$ *in K.*

Proof. This follows from 3.8(iii), as $\mathrm{Ker}(\varphi)\, H = NH$.

3.11 Lemma. *Let A, B be H-connected pseudotransversals to a subgroup H in a group G.*
(i) *If C is a subset of* $A \cup B$ *and* $K = \langle H, C\rangle$, *then* $C \subseteq \mathbf{L}_G(K)$.
(ii) $A \cap H \subseteq \mathbf{L}_G(H)$ *and* $B \cap H \subseteq \mathbf{L}_G(H)$.
(iii) $\mathbf{Z}(G) \subseteq A \cdot \mathbf{L}_G(H)$ *and* $\mathbf{Z}(G) \subseteq B \cdot \mathbf{L}_G(H)$.

Proof. (i) Let $x \in G$ and $c \in C$. We have to show that $x^{-1}cx \in K$. To this purpose, we can assume that $c \in A$ and $x = bu$, $b \in B$, $u \in H$. Then $x^{-1}c^{-1}x = u^{-1}b^{-1}c^{-1}bu = u^{-1} \cdot b^{-1}c^{-1}bc \cdot c^{-1}u \in K$, and therefore $x^{-1}cx \in K$.
(ii) By (i), $(A \cap H) \cup (B \cap H) \subseteq \mathbf{L}_G(K)$, where $K = \langle H, A \cap H, B \cap H\rangle = H$.
(iii) Let $c \in \mathbf{Z}(G)$, $c = au$, $a \in A$, $u \in H$. For every $b \in B$, $b^{-1}ub = b^{-1}a^{-1}cb = b^{-1}a^{-1}bc = b^{-1}a^{-1}ba \cdot u \in H$. Consequently, $u \in \mathbf{L}_G(H)$ and $c \in A\mathbf{L}_G(H)$. Quite similarly, $c \in B\mathbf{L}_G(H)$.

3.12 Lemma. *Let H be a subgroup of a group G such that* $\mathbf{L}_G(H) = 1$ *and let A, B be H-connected transversals to H in G.*
(i) $A \cap H = \{1\} = B \cap H$.
(ii) $\mathbf{Z}(G) \subseteq A \cap B$.
(iii) *If N is a normal subgroup of G and* $N \subseteq A \cap B$, *then* $N \subseteq \mathbf{Z}(\langle A, B\rangle)$.

Proof. (i) and (ii) follow from 3.11 (ii) and (iii), respectively.
(iii) Let $x \in N$ and $a \in A \cup B$. Then $a^{-1}x^{-1}a \in N$ and $a^{-1}x^{-1}ax \in H$. Thus $a^{-1}x^{-1}ax \in N \cap H = 1$, and hence $ax = xa$. This shows that $x \in \mathbf{Z}(\langle A, B\rangle)$.

3.13 Proposition. *Let H be a proper subgroup of a simple group G such that there exist H-connected transversals to H in G. Then H is a maximal subgroup of G.*

Proof. Let $a \in A - H$ and $K = \langle a, H \rangle$. By 3.11(i), $a \in \mathbf{L}_G(K)$, so that $\mathbf{L}_G(K) \neq 1$ and $K = G$. The assertion is now clear.

3.14 Lemma. *Let A be a left pseudotransversal to a subgroup H in a group G such that $AA \subseteq A\mathbf{L}_G(H)$ and let B be a subset of G such that A, B are H-connected. Then $[A, B] \subseteq \mathbf{L}_G(H)$.*

Proof. Let $a, c \in A$, $b \in B$. Then $ac = du$ for some $d \in A$, $u \in \mathbf{L}_G(H)$ and we have $c^{-1}a^{-1}b^{-1}acb = u^{-1}d^{-1}b^{-1}dub = u^{-1} \cdot d^{-1}b^{-1}db \cdot b^{-1}ub \in H$. However, $b^{-1}c^{-1}bc \in H$, and so $c^{-1}a^{-1}b^{-1}abc \in H$, which shows that $a^{-1}b^{-1}ab \in \mathbf{L}_G(H)$.

3.15 Proposition. *Let H be a subgroup of a group G such that $\mathbf{L}_G(H) = 1$, let A be a left or right pseudotransversal to H in G and B also a left or right pseudotransversal to H in G and suppose that $[A, B] = 1$.*

(i) *A, B are H-connected (and hence stable) transversals to H in G.*
(ii) *A, B are isomorphic subgroups of G and $A \cap B \subseteq \mathbf{Z}(G_1)$, where $G_1 = \langle A, B \rangle$. Moreover, A, B are normal subgroups of G_1.*
(iii) *$G_1 = AB \cong (A \times A)/K$, where $K = \{(a, a^{-1}); a \in A \cap B\}$.*
(iv) *$\mathbf{L}_{G_1}(H_1) = 1$, where $H_1 = H \cap G_1$.*
(v) *A, B are H_1-connected transversals to H_1 in G_1.*
(vi) *$A \cap B = \mathbf{Z}(A) \cap \mathbf{Z}(B) = \mathbf{Z}(A) = \mathbf{Z}(B) = \mathbf{Z}(G_1)$.*
(vii) *$H_1 \cong G_1/A \cong B/(A \cap B) \cong G_1/B \cong A/(A \cap B)$.*
(viii) *If H or H_1 is solvable (nilpotent), abelian), then G_1 is solvable (nilpotent, nilpotent of class at most 2).*
(ix) *If H or H_1 is cyclic, then $H_1 = 1$ and $A = B = G_1$ is an abelian group.*
(x) *If H is cyclic, then $G'' = 1$.*

Proof. (i) The only case that requires consideration is the case of A a left pseudotransversal and B a right pseudotransversal. Then $[A, B^{-1}] = 1$ as $[a, b^{-1}]^{-1} = [a, b]^{b^{-1}}$ and we can use 2.7(iv), 3.2(ii), 2.24 and 3.9.
(ii) Put $C = \langle A \rangle \cap H$. Then $bc = cb$ for all $b \in B$, $c \in C$, $C^b = C \subseteq H$ and $C \subseteq \mathbf{L}_G(H) = 1$ by 2.23(iii). Hence $C = 1$ and A is a subgroup of G by 2.23(ii). Quite similarly, B is a subgroup of G.

For each $a \in A$, there is a unique $f(a) \in B$ with $af(a) \in H$. Now, if $a, d \in A$, then $f(ad)H = (ad)^{-1}H = d^{-1}a^{-1}H = d^{-1}f(a)H = f(a)d^{-1}H = f(a)f(d)H$. Clearly, f: $A \to B$ is an isomorphism.

Finally, since $[A, B] = 1$, we have $A \cap B \subseteq \mathbf{Z}(G_1)$.
(iii) $G_1 = AB$, since A, B are subgroups and $[A, B] = 1$. Now, define a mapping g: $A \times A \to G_1$ by $g(a, d) = af(d)$, where f: $A \to B$ is as in (ii). Clearly, g is a homomorphism of $A \times A$ onto G_1 and $\mathrm{Ker}(g) = K$.
(iv) If $u \in \mathbf{L}_{G_1}(H_1)$, then $a^{-1}ua \in H_1 \subseteq H$ for every $a \in A$, and so $u \in \mathbf{L}_G(H) = 1$.
(v) and (vi). Obvious.
(vii) Let f: $A \to B$ be as in (ii). Then $H_1 = \{af(a); a \in A\}$, and so $H_1 \cong$ $\cong A/(A \cap B) \cong AB/B = G_1/B$. Quite similarly, $H_1 \cong B/(A \cap B) \cong AB/A =$ $= G_1/A$.

(viii) If H_1 is solvable, then $A/(A \cap B)$ is so. However, $A \cap B \subseteq \mathbf{Z}(A)$, so that A is solvable and finally, by (iii), G_1 is solvable. The other cases are similar.
(ix) Suppose that H_1 is cyclic. Then $A/\mathbf{Z}(A)$ is cyclic, and hence A is abelian. Consequently, G_1 is abelian, H_1 is normal in G_1 and $H_1 = 1$, since $\mathbf{L}_{G_1}(H_1) = 1$.
(x) By (ix), $A = B$ is abelian. We have $G = AH$ and $G'' = 1$ by [1].

3.16 Proposition. *Let A, B be left pseudotransversal to a subgroup H in a group G. The following conditions are equivalent:*
i) $[A, B] \subseteq \mathbf{L}_G(H)$.
(ii) *A, B are H-connected and $AA \subseteq A\mathbf{L}_G(H)$.*
(iii) *A, B are H-connected and $BB \subseteq B\mathbf{L}_G(H)$.*
(iv) *A, B are H-connected and $A\mathbf{L}_G(H)$, $B\mathbf{L}_G(H)$ are subgroups of G.*

Proof. Use 3.14 and 3.15(ii).

3.17 Proposition. *Let A, B be left pseudotransversals to a subgroup H in a group G such that $\mathbf{L}_G(H) = 1$. The following conditions are equivalent:*
(i) $[A, B] = 1$.
(ii) *A, B are H-connected and $AA \subseteq A$.*
(iii) *A, B are H-connected and $BB \subseteq B$.*
(iv) *A, B are H-connected and A, B are isomorphic subgroups of G.*

3.18 Proposition. *Let A, B be H-connected transversals to a subgroup H in a group G. Then $\mathbf{N}_G(H) = HK$, where $K/\mathbf{L}_G(H) = \mathbf{Z}(G/\mathbf{L}_G(H))$. In particular, if $\mathbf{L}_G(H) = 1$, then $\mathbf{N}_G(H) = H\mathbf{Z}(G) \cong H \times \mathbf{Z}(G)$.*

Proof. Without loss of generality, we can assume that $\mathbf{L}_G(H) = 1$. For each $x \in N = \mathbf{N}_G(H)$, we can define a permutation f_x of A by $x^{-1}ax \in f_x(a)\,H$ for each $a \in A$. If $x, y \in N$ and $a \in A$, then $x^{-1}ax = f_x(a)\,u$, $u \in H$, $y^{-1}f_x(a)\,y = f_yf_x(a)\,v$, $v \in H$, and $y^{-1}x^{-1}axy = f_{xy}(a)\,w$, $w \in H$ and $f_{xy}(a) = f_yf_x(a)$, so that $f_{xy} = f_yf_x$. Now, the mapping $F\colon x \to f_{x^{-1}}$ is a homomorphism of N into the symmetric group $\mathcal{S}(A)$. Since A, B are H-connected, we have $C = B \cap N \subseteq \mathrm{Ker}(F)$. Put $L = \mathrm{Ker}(F) \cap H$. Then $L = \{z \in H;\ a^{-1}za \in H \text{ for each } a \in A\} = \mathbf{L}_G(H) = 1$. However, $N = CH$ (since B is a transversal and $H \subseteq N$), and so $\mathrm{Ker}(F) = \langle C\rangle = C$ by 2.23(ii). Naturally, C is normal in N, and hence N is the direct product of H and C. In particular, $C \subseteq \mathbf{C}_G(H)$.

It remains to show that $C \subseteq \mathbf{Z}(G)$. For, let $D = \langle [C, G]\rangle$. Then D is a normal subgroup of G. However, if $c \in C$, $x \in G$, $x = au$, $a \in A$, $u \in H$, then $c^{-1}x^{-1}cx = c^{-1}u^{-1}a^{-1}cau = u^{-1}\cdot c^{-1}a^{-1}ca \cdot u \in H$. Thus $D \subseteq H$, $D \subseteq \mathbf{L}_G(H) = 1$, $[C, G] = 1$ and $C \subseteq \mathbf{Z}(G)$ (in fact, $C = \mathbf{Z}(G)$, since $\mathbf{Z}(G) \subseteq N$).

3.19 Lemma. *Let $H \subseteq K \subseteq L \subseteq G$ be subgroups of a group G such that K is normal in L and suppose that there exist H-connected pseudotransversals to H in G. Then the factorgroup L/K is abelian.*

Proof. Without loss of generality, we can assume that $H = K$ and $\mathbf{L}_G(H) = 1$. By 3.18, $\mathbf{N}_G(H) = H\cdot\mathbf{Z}(G)$. However, $L \subseteq \mathbf{N}_G(H)$.

3.20 Lemma. *Let H be a proper subgroup of a group G such that $H \cap H^x = 1$ for each $x \in G - H$. If A, B are H-connected pseudotransversals to H in G, then $A = B$ is an abelian subgroup of G (and a transversal).*

Proof. Clearly, $\mathbf{L}_G(H) = 1$, so that A, B are transversals.

If $a \in A$, then $b^{-1}a \in H$ for some $b \in B$. Now, $a^{-1}b^{-1}ab \in H$, hence $b^{-1}a \in aHb^{-1} = bHb^{-1}$. It follows that $b^{-1}a \in H \cap bHb^{-1}$. If $b \notin H$, then $H \cap bHb^{-1} = 1$ and $a = b$. If $b \in H$, then $b = 1$, $a \in H$ and $a = b = 1$. We conclude that $A = B$.

Now, let $a, b \in A$. Again, $c^{-1}ab \in H$ for suitable $c \in A$. Now, $c^{-1}abaH = c^{-1}aabH = c^{-1}acH = aa^{-1}c^{-1}acH = aH$. From this, $a^{-1}c^{-1}aba \in H$ and $c^{-1}ab \in H \cap aHa^{-1}$. If $a \notin H$, then $c = ab$. If $a \in H$, then $a = 1$ and $c = b$. We have shown that $AA \subseteq A$. By 3.17, A is a subgroup of G. Since A is H-selfconnected, A is abelian.

3.21 Lemma. *Let H be a non-normal subgroup of a group G such that $H \cap H^x = 1$ for each $x \in G - \mathbf{N}_G(H)$. If A, B are H-connected pseudotransversals to H in G, then $A = B$ is an abelian subgroup of G (and a transversal).*

Proof. Clearly, $\mathbf{L}_G(H) = 1$, so that A, B are transversals. Further, $\mathbf{Z}(G) \subseteq A \cap B$ by 3.12(ii) and $\mathbf{N}_G(H) = H\mathbf{Z}(G)$ by 3.18. Consequently, $A \cap \mathbf{N}_G(H) = \mathbf{Z}(G) = B \cap \mathbf{N}_G(H)$.

Let $a \in A$, $b \in B$ and $b^{-1}a \in H$. Then $b^{-1}a \in H \cap H^{b^{-1}}$ (see the proof of 3.20). If $b^{-1}a \neq 1$, then $b \in \mathbf{N}_G(H) \cap B = \mathbf{Z}(G)$, $a \in bH \subseteq \mathbf{N}_G(H)$, $a \in \mathbf{Z}(G)$, $b^{-1}a \in H \cap \mathbf{Z}(G) = 1$, a contradiction. Thus $b^{-1}a = 1$ and $b = a$. We have proved that $A = B$.

Now, let $a, b \in A$. If $a, b \in \mathbf{Z}(G)$, then $ab \in \mathbf{Z}(G) \subseteq A$. If $a \notin \mathbf{Z}(G)$, then $c^{-1}ab \in H \cap H^{a^{-1}}$ for some $a \in A$ (see the proof 3.20). Again, if $c^{-1}ab \neq 1$, then $a \in \mathbf{Z}(G)$, a contradiction. Thus $c^{-1}ab = 1$ and $c = ab$. Finally, if $a \in \mathbf{Z}(G)$ and $b \notin \mathbf{Z}(G)$, then $ab = ba$ and $ba \in A$ by the preceding part of the proof.

3.22 Remark. Let H be a subgroup of G such that $\mathbf{L}_G(H) = 1$ and there exist H-connected transversals A, B to H in G.

(i) Put $I = \bigcup_{x \in G} H^x$ and $J = G - I$. Since both A and B are stabler transversals, we have $A - \{1\} \subseteq J$ and $B - \{1\} \subseteq J$. Clearly, J is just the set of $w \in G$ such that $x^{-1}wx \notin H$ for every $x \in G$. Moreover, $J^x = J$ for every $x \in G$.

(ii) Suppose that G is finite, $\operatorname{card}(H) = m$, $\operatorname{card}(A) = \operatorname{card}(B) = n$ and $\operatorname{card}(\mathbf{Z}(G)) = r$. Then $\operatorname{card}(I) \leqq (m-1)\,n/r + 1$ and $\operatorname{card}(J) \geqq n(rm - m + 1)/r - 1$.

(iii) Now, suppose that G is finite and that $H \cap H^x = 1$ for each $x \in G - H$. By 3.20, $A = B$ is an abelian group and we have $\operatorname{card}(J) = n - 1$ by (ii). Thus $J = A - \{1\}$, $A = J \cup \{1\}$ and $A = B$ is a normal subgroup of G by (i). (A is the Frobenius kernel of G.)

3.23 Remark. Consider the situation from 3.21. We have $\mathbf{Z}(G) \subseteq A$ by 3.12(ii), and hence $\mathbf{Z}(G) \subseteq L = \mathbf{L}_G(A)$.

(i) Suppose that $L \neq \mathbf{Z}(G)$. Clearly, $A \subseteq \mathbf{C}_G(L) \subseteq \mathbf{N}_G(L) = G$. If $A \neq \mathbf{C}_G(L)$, then there are $a \in L - \mathbf{Z}(G)$ and $1 \neq u \in H$ such that $au = ua$, i.e. $1 \neq u = a^{-1}ua \in H \cap H^a$, which is not possible. Hence, $A = \mathbf{C}_G(L)$ and this implies that A is normal in G.
(ii) We have $G = AH$, and therefore there exists a homomorphism $\varphi: A \to \mathcal{S}(H)$ (the symmetric group on H) such that $\mathrm{Ker}(\varphi) = L$. In particular, if $\mathrm{card}(H)! < \mathrm{card}(A/\mathbf{Z}(G))$, then $L \neq \mathbf{Z}(G)$.
(iii) Let $\mathbf{Z}(G) = 1$ and let H be finite. If A is infinite, then $L \neq \mathbf{Z}(G)$ by (ii) and A is normal by (i). If A is finite, then $\mathbf{N}_G(H) = H\mathbf{Z}(G) = H$ by 3.18, and hence 3.22(iii) can be applied. Thus A is a normal abelian subgroup of G whenever $\mathbf{Z}(G) = 1$ and H is finite.
(iv) Suppose that $\mathbf{Z}(G) = \mathbf{L}_G(\mathbf{N}_G(H))$ and that H is finite and abelian. We are again going to show that A is normal in G. Indeed, we have $\overline{G} = G/\mathbf{Z}(G) = \overline{A}\overline{H}$, $\overline{A} = A/\mathbf{Z}(G)$, $\overline{H} = H\mathbf{Z}(G)/\mathbf{Z}(G) \cong H$, $\mathbf{L}_{\overline{G}}(\overline{H}) = 1$. If $\overline{A}$ is infinite, then the result follows from (i) and (ii). If $\overline{A}$ is finite (and non-trivial), then $\mathbf{L}_{\overline{G}}(\overline{A}) \neq 1$ by a well known result of Itô (1]. However, then $L \neq \mathbf{Z}(G)$ and we can use (i) again.

3.24 Lemma. *Let H be a subgroup of G such that $[G:H] \geq 3$ and $H \cap H^u \cap H^v = 1$ whenever $u, v \in G - H$ and $uv^{-1} \in G - H$. If A, B are H-connected pseudotransversals to H in G and if there exists an element $e \in A \cap B$ with $e \neq 1$, then $A = B$ is a transversal to H in G.*

Proof. Clearly, $\mathbf{L}_G(H) = 1$, so that A, B are transversals and $A \cap H = \{1\} = B \cap H$.

Let $a \in A$, $b \in B$ and $a^{-1}b \in H$. Then $aH = bH$ and $a^{-1}b \in H \cap H^{a^{-1}}$ (see the proof of 3.20). Further, $e^{-1}a^{-1}ea \in H$, $e^{-1}b^{-1}eb \in H$ and consequently $b^{-1}e^{-1}ba^{-1}ea \in H$, $ba^{-1} \in ebHa^{-1}e^{-1} = eaHa^{-1}e^{-1} = H^{(ea)^{-1}}$ and $a^{-1}b \in H^{(a^{-1}ea)^{-1}}$.

We have proved that $a^{-1}b \in H \cap H^{a^{-1}} \cap H^{(a^{-1}ea)^{-1}}$. Assume that $a \neq b$. If $a^{-1} \in H$, then $a = 1 = b$, a contradiction. If $(a^{-1}ea)^{-1} \in H$, then $e = a^{-1}eaa^{-1}e^{-1} \in H$, and so $e = 1$, a contradiction. Finally, if $(a^{-1}ea)^{-1}a \in H$, then $e^{-1}a = e^{-1}a^{-1}ea \cdot a^{-1}e^{-1}aa \in H$, and so $e = a$, $e = b$ and $a = b$, again a contradiction.

3.25 Lemma. *Let H be a subgroup of G.*

(i) *If $G' \subseteq H$ and A, B are (pseudo)transversals to H in G, then A, B are H-connected.*
(ii) *If K is an abelian subgroup of G such that $HK = G$, then K is an H-selfconnected pseudotransversal to H in G; it is a transversal iff $H \cap K = 1$.*
(iii) *If $H = 1$ and A, B are H-connected pseudotransversals to H in G, then $A = B = G$ is an abelian group.*
(iv) *If H is normal in G and A, B are H-connected pseudotransversals to H in G, then $G' \subseteq H$.*

3.26 Corollary. *Let H be a subgroup of a group G. Then $G' \subseteqq H$ iff H is normal in G and there exist H-connected transversals to H in G.*

4. Semiconnected transversals

4.1 Let H be a subgroup of a group G and A, B subsets of G. We shall say that A, B are *H-semiconnected* if for all $u \in A$ and $v \in B$ there exists $x \in G$ such that $[Au^{-1}, Bv^{-1}] \subseteqq H^x$ (i.e., Au^{-1}, Bv^{-1} are H^x-connected).

4.2 Lemma. *Let H be a subgroup of G and A, B subsets of G.*

(i) *If A, B are H-connected, then $[Au^{-1}, Bv^{-1}] \subseteqq H^{(uv)^{-1}}$ for all $u \in A$, $v \in B$. In particular, A, B are H-semiconnected.*

(ii) *If A, B are H-connected, then A^x, B^x are H^x-connected for every $x \in G$.*

(iii) *If A, B are H-semiconnected, then A^x, B^x are H-semiconnected for every $x \in G$.*

(iv) *If A, B are H-semiconnected, then A, B are H^x-semiconnected for every $x \in G$.*

(v) *If A, B are H-semiconnected, then A^x, B^x are H^y-semiconnected for all $x, y \in G$.*

(vi) *If A, B are H-semiconnected, then Ax, By are H-semiconnected for all $x, y \in G$.*

vii) *If A, B are H-semiconnected, then xA, xB are H-semiconnected for every $x \in G$.*

(viii) *If A, B are H-semiconnected, then $(xAy)^u$, $(xBz)^u$ are H^v-semiconnected for all $x, y, z, u, v \in G$.*

Proof. (i) $[au^{-1}, bv^{-1}] = ua^{-1}vb^{-1}au^{-1}bv^{-1} = uv \cdot v^{-1}a^{-1}va \cdot a^{-1}b^{-1}ab \cdot \cdot b^{-1}u^{-1}bu \cdot u^{-1}v^{-1}uv \cdot v^{-1}u^{-1} \in uvHv^{-1}u^{-1} = H^{(uv)^{-1}}$ for all $a \in A$, $b \in B$. The remaining assertions are clear.

4.3 Lemma. *Let H be a subgroup of a group G. Subsets A, B of G are H-semiconnected iff there exist $u \in A$, $v \in B$ and $x \in G$ such that $[Au^{-1}, Bv^{-1}] \subseteqq H^x$.*

Proof. This follows easily from 4.2.

4.4 Lemma. *Let H be a subgroup of a group G. The following conditions are equivalent:*

(i) *There exist H-connected pseudotransversals to H in G.*

(ii) *There exist H-connected transversals to H in G.*

(iii) *There exist H-semiconnected stable transversals to H in G.*

(iv) *There exist H-semiconnected stable pseudotransversals to H in G.*

Proof. (i) implies (ii) by 3.7, (ii) implies (iii) and (iii) implies (iv) trivially. (iv) implies (i). Let A, B be H-semiconnected stable pseudotransversals to H in G. Take $u \in A$, $b \in B$. Then $[Au^{-1}, Bv^{-1}] \subseteqq H^x$ for some $x \in G$. Since A, B are stable, Au^{-1} and Bv^{-1} are left pseudotransversals, and so Au^{-1}, Bv^{-1} are H^x-connected pseudotransversals. But then $(Au^{-1})^{x^{-1}}$, $(Bv^{-1})^{x^{-1}}$ are H-connected pseudotransversals.

4.5 Lemma. *Let A, B be H-semiconnected stable pseudotransversals to a subgroup H in a group G. If* $\mathbf{L}_G(H) = 1$, *then A, B are stable transversals.*

Proof. Let $u \in A$, $v \in B$ and $x \in G$ be such that $[Au^{-1}, Bv^{-1}] \subseteq H^x$. Then Au^{-1}, Bv^{-1} are stable pseudotransversals. Of course, $\mathbf{L}_G(H^x) = 1$, and therefore Au^{-1}, Bv^{-1} are stable transversals by 3.9 and 3.3. By 2.15, A and B are stable transversals to H in G.

5. Stable transversals and graphs

5.1 Here, by a graph we mean a non-empty set R together with a binary relation $r \subseteq R^{(2)}$ (the case $r = \emptyset$ being also allowed) which is symmetric and antirefelxive.

If $\emptyset \neq S \subseteq R$, then S together with $s = r \cap S^{(2)}$ is also a graph and it is called the subgraph induced by S.

If $s = \emptyset$, then S is said to be an independent subset.

5.2 Let $\mathscr{R} = (R, r)$ be a graph.

(i) Put $\operatorname{dis}(\mathscr{R}) = \max\{\operatorname{card}(S); \emptyset \neq S \subseteq R, S \text{ independent}\}$, provided that such a cardinal number exists.

(ii) For each $a \in R$, let $\deg(a) = \deg(a, \mathscr{R}) = \operatorname{card}(\{x \in R; (a, x) \in r\})$.

The graph $\mathscr{R}$ is said to be *regular* if $\deg(a) = \deg(b)$ for all $a, b \in R$. In that case, we put $\deg(\mathscr{R}) = \deg(a)$, $a \in R$.

(iii) The graph $\mathscr{R}$ is said to be *discrete* if $r = \emptyset$. Then $\operatorname{dis}(\mathscr{R}) = \operatorname{card}(R)$ and $\deg(\mathscr{R}) = 0$.

(iv) The graph $\mathscr{R}$ is said to be *complete* if $r = R^{(2)} - \mathrm{id}_R$. Then $\operatorname{dis}(R) = 1$ and $\deg(R) = \operatorname{card}(R) - 1$ ($= \operatorname{card}(R)$ if this cardinal number is infinite).

5.3 Let G be a group and A a subset of G such that $A^{-1} = A$. Now, define a graph $\mathscr{G} = \mathscr{G}(G, r_A)$ on G by $(x, y) \in r_A$ iff $x, y \in G$, $x \neq y$, $xy^{-1} \in \operatorname{cn}(A)$ (see 1.5). We put $\operatorname{dis}(G, A) = \operatorname{dis}(\mathscr{G}(G, r_A))$.

5.4 Lemma. *Let A be a subset of a group G such that $A^{-1} = A$ and let* $B = \operatorname{cn}(A)$. *Then:*

(i) $r_A = r_B$.

(ii) $(x, y) \in r_A$ *iff* $x = ya$ *(or* $x = ay$, $y = ax$, $y = xa$*) for some* $a \in B$.

Proof. Since $A \subseteq B$, we have $r_A \subseteq r_B$. If $(x, y) \in r_B$, then $xy^{-1} \in \operatorname{cn}(B) = B = \operatorname{cn}(A)$, so that $(x, y) \in r_A$.

5.5 Corollary. *Let A, B be subsets of a group G such that $A^{-1} = A$ and $B^{-1} = B$. Then $r_A = r_B$ iff* $\operatorname{cn}(A) = \operatorname{cn}(B)$.

5.6 Lemma. *Let A be a subset of a group G such that $A^{-1} = A$. Then:*

(i) $\mathscr{G} = \mathscr{G}(G, r_A)$ *is a regular graph and* $\deg(\mathscr{G}) = \operatorname{card}(\operatorname{cn}(A) - \{1\})$.

(ii) $\deg(\mathscr{G}) = \operatorname{card}(\operatorname{cn}(A))$ *if* $1 \notin A$.
(iii) $\deg(\mathscr{G}) = \operatorname{card}(\operatorname{cn}(A)) - 1$ *if* $1 \in A$.
(iv) *The following permutations are automorphisms of the graph* $\mathscr{G}$: *All the translation* $\mathscr{L}(G, u)$, $\mathscr{R}(G, u)$, $u \in G$; *the permutation* $x \to x^{-1}$; *all automorphisms f of G with* $f(\operatorname{cn}(A)) = \operatorname{cn}(A)$ (*or* $f(A) = A$).

Proof. Easy.

5.7 Lemma. *Let H be a subgroup of a group G and* $\mathscr{G} = \mathscr{G}(G, r_A)$. *Then:*
(i) *A subset A of G is independent in* $\mathscr{G}$ *iff A is a stable partial transversal to H in G.*
(ii) *If* $\operatorname{dis}(G, H)$ *exists, then* $\operatorname{dis}(G, H) \leqq [G : H]$.
(iii) *If H is normal in G, then* $\operatorname{dis}(G, H) = [G : H]$.

Proof. (i) See 2.3(i).
(ii) and (iii). Easy.

5.8 Lemma. *Let H, K be subgroups of a group G.*
(i) *If* $HK = G$ *and* $H \cap K = 1$, *then* $\operatorname{dis}(G, H) = [G : H] = \operatorname{card}(K)$ *and* $\operatorname{dis}(G, K) = [G : K] = \operatorname{card}(H)$.
(ii) *If* $H \subseteqq K$ *and K is normal in G, then* $\operatorname{dis}(G, H) = [G : K] \cdot \operatorname{dis}(K, \operatorname{cn}_G(H)) = \operatorname{dis}(G, K) \cdot \operatorname{dis}(K, \operatorname{cn}_G(H))$ *and* $\operatorname{dis}(K, \operatorname{cn}_G(H)) \leqq \operatorname{dis}(K, H)$.
(iii) *If* $H \subseteqq K$ *and H is normal in G, then* $\operatorname{dis}(G, K) = \operatorname{dis}(G/H, K/H)$.
(iv) *If* $\operatorname{card}(H) < \operatorname{card}(K)$, $\operatorname{cn}(H) = \operatorname{cn}(K)$ *and G is finite, then* $\operatorname{dis}(G, H) < [G : H]$.

Proof. (i) This follows from 5.7(i) and 2.20.
(ii) First, let A be a left transversal to K in G; then A is a stable transversal and $\operatorname{card}(A) = [G : K]$. Let B be a subset of K independent in $\mathscr{G}(K, r_R)$, $R = \operatorname{cn}_G(H)$. We are going to show that AB is independent in $\mathscr{G}(G, r_R)$. For, let $a, c \in A$, $b, d \in B$ and $abd^{-1}c^{-1} = ab \cdot (cd)^{-1} \in R$. Then $abd^{-1}c^{-1} = x^{-1}ux$ for some $x \in G$, $u \in H$, so that $xax^{-1} \cdot xbd^{-1}x^{-1} \cdot xc^{-1}x^{-1} \in H$. But $H \subseteqq K$, K is normal in G, hence $xbd^{-1}x^{-1} \in K$ and $xax^{-1} \cdot xc^{-1}x^{-1} = xac^{-1}x^{-1} \in K$, $ac^{-1} \in K$, $a = c$, $bd^{-1} = a^{-1}x^{-1}uxa \in R$ and, finally, $b = d$. We have proved that AB is independent in $\mathscr{G}(G, r_R)$ and that $\operatorname{card}(AB) \leqq \operatorname{card}(A) \cdot \operatorname{card}(B)$. From this, $\operatorname{dis}(G, H) \geqq [G : K] \cdot \operatorname{dis}(K, R)$. Since $\operatorname{cn}_K(H) \subseteqq \operatorname{cn}_G(H)$, we have $\operatorname{dis}(K, R) \leqq \operatorname{dis}(K, H)$.

Now, let C be a subset of G such that C is independent in $\mathscr{G}(G, r_R)$. The set C is divided in pair-wise disjoint blocks of elements congruent modulo K. The number of all such blocks is at most $[G : K]$. Let D be one of these blocks, $d \in D$ and $E = Dd^{-1}$. Then $E \subseteqq K$ and E is independent in $\mathscr{G}(K, r_K)$. Consequently, $\operatorname{card}(E) \leqq \operatorname{dis}(K, R)$ and we see that $\operatorname{card}(C) \leqq [G : K] \operatorname{dis}(K, R)$.
(iii) Easy.
(iv) We have $r_H = r_K$ and $\operatorname{dis}(G, H) = \operatorname{dis}(G, K) \leqq [G : K] < [G : H]$.

5.9 Corrollary. *Let H be a subgroup of a finite group G. Then* $\operatorname{dis}(G, H) = [G : H]$ *iff there exists a stable transversal to H in G.*

References

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