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Trimedial Quasigroups and Generalized Modules I.

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V článku jsou vyloženy základy teorie zobecněných modulů.

In the paper, the essentials of the theory of generalized modules are presented.

В статье излагаются основы теории обобщенных модулей.

This is the first from a series of expository papers devoted to trimedial quasigroups and corresponding generalized modules. Various more or less known results are collected here, often equipped with new proofs and, moreover, some new material is added. An attempt is made to get a unifying overview of this field. To this end, an extensive (however by no means exhaustive) bibliography is also added at the end of this first part.

1. Preliminaries on commutative Moufang loops

Every loop (i.e. a quasigroup with unit) satisfying the identity $xx \cdot yz = xy \cdot xz$ is commutative (the identity implies $xy \cdot x = xx \cdot y = x \cdot xy$ and the commutativity follows) and is called a commutative Moufang loop.

As there is a close connection between trimedial quasigroups and commutative Moufang loops with operators, we have included this introductory section containing some necessary facts about these loops which will be needed in the sequel. The standard reference for commutative Moufang loops is [27]. In this book, the reader can also find all proofs which are not included in this section.

Convention. In the whole treatment, the basic operation of a commutative Moufang loop will always be denoted additively and 0 will mean the neutral element of such a loop. Further, $\mathbf{Z}$ will always denote the ring of integers and $\mathbf{Z}_3 = \{0, 1, 2\}$ the field of integers modulo 3.

Throughout this section, let Q be a commutative Moufang loop. For all $a, b, c \in Q$

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we put $[a, b, c] = ((a + b) + c) - (a + (b + c))$, so called associator of the elements a, b, c. If A, B, C are non-empty subsets of Q, $[A, B, C]$ is the subloop generated by all $[a, b, c]$, $a \in A$, $b \in B$, $c \in C$. The subloop $A(Q) = [Q, Q, Q]$ is called the associator subloop.

1.1. Lemma. *Let $a, b, c \in Q$. Then $[a, b, c] = -[b, a, c] = [b, c, a] = -[a, c, b] = [c, a, b] = -[c, b, a]$ and $[a, b, c] = [a, a + b, c]$.*

1.2. Lemma. *Let f be an endomorphism of Q and $a, b, c \in Q$. Then $f([a, b, c]) = [f(a, f(b), f(c)]$.*

1.3. Lemma. *Let $a, b, c \in Q$ be such that $[a, b, c] = 0$. Then the subloop generated by these elements is a group.*

1.4. Lemma. *Every commutative Moufang loop is diassociative.*

For every $n \geqq 0$, we define $A_n(Q)$ by $A_0(Q) = Q$, $A_{n+1}(Q) = [A_n(Q), Q, Q]$. In this way, we obtain a series of subloops $Q = A_0(Q) \supseteq A(Q) = A_1(Q) \supseteq A_2(Q) \supseteq \ldots \supseteq A_n(Q) \supseteq \ldots$, which is called the lower central series of Q. Notice that all these subloops are fully characteristic, i.e. invariant under endomorphisms of Q.

Further, put $C(Q) = \{a \in Q; [a, b, c] = 0$ for all $a, b, c \in Q\}$. Then $C(Q)$ is a normal subloop of Q called the centre of Q. Now, the upper central series of Q, $0 = C_0(Q) \subseteq C(Q) = C_1(Q) \subseteq C_2(Q) \subseteq \ldots \subseteq C_n(Q) \subseteq \ldots$, is defined by $C_{n+1}(Q)/C_n(Q) = C(Q/C_n(Q))$. Notice that all these subloops are normal and characteristic, i.e. invariant under automorphisms of Q.

1.5. Lemma. *Let $n \geqq 0$. Then $A_n(Q) = 0$ iff $C_n(Q) = Q$.*

A commutative Moufang loop Q with $A_n(Q) = 0$ is said to be nilpotent of class at most n. Q is said to be 3-elementary if it satisfies the identity $3x = 0$.

1.6. Lemma. *Both $A(Q)$ and $Q/C(Q)$ are* 3*-elementary.*

1.7. Lemma. *Every commutative Moufang loop which can be generated by n elements is nilpotent of class at most* $\max(1, n - 1)$.

1.8. Lemma. *Every simple commutative Moufang loop is associative.*

For all $a, b \in Q$ we define a transformation $i_{a,b}$ of Q by $i_{a,b}(x) = ((x + a) + b) - (a + b)$ for all $x \in Q$.

1.9. Lemma. *Let $a, b, c \in Q$. Then:*

(i) *$i_{a,b}$ is an automorphism of Q.*
(ii) $i_{a,b}(c) = c + [c, a, b]$.
(iii) *$fi_{a,b} = i_{f(a),f(b)}f$ for every endomorphism of f Q.*

The automorphisms $i_{a,b}$, $a, b \in Q$, generate a subgroup $\mathrm{Inn}(Q)$ of the automorphism group $\mathrm{Aut}(Q)$. This subgroup is normal and is called the group of inner automorphisms.

1.10. Lemma. *Let P be a subloop of Q. The following conditions are equivalent:*

(i) *P is a normal subloop of Q.*
(ii) $i_{a,b}(P) \subseteq P$ *for all* $a, b \in Q$.
(iii) $i_{a,b}(P) = P$ *for all* $a, b \in Q$.
(iv) $[x, a, b] \in P$ *for all* $x \in P$, $a, b \in Q$.

1.11. Lemma. *Let P be a normal subloop of Q and K be a characteristic subloop of P. Then K is a normal subloop of Q.*

1.12. Lemma. *Let P, K be subloops of Q and L be the subloop generated by $P \cup K$. If $A(Q) \subseteq P$ or $P \subseteq C(Q)$ then $L = \{a + b; a \in P, b \in K\}$.*

1.13. Lemma. *Let $a, b \in Q$. Then $(a + b) + c = -a + (b + c)$ iff $2a = 0$ and $a \in C(Q)$.*

Proof. We shall prove only the direct implication. We have $(a + b) + (3a + c) = ((a + b) + c) + 3a = (-a + (b + c)) + 3a = 2a + (b + c) = (a + b) + (a + c)$. From this, $3a = a$ and $2a = 0$. However $3a \in C(Q)$ and hence $a \in C(Q)$.

1.14. Lemma. *Let $a, b, c, d \in Q$. The following conditions are equivalent:*

(i) $(a + b) + (c + d) = (a + c) + (b + d)$.
(ii) $[a - b, c - b, d - b] = 0$.
(iii) $[a - c, b - c, d - c] = 0$.
(iv) $[b - a, c - a, d - a] = 0$.
(v) $[a - d, b - d, c - d] = 0$.

Proof. (i) $\Leftrightarrow$ (ii) Clearly, (i) is equivalent to $a + ((c + d) - b) = ((a + c) - b) + d$ and this is equivalent to $(a - b) + ((c - b) + (d - b)) = ((a - b) + (c - b)) + (d - b)$.
The rest is similar.

1.15. Lemma. *Let $a, b, c, d \in Q$ be such that $(a + b) + (c + d) = (a + c) + (b + d)$. Then $(x + y) + (u + v) = (x + u) + (y + v)$ for all $x, y, u, v \in \{a, b, c, d\}$.*

Proof. An easy consequence of 1.14.

1.16. Lemma. *Let $a, b \in Q$. The following conditions are equivalent:*

(i) $(a + b) + (x + y) = (a + x) + (b + y)$ *for all* $x, y \in Q$.
(ii) $(a + x) + (b + y) = (a + y) + (b + x)$ *for all* $x, y \in Q$.
(iii) $a - b \in C(Q)$.

Proof. It follows from 1.14.

1.17. Lemma. $((x + y) + z) + y = x + (2y + z)$ *for all* $x, y, z \in Q$.

Proof. Put $u = ((x + y) + z) + y$ and $v = x + (2y + z)$. Then $u - 3y = ((x +$

$+ y) + z) - 2y = x + (z - y) = (x + ((z - y) + 3y)) - 3y = (x + (2y + z)) - 3y = v - 3y$. Consequently, $u = v$.

1.18. Lemma. *Let $a, b \in Q$ be such that $a + b \in C(Q)$. Then $(a + b) + x) = a + (b + x)$ for all $x \in Q$.*

Proof. We have $x = (-b) + (b + x) = (a + (-a - b)) + (b + x) = (-a - b) + (a + (b + x))$ and hence $(a + b) + x = a + (b + x)$.

2. Central mappings of commutative Moufang loops

Throughout this section, Q will always denote an additively written commutative Moufang loop.

A transformation f of Q is said to be n-central ($n \in \mathbf{Z}$) if $f(x) + nx \in C(Q)$ for every $x \in Q$. Further, f is called central if it is n-central for some $n \in \mathbf{Z}$.

2.1. Lemma. *Let $n \in \mathbf{Z}$, $n = 3k + m$, $m \in \{0, 1, 2\}$, and f be a transformation of Q. Then:*

(i) *f is a-central iff it is m-central.*

(ii) *If f is central and Q is non-associative then there is exactly one $r \in \{0, 1, 2\}$ such that f is r-central.*

Proof. (i) follows immediately from the fact that $3x \in C(Q)$ for every $x \in Q$ and (ii) is obvious.

Obviously, for any $k \in \mathbf{Z}$, the mapping $x \to kx$ is $(-k)$-central. Hence $x \to x$ is 2-central, $x \to 2x$ is 1-central etc.

2.2. Lemma. *Let $m, n \in \mathbf{Z}$ and f, g be endomorphisms of Q such that f is m-central and g is n-central. Then:*

(i) *fg is $(-mn)$-central.*

(ii) *The mapping $f + g$ (defined by $(f + g)(x) = f(x) + g(x)$ for every $x \in Q$) is a $(m + n)$-central endomorphism of Q.*

(iii) *If f is an automorphism of Q then f^{-1} is a m-central automorphism of Q.*

Proof. Let $x, y \in Q$ be arbitrary.

(i) Since $fg(x) + mg(x) \in C(Q)$ and $-g(mx) - nmx \in C(Q)$, we have $fg(x) - nmx \in C(Q)$.

(ii) Define $a = -f(x) - nx$, $b = -g(x) - nx$, $c = -f(y) - my$, $d = -g(y) - ny$. Then $a, b, c, d \in C(Q)$ and $((f(x) + g(x)) + (f(y) + g(y))) + (a + b + c + d) = -mx - nx - my - ny = -mx - my - nx - ny = ((f(x) + f(y)) + (g(x) + g(y))) + (a + b + c + d)$. Hence $(f + g)(x) + (f + g)(y) = (f + g)(x + y)$. Further, $((f(x) + g(x)) + (m + n)x) + (a + b) = 0$, however $a + b \in C(Q)$ and so $(f + g)(x) + (m + n)x \in C(Q)$.

(iii) With respect to 2.1 (i), we may assume that $m \in \{0, 1, 2\}$. If $m = 0$ then Q is associative and there is nothing to prove. If $m = 1$ then $f^{-1}(x) + x = f(f^{-1}(x)) + f^{-1}(x) \in C(Q)$. Finally, if $m = 2$ then $x + 2f^{-1}(x) \in C(Q)$, hence $-x + f^{-1}(x) \in C(Q)$ and consequently $f^{-1}(x) + 2x \in C(Q)$.

Denote by Cent (Q) the set of all central endomorphisms of Q and by Caut (Q) the set of all central automorphisms of Q.

2.3. Lemma. Cent (Q) *is an associative ring with unit and* Caut (Q) *is a normal subgroup of* Aut (Q).

Proof. Let f, g, h be endomorphisms of Q such that f is m-central, g is n-central and h is k-central. By 2.2, $f + g \in$ Cent (Q) and $fg \in$ Cent (Q). For arbitrary $x \in Q$ define $a = -f(x) - mx$, $b = -g(x) - nx$, $c = -h(x) - kx$. Then $a, b, c \in C(Q)$ and $((f(x) + g(x)) + h(x)) + (a + b + c) = -mx - nx - kx = (f(x) + (g(x) + h(x))) + (a + b + c)$. Hence $((f + g) + h)(x) = (f + (g + h))(x)$. The distributive laws are obvious. Finally, if f is an automorphism and $t \in$ Aut (Q) is arbitrary then, for every $x \in Q$, $ft(x) + nt(x) \in C(Q)$ and hence $t^{-1}ft(x) + nx = t^{-1}(ft(x) + nt(x)) \in C(Q)$.

2.4. Remark. Suppose that Q is non-associative. For every $f \in$ Cent (Q) there is uniquely determined $\Psi(f) \in \{0, 1, 2\} = \mathbf{Z}_3$ such that f is $\Psi(f)$-central. Now the mapping Φ: Cent $(Q) \to \mathbf{Z}_3$, defined by $\Phi(f) = -\Psi(f)$, is a projective ring homomorphism.

3. Quasimodules

Throughout this section, R will always denote a non-trivial associative ring with unit $1 = 1_R$.

A quasimodule (more precisely, a unitary left R-quasimodule) Q is a universal algebra $Q = Q(+, \varrho)$ with one binary operation $+$ and a set of unary operations $x \to \varrho x$, $\varrho \in R$, satisfying the following identities:

(1) $$(x + x) + (y + z) = (x + y) + (x + z) .$$
(2) $$(-1)\,x + (x + y) = y .$$
(3) $$\varrho(x + y) = \varrho x + \varrho y \quad \text{for all} \quad \varrho \in R .$$
(4) $$(\varrho + \sigma)\,x = \varrho x + \sigma x \quad \text{for all} \quad \varrho, \sigma \in R .$$
(5) $$\varrho(\sigma x) = (\varrho\sigma)\,x \quad \text{for all} \quad \varrho, \sigma \in R .$$
(6) $$1x = x .$$
(7) $$0x = 0y .$$

For $\varrho \in R$, the mapping $x \to \varrho x$ will often be denoted by f_ϱ. Clearly, f_ϱ is an endomorphism of $Q(+)$ for every $\varrho \in R$.

3.1. Lemma. *Let Q be a quasimodule. Then*:

(i) $Q(+)$ *is a commutative Moufang loop.*
(ii) $0 = 0a$, $a \in Q$, *is the neutral element of* $Q(+)$.

Proof. By (4), (6) and (7), 0 is the neutral element of $Q(+)$. Further, (2) implies that $Q(+)$ is a left cancellation groupoid. Since $(-1)(-1x) = x$ by (5), from (2) follows that $Q(+)$ is a left quasigroup. By (1), $(x + x) + y = (x + y) + x$ and $(x + x) + z = x + (x + z)$. Hence $x + (x + y) = (x + y) + x$ and the operation $+$ is commutative. Thus $Q(+)$ is a commutative Moufang loop.

The class $\mathcal{G} = {}_R\mathcal{G}$ of all quasimodules is a variety of universal algebras and contains as a subvariety the class $\mathcal{M} = {}_R\mathcal{M}$ of all modules (more precisely, unitary left R-modules). $\mathcal{M}$ is in $\mathcal{G}$ defined by the identity

(8) $$x + (y + z) = (x + y) + z .$$

Now, let Φ: $R \to \mathbf{Z}_3$ be a non-zero homomorphism (if such a homomorphism exists then $3R \neq R$, $3R \subseteq \operatorname{Ker} \Phi$, $\operatorname{Ker} \Phi$ is a maximal left and right ideal and $\Phi(1) = 1$).

A quasimodule Q is said to be Φ-central if it satisfies the identity

(9) $$\varrho x + (y + z) = (\varrho x + y) + z \quad \text{for all} \quad \varrho \in \operatorname{Ker} \Phi .$$

This identity says that $\varrho Q \subseteq C(Q)$ for all $\varrho \in \operatorname{Ker} \Phi$. The variety of all Φ-central quasimodules clearly contains $\mathcal{M}$ as a subvariety and will be denoted by $\mathcal{C}_\Phi = {}_R\mathcal{C}_\Phi$.

A quasimodule Q is said to be Φ-primitive if it satisfies the identity

(10) $$\varrho x = 0 \quad \text{for all} \quad \varrho \in \operatorname{Ker} \Phi .$$

Clearly, every Φ-primitive quasimodule is Φ-central. The class of all Φ-primitive quasimodules is a variety which will be denoted by $\mathcal{P}_\Phi = {}_R\mathcal{P}_\Phi$.

3.2. Remark. Let Q be Φ-central and $\sigma \in R$. Since $\Phi(\sigma - m\,1_R) = 0$, where $m = \Phi(\sigma) \in \{0, 1, 2\}$, it is clear that the mapping f_σ is a $(-m)$-central endomorphism of the loop $Q(+)$.

3.3. Remark. Let Q be a non-associative quasimodule such that every f_σ is $\Psi(\sigma)$-central, $\Psi(\sigma) \in \mathbf{Z}_3$. Then the mapping φ: $R \to \operatorname{Cent}(Q)$, defined by $\varphi(\sigma) = f_\sigma$ for every $\sigma \in R$, is a ring homomorphism and Q is $\Phi\varphi$-central, where Φ: $\operatorname{Cent}(Q) \to \mathbf{Z}_3$ is the ring homomorphism defined in 2.4.

A quasimodule is said to be central if all the endomorphisms f_σ, $\sigma \in R$, are central. The class of all central quasimodules will be denoted by $\mathcal{C} = {}_R\mathcal{C}$. From the preceding remarks immediately follows that $\mathcal{C} = \cup \mathcal{C}_\Phi$, Φ ranging over all projective homomorphisms of R onto $\mathbf{Z}_3$. Moreover, $\mathcal{C}_{\Phi_1} \cap \mathcal{C}_{\Phi_2} = \mathcal{M}$, whenever $\Phi_1 \neq \Phi_2$.

3.4. Remark. Suppose that Φ, Ψ, Γ are projective homomorphisms of R onto $\mathbf{Z}_3$ and P, Q are non-associative quasimodules such that P is Φ-central, Q is Ψ-central and $P \times Q$ is Γ-central. Let $\sigma \in R$ be arbitrary and denote $n = -\Phi(\sigma)$, $m = -\Psi(\sigma)$ and $k = -\Gamma(\sigma)$. Since $C(P + Q) = C(P) \times C(Q)$, we have $\sigma x + kx \in C(P)$ and

$\sigma y + ky \in C(Q)$ for all $x \in P$, $y \in Q$, however $\sigma x + nx \in C(P)$, $\sigma y + my \in C(Q)$ and so $n = m = k$. Thus we have proved that $\Phi = \Psi = \Gamma$.

3.5. Remark. Given Φ, the variety of Φ-primitive quasimodules is equivalent to the variety of 3-elementary commutative Moufang loops. The equivalence is given by $Q(+, \varrho) \to Q(+)$ and $P(+) \to P(+, \varrho_P)$, where, for every 3-elementary commutative Moufang loop $P(+)$, $\varrho \in R$ and $x \in P$, we put $\varrho_P x = 0$ if $\Omega(\varrho) = 0$, $\varrho_P(x) = x$ if $\Phi(\varrho) = 1$ and $\varrho_P(x) = -x$ if $\Phi(\varrho) = 2$.

4. Examples

4.1. Example. Every commutative Moufang loop can be considered as a Φ-central $\mathbf{Z}$-quasimodule, where Φ: $\mathbf{Z} \to \mathbf{Z}_3$ is the natural projection.

4.2. Example. Every non-associative commutative Moufang loop Q is a Φ-central Cent (Q)-quasimodule, where Φ: Cent $(Q) \to \mathbf{Z}_3$ is defined as in 2.4.

4.3. Example. Let Φ: $R \to \mathbf{Z}_3$ be a projective homomorphism. Put $Q = \mathbf{Z}_3^4$ and define $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 + (x_1 y_2 - x_2 y_1)(x_3 - y_3))$. It is well known and easy to check that Q under this addition is a non-associative commutative Moufang loop. Putting $\sigma x = 0$ for all $\sigma \in \operatorname{Ker} \Phi$, $\sigma x = x$ if $\Phi(\sigma) = 1$ and $\sigma x = -x$ if $\Phi(\sigma) = 2$, we obtain a Φ-primitive R-quasimodule.

4.4. Example. Put $R = \mathbf{Z}_3^2$ and let Φ, Ψ denote the first and the second projection of R onto $\mathbf{Z}_3$, respectively. If Q and P are the Φ-central and Ψ-central quasimodule constructed in 4.3, respectively, then 3.4 implies that the quasimodule $Q \times P$ is not central.

4.5. Remark. It is not clear under which conditions there is a nonassociative quasimodule. However, if $3R = R$ (i.e. $1_R = 3\varrho$ for some $\varrho \in R$) then every quasimodule is a module. This follows from the fact that $3a = 0$ for every $a \in A(Q)$.

5. Basic properties of quasimodules

Throughout this section, let R be a non-trivial associative ring with unit.

Most of the notions concerning modules immediately carry over to quasimodules. Here we shall only recall those concepts where some difficulties arise.

A subquasimodule P of a quasimodule Q is said to be normal if it is a block of some congruence of Q (considered as universal algebra).

5.1. Lemma. *A subquasimodule P of a quasimodule Q is normal iff $P(+)$ is a normal subloop of the additive loop $Q(+)$.*

Proof. The direct implication is trivial. For the converse, let $P(+)$ be normal in $Q(+)$ and r be the corresponding congruence. Then $(a, b) \in r$ iff $a - b \in P$ and hence r is a congruence of the quasimodule Q.

5.2. Lemma. *Let P be a subquasimodule of a quasimodule Q and S be the least normal subloop of $Q(+)$ containing P. Then S is a normal subquasimodule of Q.*

Proof. From 1.10 easily follows that S is just the subloop generated by all $f(x)$, $f \in \text{Inn}\,(Q(+))$, $x \in P$. However, it is easy to see that for every $\varrho \in R$ we have $\varrho f(x) = g(\varrho x)$ for suitable $g \in \text{Inn}\,(Q(+))$. Hence S is a subquasimodule which is normal by 5.1.

Let P_i, $i \in I$, be a collection of subquasimodules of a quasimodule Q. By $\sum P_i$ (resp. $\sum^0 P_i$) we shall denote the subquasimodule (resp. the normal subquasimodule) generated by $\cup P_i$. Clearly, this (normal) subquasimodule coincides with the (normal) subloop of $Q(+)$ generated by $\cup P_i$.

5.3. Proposition. *Let Q be a quasimodule. Then all $A_n(Q)$ are normal subquasimodules and $Q/A(Q)$ is a module.*

Proof. The subloops $A_n(Q)$ are fully characteristic and hence they are subquasimodules. The normality follows from 5.1.

5.4. Proposition. *Let Q be a central quasimodule. Then*:

(i) *$A(Q)$ is a primitive quasimodule.*
(ii) *Every subloop of $Q(+)$ contained in $A(Q)$ is a subquasimodule.*
(iii) *If Q is non-associative then* card $(A(Q)) \geqq 3$.

Proof. For every $\varrho \in \text{Ker}\,\Phi$, $x \to \varrho x$ is an endomorphism of $Q(+)$ whose image is contained in $C(Q)$ and hence is associative. Thus $\varrho A(Q) = 0$. The rest is clear.

5.5. Proposition. *Let Q be a central quasimodule. Then*:

(i) *$C(Q)$ is a normal subquasimodule of Q and the quasimodule $Q/C(Q)$ is primitive.*
(ii) *Every subloop of $Q(+)$ containing $C(Q)$ is a subquasimodule.*
(iii) *If Q is non-associative then* card $(Q/C(Q)) \geqq 27$.

Proof. (i) For every $\varrho \in R$, $x \in C(Q)$ we have either $\varrho x \in C(Q)$ or $\varrho x - x \in C(Q)$ or $\varrho x + x \in C(Q)$ and consequently $\varrho x \in C(Q)$. (ii) By (i), $Q/C(Q)$ is primitive, however in a primitive quasimodule subloops and subquasimodules coincide. (iii) This is true for every non-associative commutative Moufang loop.

5.6. Corollary. *If Q is a central quasimodule then all $C_n(Q)$ are normal subquasimodules.*

5.7. Proposition. *Every simple central quasimodule is a module.*

Proof. Let Q be a non-zero simple central quasimodule. Suppose that $C(Q) \neq Q$. Since $C(Q)$ is a normal subquasimodule, we have $C(Q) = 0$ and so Q is primitive by 5.5. However then $Q(+)$ is a simple 3-elementary commutative Moufang loop and consequently an Abelian group.

5.8. Proposition. *Let Q be a central quasimodule. Then every inner automorphism of the loop $Q(+)$ is an automorphism of the quasimodule Q.*

Proof. Let $a, b \in Q$, $\varrho \in R$ and $\sigma = -\Phi(\varrho)\, 1_R \in R$. Then $(\varrho + \sigma)\, x \in C(Q)$ for every $x \in Q$. Put $c = (\varrho + \sigma)\, a$, $d = (\varrho + \sigma)\, b$. We have $\sigma i_{a,b}(x) + i_{a,b}(\varrho x) = i_{a,b}((\varrho + \sigma)\, x) = (\varrho + \sigma)\, x$ and $(\varrho + \sigma)\, i_{a,b}(x) = i_{c,d}((\varrho + \sigma)\, x) = (\varrho + \sigma)\, x$. Hence $i_{a,b}(\varrho x) = \varrho i_{a,b}(x)$.

5.9. Lemma. *Let Q be a central quasimodule generated by a set M. Then*:

(i) *If $N \subseteq Q$ is such that $[N, M, M] = 0$ then $N \subseteq C(Q)$.*
(ii) If $[M, M, M] = 0$ *then Q is a module.*

Proof. (i) Repeatedly using 5.8, we obtain $[N, M, Q] = 0$ and $[N, Q, Q] = 0$. (ii) It follows from (i) and 5.5(i).

Let Q be a quasimodule. By o(Q) we denote the least cardinality of a generator set of Q.

5.10. Lemma. *Every cyclic quasimodule Q (i.e. $o(Q) \leqq 1$) is a module.*

Proof. If Q is generated by $\{a\}$ then $Q = Ra = \{\varrho a; \varrho \in R\}$.

5.11. Lemma. *Every central quasimodule with $o(Q) \leqq 2$ is a module.*

Proof. It follows from 5.9.

5.12. Corollary. *Every central quasimodule is diassociative.*

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