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A Representation of Orthomodular Lattices

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By analyzing central ultrafiltres on an orthomodular lattice L we construct a closure space $\mathscr{L}$ such that L is orthoisomorphic to the orthomodular lattice $CO(\mathscr{L})$ of all clopen sets in $\mathscr{L}$. This orthoisomorphism becomes the Stone representation on the centre of L. Although the lattice-theoretic operations are generally not set-theoretic in $CO(\mathscr{L})$ — this cannot be done at all (see e.g. [2], [5]) — we show that it is so for couples containing at least one central element. This generalizes and complements the representation by L. Iturrioz [4] and R. Mayet [6].

V článku je ukázáno, že ke každému ortomodulárnímu svazu L existuje uzávěrový prostor $\mathscr{L}$ tak, že L je izomorfní s ortomodulárním svazem $CO(\mathscr{L})$ všech uzavřeně-otevřených množin v $\mathscr{L}$. Tento izomorfismus přejde na centru svazu L ve Stoneovu reprezentaci a navíc vytváření svazových operací na $CO(\mathscr{L})$ je množinové pro každou dvojici, která obsahuje alespoň jeden centrální prvek.

В статье доказывается, что к любой ортомодулярной решетке L существует пространство с замыканием $\mathscr{L}$ так, что L изоморфна ортомодулярной решётке $CO(\mathscr{L})$ всех открыто-замкнутых множеств в $\mathscr{L}$. Этот изоморфизм переходит на центре решётки в представление Стоуна и решёточные операции на $CO(\mathscr{L})$ совпадают с теоретикомножествеными для каждой пары, которая содержит центральный элемент.

1. Preliminaries on orthomodular lattices

Definition 1.1. A triple $(P \leqq, ')$ is called an orthomodular lattice (abbr. OML) if $(P, \leqq)$ is a partially ordered set with an orthocomplementation $'$ such that

1) P is a lattice with respect to the ordering $\leqq$,
2) there is a least and a greatest element in P, 0, 1,
3) if $a, b \in P$ and $a \leqq b$ then $a' \geqq b'$,
4) if $a \in P$ then $(a')' = a$,
5) if $a \leqq b$ then $b = a \vee (a' \wedge b)$.

In what follows, let L always mean an OML.

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Let us call the set $C(L) = \{a \in L \mid a = (a \wedge b) \vee (a \wedge b')$ for each $b \in L\}$ the centre of L. Let us recall (see [4]) that the set $C(L)$ as the set of absolutely commutative elements of L is a Boolean subalgebra of L and $C(L) = L$ exactly in case L is Boolean.

In what follows we shall deal with so called central ultrafilters (see also [8]) which will play an essential role in the representation theorem. (It should be observed that there is no straightforward generalization of Boolean technique because OML's do not generally possesses enough ultrafilteres. For instance the lattice $L(H)$ of projections in a Hilbert space H possesses none ultrafilter at all (see [1]).

Definition 1.2. Let F be a subset of L. Then F is called a central filter (abbr. c-filter) on L if the following conditions are satisfied:

1) if $a \in F$, $b \in L$ and $b \geqq a$ then $b \in F$,
2) if $a \in F \cap C(L)$ and $b \in F$ then $a \wedge b \in F$.

If F is a filter and the condition $a \in F$ implies $a' \notin F$, then F is called proper. Finally, if F is a proper filter and if for any $a \in L$ either $a \in F$ or $a' \in F$, then F is called a central ultrafilter (c-ultrafilter).

Proposition 1.3. *Let $\{F_\alpha \mid \alpha \in I\}$ be a collection of c-filters on L. Then the set $G = \{a \in L \mid a \geqq a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge \tilde{a},\ a_{\alpha_i} \in F_{\alpha_i} \cap C(L)\ (i \leqq k)$ and $\tilde{a} \in \bigcup_{\alpha \in I} F_\alpha\}$ is the least c-filter on L containing all $F_\alpha(\alpha \in I)$.*

Proof. By the definition of G, if $a \in G$ and $a \leqq b$ then $b \in G$. Suppose now that $a \in G \cap C(L)$, $b \in G$. We have to show $a \wedge b \in G$. Since $a \in G$, we have $a \geqq a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge \tilde{a}$, where $a_{\alpha_i} \in F_{\alpha_i} \cap C(L)$ $(i \leqq k)$ and $\tilde{a} \in F_\gamma$. Obviously, there is a Boolean subalgebra of L containing the set $\{a_{\alpha_1}, a_{\alpha_2}, \dots, a_{\alpha_k}, \tilde{a}\}$. Therefore we may write $a = a \vee (a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge \tilde{a}) = (a \vee a_{\alpha_1}) \wedge (a \vee a_{\alpha_2}) \wedge \dots \dots \wedge (a \vee \tilde{a})$. Thus, changing $a \vee a_{\alpha_i}$ for a_{α_i} and $a \vee \tilde{a}$ for $\tilde{a}$, we may assume that $a = a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge \tilde{a}$, where $a_{\alpha_i} \in F_{\alpha_i} \cap C(L)\ (i \leqq k)$ and $\tilde{a} \in F_\gamma$.

Put $c = a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge a'$. Then $c \in C(L)$ and therefore $c' \in C(L)$. Moreover, $c' \geqq \tilde{a}$ and therefore $c' \in F_\gamma$. Thus, $c' \in F_\gamma \cap C(L)$ and moreover, $a_{\alpha_1} \wedge \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge c' = (a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k}) \wedge ((a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k})' \vee \vee a) = 0 \vee (a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge a) = a$. Summarizing what we have showed so far, we have obtained the expression $a = a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_k} \wedge a_{\alpha_{k+1}}$, where $a_{\alpha_i} \in F_{\alpha_i} \cap C(L)\ (i \leqq k + 1)$.

The rest is easy. If $b \geqq b_{\beta_1} \wedge b_{\beta_2} \wedge \dots \wedge b_{\beta_p} \wedge \tilde{b}$, where $b_{\beta_i} \in F_{\beta_i} \cap C(L)\ (i \leqq p)$ and $\tilde{b} \in F_\delta$, we have $a \wedge b \geqq a_{\alpha_1} \wedge a_{\alpha_2} \wedge \dots \wedge a_{\alpha_{k+1}} \wedge b_{\beta_1} \wedge b_{\beta_2} \wedge \dots \wedge b_{\beta_p} \wedge \tilde{b}$ and therefore $a \wedge b \in G$. The proof of Proposition 1.3. is complete.

Proposition 1.4. *Suppose that $a, b \in L$. Then either $a \leqq b$ or there exists a central ultrafilter F on L such that $a \in F$ and $b \notin F$.*

Proof. Suppose that $a \nleqq b$. Denote by $\mathscr{F}_{a,b}$ the collection of all proper c-filters which contain a and do not contain b. Since the c-filter $F_a = \{x \in L \mid x \geqq a\}$ belongs to $\mathscr{F}_{a,b}$, we see that $\mathscr{F}_{a,b} \neq \emptyset$. Let us order the set $\mathscr{F}_{a,b}$ by inclusion and take a maximal element in $\mathscr{F}_{a,b}$, F. A maximal element obviously exists by Zorn's lemma. We are going to show that F is a c-ultrafilter.

Suppose that it is not the case. Then there is an element $c \in L$ such that $\{c, c'\} \cap F = \emptyset$. Put $F_c = \{x \in L \mid x \geqq c\}$ and denote by G the c-filter generated by F_c and F. We shall show first that G is proper. Suppose on the contrary that there is an element $d \in L$ such that $\{d, d'\} \subset G$. Applying Proposition 1.3. we may assume that $d \geqq m \wedge k$ and $d' \geqq n \wedge s$, where $m \in F \cap C(L)$, $n \in F_c \cap C(L)$, $k \in F_c$ and $s \in F$ (The other cases argue similarly) Moreover, we may assume that in the latter expression we have $m \geqq k'$ and $n \geqq s'$ (Indeed, we can write $m \wedge k = m \wedge (k \vee m')$ and take $k \vee m'$ for k if necessary.). Since m, n are central, we can write $d = (m \vee d) \wedge (m' \vee d)$ and $d' = (n \vee d') \wedge (n' \vee d')$. Therefore $0 = d \wedge d' = (m \vee d) \wedge (m' \vee d) \wedge (n \vee d') \wedge (n' \vee d')$. As $d \geqq m \wedge k$, we obtain $d \vee m' \geqq (m \wedge k) \vee m' = k \vee m' \geqq k$. Thus, $d \vee m' \in F_c$. Analogously, $n' \vee d' \in F$. Since $(d \vee m') \wedge (n \vee d') \geqq c$, the equality $(m \vee d) \wedge (m' \vee d) \wedge (n \vee d') \wedge (n' \vee d') = 0$ consisting of mutually compatible elements implies $c' \geqq (m \vee d) \wedge (n' \vee d') \geqq m \wedge (n' \vee d') \in F$. It follows that $c' \in F$ and this is a contradiction. We have checked that F is proper. We may suppose that $b \notin G$. (Indeed, this follows automatically if $b' \in F$. If $b' \notin F$, then $\{b, b'\} \cap F = \emptyset$ and we could take b' for c in the former construction.) We therefore have $G \in \mathscr{F}_{a,b}$ and G extends F. This is absurd since F was maximal. Thus F is a c-ultrafilter. The proof of Proposition 1.4. is complete.

Corollary 1.5. *Let L be an orthomodular lattice and let F be a Boolen ultrafilter on $C(L)$. Then F can be extended to a c-ultrafilter on L.*

Proof. Put $F_1 = \{x \in L \mid x \geqq a$ for any $a \in F\}$. Then F_1 is obviously a proper c-filter on L and the extension can be obtained from Proposition 1.4.

Proposition 1.6. *Let $\mathscr{F}$ be the set of all central ultrafilters on L. Let $P(\mathscr{L})$ denote the set of all subsets of $\mathscr{L}$ and let $\varphi: L \to P(\mathscr{L})$ be the mapping defined by the equality $\varphi(a) = \{F \in \mathscr{L} \mid a \in F\}$. Then φ has the following properties:*

1) $\varphi(0) = \emptyset$,
2) $\varphi(a') = \mathscr{L} - \varphi(a)$,
3) *if* $a \in C(L)$, *then* $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$ *for any* $b \in L$,
4) *if* $a, b \in L$ *then* $a \leqq b \Leftrightarrow \varphi(a) \subset \varphi(b)$.

Proof. The conditions 1), 2), 4) follows from the definition of φ and Proposition 1.4. As for the condition 3), suppose that $a \vee b \in F$ for a c-ultrafilter F and $a \in C(L)$. Then $a \vee b = a \vee (b \wedge a') \in F$. If both $a, b \wedge a'$ do not belong to F, then both a', $b' \vee a$ belong to F and so does $a' \wedge (b' \vee a)$. But $a' \wedge (b' \vee a) = a' \wedge b' = (a \vee b)' \in F$, which is absurd. This completes the proof of Proposition 1.6.

2. Preliminaries on closure spaces

Following [3], a nonvoid set X together with a closure operation $^-$ is called a closure space if the following four conditions are satisfied:

1) $\bar{\emptyset} = \emptyset$,
2) $A \subset \bar{A}$ for *any* $A \subset X$,
3) $A \subset B \Rightarrow \bar{A} \subset \bar{B}$ for any $A, B \subset X$,
4) $\bar{\bar{A}} = \bar{A}$ for any $A \subset X$.

A set $A \subset X$ is called closed (resp. open) if $\bar{A} = A$ (resp. $\overline{X - A} = X - A$). Obviously, the intersection of closed sets in X is a closed set. Further, X is called Hausdorff if any pair of distinct points in X separates by open sets, and X is called compact if any collection $\{C_\alpha \mid \alpha \in I\}$ of closed sets in X fulfils the following property: If $\bigcap_{\alpha \in I} C_\alpha = \emptyset$, then there is a finite collection $C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}$ such that $\bigcap_{k \leq n} C_n = \emptyset$. (A closure space is a topological space if a union of any pair of closed sets is a closed set.)

3. A representation theorem

Let L be an orthomodular lattice and let $\varphi: L \to P(\mathscr{L})$ be the mapping defined in Proposition 1.6. For any $A \in P(\mathscr{L})$, put $\bar{A} = \cap\{\varphi(a) \mid a \in L, \varphi(a) \supset A\}$. This operation converts $\mathscr{L}$ to a closure space which we denote again by $\mathscr{L}$. Let $CO(\mathscr{L})$ denote the set of all sets which are simultaneously closed and open in $\mathscr{L}$.

Proposition 3.1. *Let $\varphi: L \to P(\mathscr{L})$ be the mapping defined in Proposition 1.6. Then φ has the following properties:*

1) *the set $\varphi(L) = \{A \subseteq \mathscr{L} \mid A = \varphi(a)$ for any $a \in L\}$ is a subset of $CO(\mathscr{L})$,*

2) *every set closed in $\mathscr{L}$ is an intersection of elements of $\varphi(L)$ (and dually for open sets),*

3) *φ is an order isomorphism of L and $(\varphi(L), \subset)$,*

4) *the set $\varphi(L)$ endowed with the inclusion relation and the set-theoretic orthocomplementation is an orthomodular lattice and moreover, if A, $B \in \varphi(L)$ then $A \vee B = \overline{A \cup B}$ and $A \wedge B = (A \cap B)^\circ$ (here $^\circ$ stands for the operation of taking the interior),*

5) *if in the above condition 4) we have B central in $CO(\mathscr{L})$, the $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.*

Proof.

1) Suppose that $a \in L$. Then $\varphi(a)$ is closed by the definition of the closure in X. Further, $\mathscr{L} - \varphi(a) = \varphi(a') = \overline{\varphi(a')} = \overline{\mathscr{L} - \varphi(a)}$ and therefore $\varphi(a)$ is open. Thus, $\varphi(a) \in CO(\mathscr{L})$.

2) It follows from the property 1) and from the definition of the closure in X.

3) Obvious.

4) The first part follows from the property 3) and from the fact that $(\varphi(a))' = \varphi(a')$. Take elements $A, B \in \varphi(L)$. Since $\varphi(L)$ is a lattice, there exists $C \in \varphi(L)$ such that $C = A \vee B$. By the definition of the closure in X, we have $\overline{A \vee B} = \cap\{\tilde{C} \in \varphi(L) \mid A \cup B \subset \tilde{C}\}$. Since for any $\tilde{C}$ from the latter formula we have $C \subset \tilde{C}$, we infer that $\cap\{\tilde{C} \in \varphi(L) \mid A \cup B \subset \tilde{C}\} = \cap\{\tilde{C} \in \varphi(L) \mid C \subset \tilde{C}\} = C$. The rest derives dually.

5) We apply Proposition 1.6. 3).

Proposition 3.2. *Let $\mathscr{L}$ be the closure space associated with L. Then $\mathscr{L}$ is compact Hausdorff and $\varphi(L) = CO(\mathscr{L})$.*

Proof. Let us check first that $\mathscr{L}$ is Hausdorff. Take $F_1, F_2 \in \mathscr{L}$ such that $F_1 \neq F_2$. Suppose that $a \in F_1 - F_2$. Then $a' \in F_2 - F_1$ and the sets $\varphi(a)$, $\varphi(a')$ separate F_1, F_2.

Let us show now that $\varphi(L)$ is compact. Let $\{C_\alpha \mid \alpha \in I\}$ be such a system of closed subsets of $\mathscr{L}$ that $\bigcap_{\alpha \in I} C_\alpha = \emptyset$. Since any closed set is an intersection of elements of $\varphi(L)$, it suffices to establish the following implication: If $\bigcap_{j \in J} \varphi(a_j) = \emptyset$ then there exists a finite subset $\{j_1, j_2, \ldots, j_n\}$ of J such that $\bigcap_{k \leqq n} \varphi(a_{j_k}) = \emptyset$.

If $\bigcap_{j \in J} \varphi(a_j) = \emptyset$ then there is no such c-ultrafilter F that $F \in \varphi(a_j)$ $(j \in J)$. Since $F \in \varphi(a_j)$ if and only if $a_j \in F$, we see that there is no c-ultrafilter containing each c-filter $F_j = \{a \in L \mid a \geqq a_j\}$ $(j \in J)$. Therefore the c-filter generated by all F_j $(j \in J)$ cannot be proper. By Proposition 1.3, there exists $x \in L$ such that $x \geqq b_{j_1} \wedge \ldots \wedge b_{j_r} \wedge \tilde{b}_p$, where $b_{j_k} \in F_{j_k} \cap C(L)$ and $\tilde{b}_p \in F_p$, and $x' \geqq b_{m_1} \wedge \ldots \wedge b_{m_s} \wedge \tilde{b}_q$, where $b_{m_t} \in F_{m_t} \cap C(L)$ $(t \leqq s)$ and $\tilde{b}_q \in F_q$. Then $\varphi(x) \supset \varphi(b_{j_1} \wedge \ldots \wedge b_{j_r} \wedge \tilde{b}_p) = \varphi(b_{j_1}) \cap \ldots \cap \varphi(b_{j_r}) \cap \varphi(b_p) \supset \varphi(a_{j_1}) \cap \ldots \cap \varphi(a_{j_r}) \cap \varphi(a_p)$ and similarly, $\varphi(x') \supset \varphi(b_{m_1} \wedge \ldots \wedge b_{m_s} \wedge \tilde{b}_q) \supset \varphi(a_{m_1}) \cap \ldots \cap \varphi(a_{m_s}) \cap \varphi(a_q)$. It follows that $\emptyset = \varphi(x) \cap \varphi(x') \supset \varphi(a_{j_1}) \cap \ldots . \cap \varphi(a_{j_r}) \cap \varphi(a_p) \cap \varphi(a_{m_1}) \cap \ldots \cap \varphi(a_{m_s}) \cap \varphi(a_q)$ So $\mathscr{L}$ is compact.

Finally, let A belong to $CO(\mathscr{L})$. According to Proposition 3.1. 2) we have $A = \bigcap_{i \in I} A_i$ and $A = \bigcup_{j \in J} B_j$, where $A_i = \varphi(a_i)$ $(i \in I)$ and $B_j = \varphi(b_j)$ $(j \in J)$. Thus, $\emptyset = A \cap A' = \bigcap_{i \in I} A_i \cap \bigcap_{j \in J} B'_j$. Since $\mathscr{L}$ is compact, there are indices $i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_n$ such that $\emptyset = A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m} \cap (B_{j_1} \cup B_{j_2} \cup \ldots \cup B_{j_n})'$. It follows that $A = A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}$. Since A is open, we have $A = A^0 = (A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m})^0 = A_{i_1} \wedge A_{i_2} \wedge \ldots \wedge A_{i_m} = \varphi(a_{i_1}) \wedge \ldots \wedge \varphi(a_{i_m}) = \varphi(a_{i_1} \wedge \ldots \wedge a_{i_m}) \in \varphi(L)$. The proof is complete.

The foregoing proposition plus Proposition 1.6. gives us the following theorem. (Recall that a mapping $\varphi\colon L_1 \to L_2$ between two orthomodular lattices is called an orthoisomorphism if φ is an orderisomorphism and if $\varphi(a') = \varphi(a)'$ for any $a \in L_1$. Obviously, any orthoisomorphism is necessary a lattice isomorphism.)

Theorem 3.3. *Let L be an orthomodular lattice. Then there exists a compact Hausdorff closure space $\mathscr{L}$ such that the orthomodular lattice $CO(\mathscr{L})$ of all clopen sets in $\mathscr{L}$ is orderisomosphic to L. Moreover, the lattice operations in $CO(\mathscr{L})$ are set-theoretic on the couples which contain a central element.*

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