Acta Universitatis Carolinae. Mathematica et Physica

M. Matyáš; Vladislav Čápek
Derivation of explicit dispersion relation for gyrotropic waveguide

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 1, 45--54

Persistent URL: `http://dml.cz/dmlcz/142605`

Derivation of Explicit Dispersion Relation for Gyrotropic Waveguide

M. MATYÁŠ, JR., V. ČÁPEK

Institute of Physics, Charles University, Prague*)

Received 12 September 1988

Mathematical details of derivation of the dispersion relation for a three-layers gyrotropic waveguide are presented.

Jsou uvedeny matematické detaily odvození dispersních vztahů pro třívrstvový gyrotropní vlnovod.

Математические детали вывода соотношений дисперсии представляются для случая гиротропического трех-слойного волновода.

Introduction

Let us assume a waveguide formed by a two-dimensional structure consisting of three layers: the substrate (medium I), the film (medium II) and the superstratum (medium III) – Fig. 1. In [1], Yamamoto et al. derived the dispersion relation for guided modes in the isotropic case. In [2], we extended the treatment to the gyrotropic case and discussed it physically; mathematical details of the complicated derivation have, however, not been published in detail. This is the aim of the present work.

Let us as usual assume the substrate and superstratum isotropic with the scalar permitivity

(1a) $$\varepsilon_{\mathrm{I}} = \varepsilon_0 n_{\mathrm{I}}^2 \,,$$

(1b) $$\varepsilon_{\mathrm{III}} = \varepsilon_0 n_{\mathrm{III}}^2 \,.$$

The film is gyrotropic. In the longitudinal configuration (with the magnetization parallel to the direction of light propagation, i.e. the z-axis), its permitivity tensor has the form

(2) $$[\varepsilon] = \varepsilon_0[K] = \varepsilon_0 \begin{bmatrix} K_{\mathrm{II}xx} & K_{\mathrm{II}xy} & 0 \\ K_{\mathrm{II}xy}^* & K_{\mathrm{II}yy} & 0 \\ 0 & 0 & K_{\mathrm{II}zz} \end{bmatrix}.$$

*) Ke Karlovu 5, 121 16 Praha 2, Czechoslovakia

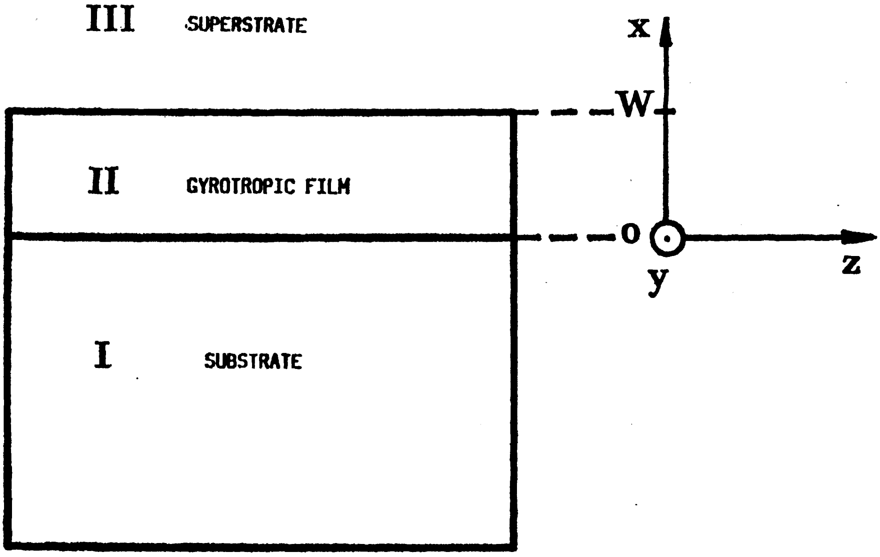

Fig. 1. Waveguide configuration.

Mode Equation

Solution of the guided wave problem is obtained from Maxwell's curl equations

(3a) $$\nabla \times \mathbf{E} = -\mu_0 \, \partial \mathbf{H}/\partial t ,$$

(3b) $$\nabla \times \mathbf{H} = \varepsilon_0 [K] \, \partial \mathbf{E}/\partial t .$$

Combination of (3a) and (3b) gives the wave equation for the electric field

(4) $$\nabla(\nabla \mathbf{E}) - \nabla^2 \mathbf{E} + \mu_0 \, \varepsilon_0 [K] \, \partial^2 \mathbf{E}/\partial t^2 = 0 .$$

We consider a two-dimensional problem having no variation along the y-axis, i.e. all $\partial/\partial y = 0$. The electric and magnetic fields can be written as

(5) $$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{e}(x) \\ \mathbf{h}(x) \end{bmatrix} \exp\left[i(\omega t - \beta z)\right] ,$$

where ω is the angular frequency and β is the propagation constant in the z-direction. We take $\mathbf{e}(x)$ as

(6) $$\begin{bmatrix} e_x(x) \\ e_y(x) \\ e_z(x) \end{bmatrix} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \exp\left(i\lambda x\right) .$$

Then Eq. (4) can be written as ($K_{Lij} = \delta_{ij} n_L^2$, $L =$ I, III)

$$\begin{bmatrix} \beta^2 - k_0^2 K_{xx} & -k_0^2 K_{xy} & \beta\lambda \\ -k_0^2 K_{xy}^* & \beta^2 - k_0^2 K_{yy} + \lambda^2 & 0 \\ \beta\lambda & 0 & -k_0^2 K_{zz} + \lambda^2 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = 0\,, \tag{7}$$

where $k_0^2 = \omega^2 \varepsilon_0 \mu_0$. Distributions of the electric and magnetic fields result from Eq. (7) using (6), (5) and (3a):

a) distribution of the electric field in the x-direction in all three regions:

region I:

$$e_x(x) = \frac{\beta}{\omega \varepsilon_0 n_{\rm I}^2} C_{\rm I}^M \exp(p_{\rm I} x)\,, \tag{8a}$$

$$e_y(x) = C_{\rm I}^E \exp(p_{\rm I} x)\,, \tag{8b}$$

$$e_z(x) = \frac{p_{\rm I}}{i\omega \varepsilon_0 n_{\rm I}^2} C_{\rm I}^M \exp(p_{\rm I} x)\,, \tag{8c}$$

where

$$p_{\rm I}^2 = \beta^2 - k_0^2 n_{\rm I}^2 \tag{8d}$$

and $C_{\rm I}^M$ and $C_{\rm I}^E$ are constants.

region II:

$$e_x(x) = D_1 \cos(h_{\rm II}^{(m)} x + \varphi_1) + D_2 \cos(h_{\rm II}^{(e)} x + \varphi_2)\,, \tag{9a}$$

$$e_y(x) = \frac{k_0^2 K_{{\rm II}xy}^*}{\beta^2 - k_0^2 K_{{\rm II}yy} + h_{\rm II}^{(m)2}} D_1 \cos(h_{\rm II}^{(m)} x + \varphi_1) + \tag{9b}$$
$$+ \frac{k_0^2 K_{{\rm II}xy}^*}{\beta^2 - k_0^2 K_{{\rm II}yy} + h_{\rm II}^{(e)2}} D_2 \cos(h_{\rm II}^{(e)} x + \varphi_2)\,,$$

$$e_z(x) = \frac{i\beta h_{\rm II}^{(m)}}{k_0^2 K_{{\rm II}zz} - h_{\rm II}^{(m)2}} D_1 \sin(h_{\rm II}^{(m)} x + \varphi_1) + \tag{9c}$$
$$+ \frac{i\beta h_{\rm II}^{(e)}}{k_0^2 K_{{\rm II}zz} - h_{\rm II}^{(e)2}} D_2 \sin(h_{\rm II}^{(e)} x + \varphi_2)\,,$$

where for TM modes

$$\begin{aligned} h_{\rm II}^{(m)2} = {} & (k_0^2 K_{{\rm II}xx}(K_{{\rm II}zz} + K_{{\rm II}yy}) - \beta^2(K_{{\rm II}xx} + K_{{\rm II}zz}) + \\ & + \{[\beta^2(K_{{\rm II}xx} - K_{{\rm II}zz}) - k_0^2 K_{{\rm II}xx}(K_{{\rm II}yy} - K_{{\rm II}zz}) + \\ & + k_0^2 K_{{\rm II}xy} K_{{\rm II}xy}^*]^2 + 4k_0^2 \beta^2 K_{{\rm II}xy} K_{{\rm II}xy}^* K_{{\rm II}zz}\}^{1/2})/(2K_{{\rm II}xx}) \end{aligned} \tag{9d}$$

and for TE modes

$$h_{II}^{(e)2} = (k_0^2 K_{IIxx}(K_{IIzz} + K_{IIyy}) - \beta^2(K_{IIxx} + K_{IIzz}) - \tag{9e}$$
$$- \{[\beta^2(K_{IIxx} - K_{IIzz}) - k_0^2 K_{IIxx}(K_{IIyy} - K_{IIzz}) +$$
$$+ k_0^2 K_{IIxy} K_{IIxy}^*]^2 + 4k_0^2 \beta^2 K_{IIxy} K_{IIxy}^* K_{IIzz}\}^{1/2})/(2K_{IIxx})$$

and D_1, D_2, φ_1 and φ_2 are constants.

$$\text{region III:} \quad e_x(x) = \frac{\beta}{\omega\varepsilon_0 n_{III}^2} C_3^M \exp[p_{III}(W - x)], \tag{10a}$$

$$e_y(x) = C_3^E \exp[p_{III}(W - x)], \tag{10b}$$

$$e_z(x) = -\frac{p_{III}}{i\omega\varepsilon_0 n_{III}^2} C_3^M \exp[p_{III}(W - x)] \tag{10c}$$

where

$$p_{III}^2 = \beta^2 - k_0^2 n_{III}^2, \tag{10d}$$

W is the thin gyrotropic film thickness and C_3^E and C_3^M are constants.

b) distribution of magnetic field in x-direction in all three regions:

$$\text{region I:} \quad h_x(x) = -\frac{\beta}{\omega\mu_0} C_1^E \exp(p_I x), \tag{11a}$$

$$h_y(x) = C_1^M \exp(p_I x), \tag{11b}$$

$$h_z(x) = -\frac{p_I}{i\omega\mu_0} C_1^E \exp(p_I x). \tag{11c}$$

$$\text{region II:} \quad h_x(x) = -\frac{\beta}{\omega\mu_0}\left[\frac{k_0^2 K_{IIxy}^*}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)2}} D_1 \cos(h_{II}^{(m)}x + \varphi_1) + \right. \tag{12a}$$
$$\left. + \frac{k_0^2 K_{IIxy}^*}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)2}} D_2 \cos(h_{II}^{(e)}x + \varphi_2)\right],$$

$$h_y(x) = \frac{\beta}{\omega\mu_0}\left[\frac{k_0^2 K_{IIzz}}{k_0^2 K_{IIzz} - h_{II}^{(m)2}} D_1 \cos(h_{II}^{(m)}x + \varphi_1) + \right. \tag{12b}$$
$$\left. + \frac{k_0^2 K_{IIzz}}{k_0^2 K_{IIzz} - h_{II}^{(e)2}} D_2 \cos(h_{II}^{(e)}x + \varphi_2)\right],$$

$$h_z(x) = -\frac{i}{\omega\mu_0}\left[\frac{k_0^2 K_{IIxy}^* h_{II}^{(m)}}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)2}} D_1 \cos(h_{II}^{(m)}x + \varphi_1) + \right. \tag{12c}$$
$$\left. + \frac{k_0^2 K_{IIxy}^* h_{II}^{(e)}}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)2}} D_2 \cos(h_{II}^{(e)}x + \varphi_2)\right].$$

(13a) region III: $h_x(x) = -\frac{\beta}{\omega\mu_0} C_3^E \exp\left[p_{III}(W - x)\right]$,

(13b) $h_y(x) = C_3^M \exp\left[p_{III}(W - x)\right]$,

(13c) $h_z(x) = \frac{p_{III}}{i\omega\mu_0} C_3^E \exp\left[p_{III}(W - x)\right]$.

We express constants C_j^E and C_j^M $(j = 1, 3)$ from the continuity requirement of tangential components of the electric (e_y, e_z) and magnetic (h_y, h_z) fields at boundaries of regions I, II and III:

a) boundary between regions I and II $(x = 0)$:

(14a) e_y components: $$C_1^E = k_0^2 K_{IIxy}^* \left(\frac{D_1 \cos \varphi_1}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)2}} + \frac{D_2 \cos \varphi_2}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)2}}\right).$$

(14b) e_z components: $$C_1^M = -\frac{\omega\varepsilon_0 \beta n_I^2}{p_I}\left(\frac{h_{II}^{(m)} D_1 \sin \varphi_1}{k_0^2 K_{IIzz} - h_{II}^{(m)2}} + \frac{h_{II}^{(e)} D_2 \sin \varphi_2}{k_0^2 K_{IIzz} - h_{II}^{(e)2}}\right).$$

(15a) h_y components: $$C_1^M = \frac{\beta k_0^2 K_{IIzz}}{\omega\mu_0}\left(\frac{D_1 \cos \varphi_1}{k_0^2 K_{IIzz} - h_{II}^{(m)2}} + \frac{D_2 \cos \varphi_2}{k_0^2 K_{IIzz} - h_{II}^{(e)2}}\right).$$

(15b) h_z components: $$C_1^E = -\frac{k_0^2 K_{IIxy}^*}{p_I}\left(\frac{h_{II}^{(m)} D_1 \sin \varphi_1}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)2}} + \frac{h_{II}^{(e)} D_2 \sin \varphi_2}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)2}}\right).$$

b) boundary between regions II and III $(x = W)$:

(16a) e_y components: $$C_3^E = k_0^2 K_{IIxy}^* \left(\frac{D_1 \cos\left(h_{II}^{(m)} W + \varphi_1\right)}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)2}} + \frac{D_2 \cos\left(h_{II}^{(e)} W + \varphi_2\right)}{\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)2}}\right).$$

(16b) e_z components: $$C_3^M = \frac{\omega\varepsilon_0 \beta n_{III}^2}{p_{III}}\left(\frac{h_2^{(m)} D_1 \sin\left(h_{II}^{(m)} W + \varphi_1\right)}{k_0^2 K_{IIzz} - h_{II}^{(m)2}} + \right.$$

$$+\frac{h_{II}^{(e)} D_2 \sin\left(h_{II}^{(e)} W+\varphi_2\right)}{k_0^2 K_{IIzz}-h_{II}^{(e)2}}\Bigg).$$

(17a) h_y components: $$C_3^M=\frac{\beta k_0^2 K_{IIzz}}{\omega \mu_0}\left(\frac{D_1 \cos\left(h_{II}^{(m)} W+\varphi_1\right)}{k_0^2 K_{IIzz}-h_{II}^{(m)2}}+\right.$$

$$\left.+\frac{D_2 \cos\left(h_{II}^{(e)} W+\varphi_2\right)}{k_0^2 K_{IIzz}-h_{II}^{(e)2}}\right).$$

(17b) h_z components: $$C_3^E=\frac{k_0^2 K_{IIxy}^*}{p_{III}}\left(\frac{h_{II}^{(m)} D_1 \sin\left(h_{II}^{(m)} W+\varphi_1\right)}{\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}}+\right.$$

$$\left.+\frac{h_{II}^{(e)} D_2 \sin\left(h_{II}^{(e)} W+\varphi_2\right)}{\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}}\right).$$

From equations (14a), (14b), (15a) and (15b) we determine $D_i \cos\varphi_i$ and $D_i \sin\varphi_i$ $(i=1,2)$ as

(18a) $D_1 \cos\varphi_1=$

$$=\frac{\left(k_0^2 K_{IIzz}-h_{II}^{(m)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}\right)}{\left(k_0^2 K_{IIzz}-h_{II}^{(m)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}\right)-\left(k_0^2 K_{IIzz}-h_{II}^{(e)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}\right)}\times$$

$$\times\left(\frac{\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}}{k_0^2 K_{IIxy}^*} C_1^E-\frac{\omega\mu_0\left(k_0^2 K_{IIzz}-h_{II}^{(e)2}\right)}{\beta k_0^2 K_{IIzz}} C_1^M\right),$$

(18b) $D_2 \cos\varphi_2=$

$$=\frac{\left(k_0^2 K_{IIzz}-h_{II}^{(e)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}\right)}{\left(k_0^2 K_{IIzz}-h_{II}^{(e)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}\right)-\left(k_0^2 K_{IIzz}-h_{II}^{(m)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}\right)}\times$$

$$\times\left(\frac{\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}}{k_0^2 K_{IIxy}^*} C_1^E-\frac{\omega\mu_0\left(k_0^2 K_{IIzz}-h_{II}^{(m)2}\right)}{\beta k_0^2 K_{IIzz}} C_1^M\right),$$

(18c) $D_1 \sin\varphi_1=\frac{p_I}{h_{II}^{(m)}}\times$

$$\times\frac{\left(k_0^2 K_{IIzz}-h_{II}^{(m)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}\right)}{\left(k_0^2 K_{IIzz}-h_{II}^{(e)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}\right)-\left(k_0^2 K_{IIzz}-h_{II}^{(m)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}\right)}\times$$

$$\times\left(\frac{\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}}{k_0^2 K_{xy}^*} C_1^E-\frac{\omega\mu_0\left(k_0^2 K_{IIzz}-h_{II}^{(e)}\right)}{\beta k_0^2 n_I^2} C_1^M\right).$$

(18d) $D_2 \sin\varphi_2=\frac{p_I}{h_{II}^{(e)}}\times$

$$\times\frac{\left(k_0^2 K_{IIzz}-h_{II}^{(e)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}\right)}{\left(k_0^2 K_{IIzz}-h_{II}^{(m)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(e)2}\right)-\left(k_0^2 K_{IIzz}-h_{II}^{(e)2}\right)\left(\beta^2-k_0^2 K_{IIyy}+h_{II}^{(m)2}\right)}\times$$

$$\times \left(\frac{\beta^2 - k_0^2 K_{IIzz} + h_{II}^{(m)^2}}{k_0^2 K_{IIxy}^*} C_1^E - \frac{\omega\mu_0 (k_0^2 K_{IIzz} - h_{II}^{(m)^2})}{\beta k_0^2 n_I^2} C_1^M \right).$$

Introducing equations (18a), (18b), (18c) and (18d) to (16a), (16b), (17a) and (17b) gives

(19a)
$$A \Bigg\{ \Bigg[(k_0^2 K_{IIzz} - h_{II}^{(m)^2})(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)^2}) \left(\cos h_{II}^{(m)} W + \frac{p_I \sin h_{II}^{(m)} W}{h_{II}^{(m)}} \right) -$$
$$- (k_0^2 K_{IIzz} - h_{II}^{(e)^2})(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)^2}) \left(\cos h_{II}^{(e)} W + \frac{p_I \sin h_{II}^{(e)} W}{h_{II}^{(e)}} \right) \Bigg] C_1^E -$$
$$- \frac{(k_0^2 K_{IIzz} - h_{II}^{(m)^2})(k_0^2 K_{IIzz} - h_{II}^{(e)^2}) k_0 K_{IIxy}^*}{\beta\omega\varepsilon_0} \Bigg[\frac{\cos h_{II}^{(m)} W - \cos h_{II}^{(e)} W}{K_{IIzz}} +$$
$$+ \frac{p_I}{n_I^2} \left(\frac{\sin h_{II}^{(m)} W}{h_{II}^{(m)}} - \frac{\sin h_{II}^{(e)} W}{h_{II}^{(e)}} \right) \Bigg] C_1^M \Bigg\} = C_3^E,$$

(19b)
$$A\beta \Bigg\{ \frac{(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)^2})(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)^2})}{k_0^2 K_{IIxy}^*} \times$$
$$[(h_{II}^{(m)} \sin h_{II}^{(m)} W - p_I \cos h_{II}^{(m)} W) - (h_{II}^{(e)} \sin h_{II}^{(e)} W - p_I \cos h_{II}^{(e)} W)] C_1^E -$$
$$- \Bigg[\frac{(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)^2})(k_0^2 K_{IIzz} - h_{II}^{(e)^2})}{\beta\omega\varepsilon_0} \left(\frac{h_{II}^{(m)} \sin h_{II}^{(m)} W}{K_{IIzz}} - \frac{p_I \cos h_{II}^{(m)} W}{n_I^2} \right) -$$
$$- \frac{(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)})(k_0^2 K_{IIzz} - h_{II}^{(m)^2})}{\beta\omega\varepsilon_0} \left(\frac{h_{II}^{(e)} \sin h_{II}^{(e)} W}{K_{IIzz}} - \frac{p_I \cos h_{II}^{(e)} W}{n_I^2} \right) \Bigg] C_1^M \Bigg\} =$$
$$\frac{p_{III}}{\omega\varepsilon_0 K_{IIIzz}} C_3^M,$$

(19c)
$$A\beta\omega\varepsilon_0 K_{IIzz} \Bigg\{ \frac{(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)^2})(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)^2})}{k_0^2 K_{IIxy}^*} \times$$
$$\times \Bigg[\left(\cos h_{II}^{(m)} W + \frac{p_I \sin h_{II}^{(m)} W}{h_{II}^{(m)}} \right) - \left(\cos h_{II}^{(e)} W + \frac{p_I \sin h_{II}^{(e)} W}{h_{II}^{(e)}} \right) \Bigg] C_1^E -$$
$$- \Bigg[\frac{(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(m)^2})(k_0^2 K_{IIzz} - h_{II}^{(e)^2})}{\omega\varepsilon_0\beta} \left(\frac{\cos h_{II}^{(m)} W}{K_{IIzz}} + \frac{p_I \sin h_{II}^{(m)} W}{h_{II}^{(m)} n_I^2} \right) -$$
$$- \frac{(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)^2})(k_0^2 K_{IIzz} - h_{II}^{(m)^2})}{\omega\varepsilon_0\beta} \left(\frac{\cos h_{II}^{(e)} W}{K_{IIzz}} + \frac{p_I \sin h_{II}^{(e)} W}{h_{II}^{(e)} n_I^2} \right) \Bigg] C_1^M \Bigg\} = C_3^M,$$

(19d)
$$A \Bigg\{ \Bigg[h_{II}^{(m)} (k_0^2 K_{IIzz} - h_{II}^{(m)^2})(\beta^2 - k_0^2 K_{IIyy} + h_{II}^{(e)^2}) \left(\sin h_{II}^{(m)} W - \frac{p_I \cos h_{II}^{(m)} W}{h_{II}^{(m)}} \right) -$$

$$- h_{\rm II}^{(e)}(k_0^2K_{{\rm II}zz} - h_{\rm II}^{(e)2})(\beta^2 - k_0^2K_{{\rm II}yy} + h_{\rm II}^{(m)2})\left(\sin h_{\rm II}^{(e)} W - \frac{p_{\rm I}\cos h_{\rm II}^{(e)} W}{h_{\rm II}^{(e)}}\right)\Bigg] \times$$

$$\times\, C_1^E + \frac{(k_0^2K_{{\rm II}zz} - h_{\rm II}^{(m)2})(k_0^2K_{{\rm II}zz} - h_{\rm II}^{(e)2})\, k_0^2K_{{\rm II}xy}^*}{\omega\varepsilon_0\beta} \times$$

$$\times \left[\frac{p_{\rm I}(\cos h_{\rm II}^{(m)} W - \cos h_{\rm II}^{(e)} W)}{n_{\rm I}^2} - \frac{h_{\rm II}^{(m)}\sin h_{\rm II}^{(m)} W - h_{\rm II}^{(e)}\sin h_{\rm II}^{(e)} W}{K_{{\rm II}zz}}\right] C_1^M\Bigg\} = p_3 C_3^E\,,$$

where

$$A = 1 \,/\, [(k_0^2K_{{\rm II}zz} - h_{\rm II}^{(m)2})(\beta^2 - k_0^2K_{{\rm II}yy} + h_{\rm II}^{(e)2}) -$$
$$- (\beta^2 - k_0^2K_{{\rm II}yy} + h_{\rm II}^{(m)2})(k_0^2K_{{\rm II}zz} - h_{\rm II}^{(e)2})]\,.$$

Combining (19a) with (19d) and (19b) with (19c) leads to

(20a)

$$C_1^E \Bigg\{(k_0^2K_{{\rm II}zz} - h_{\rm II}^{(m)2})(\beta^2 - k_0^2K_{{\rm II}yy} + h_{\rm II}^{(e)2})\left[p_{\rm III}\left(\cos h_{\rm II}^{(m)} W + \frac{p_{\rm I}\sin h_{\rm II}^{(m)} W}{h_{\rm II}^{(m)}}\right) +\right.$$
$$\left. + (p_{\rm I}\cos h_{\rm II}^{(m)} W - h_{\rm II}^{(m)}\sin h_{\rm II}^{(m)} W)\right] -$$
$$- (k_0^2K_{{\rm II}zz} - h_{\rm II}^{(e)2})(\beta^2 - k_0^2K_{{\rm II}yy} + h_{\rm II}^{(m)2})\left[p_{\rm III}\left(\cos h_{\rm II}^{(e)} W + \frac{p_{\rm I}\sin h_{\rm II}^{(e)} W}{h_{\rm II}^{(e)}}\right) +\right.$$
$$\left. + (p_{\rm I}\cos h_{\rm II}^{(e)} W - h_{\rm II}^{(e)}\sin h_{\rm II}^{(e)} W)\right]\Bigg\} = C_1^M \frac{\omega\mu_0 K_{{\rm II}xy}^*}{\beta} \times$$
$$\times (k_0^2K_{{\rm II}zz} - h_{\rm II}^{(m)2})(k_0^2K_{{\rm II}zz} - h_{\rm II}^{(e)2})\Bigg\{\left[p_{\rm III}\left(\frac{\cos h_{\rm II}^{(e)} W}{K_{{\rm II}zz}} + \frac{p_{\rm I}\sin h_{\rm II}^{(e)} W}{h_{\rm II}^{(e)} n_{\rm I}^2}\right) +\right.$$
$$\left. + \left(\frac{p_{\rm I}\cos h_{\rm II}^{(e)} W}{n_{\rm I}^2} - \frac{h_{\rm II}^{(e)}\sin h_{\rm II}^{(e)} W}{K_{{\rm II}zz}}\right)\right] - \left[p_{\rm III}\left(\frac{\cos h_{\rm II}^{(m)} W}{K_{{\rm II}zz}} + \frac{p_{\rm I}\sin h_{\rm II}^{(m)} W}{h_{\rm II}^{(m)} n_{\rm I}^2}\right) +\right.$$
$$\left. + \left(\frac{p_{\rm I}\cos h_{\rm II}^{(m)} W}{n_{\rm I}^2} - \frac{h_{\rm II}^{(m)}\sin h_{\rm II}^{(m)} W}{K_{{\rm II}zz}}\right)\right]\Bigg\}\,,$$

(20b) $$C_1^E(\beta^2 - k_0^2K_{{\rm II}yy} + h_{\rm II}^{(m)2})(\beta^2 - k_0^2K_{{\rm II}yy} + h_{\rm II}^{(e)2})\Bigg\{\left[\frac{p_{\rm III}K_{{\rm II}zz}}{n_{\rm III}^2}\left(\cos h_{\rm II}^{(m)} W +\right.\right.$$
$$\left.\left. + \frac{p_{\rm I}\sin h_{\rm II}^{(m)} W}{h_{\rm II}^{(m)}}\right) + (p_{\rm I}\cos h_{\rm II}^{(m)} W - h_{\rm II}^{(m)}\sin h_{\rm II}^{(m)} W)\right] - \left[\frac{p_{\rm III}K_{{\rm II}zz}}{n_{\rm III}^2} \times\right.$$
$$\left.\times \left(\cos h_{\rm II}^{(e)} W + \frac{p_{\rm I}\sin h_{\rm II}^{(e)} W}{h_{\rm II}^{(e)}}\right) + (p_{\rm I}\cos h_{\rm II}^{(e)} W - h_{\rm II}^{(e)}\sin h_{\rm II}^{(e)} W)\right]\Bigg\} =$$

$$= C_1^M \frac{\omega\mu_0 K^*_{\mathrm{II}xy}}{\beta} \Big\{ (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(m)^2}) (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(e)^2}) \Big[\frac{p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{III}}^2} \times$$

$$\times \left(\frac{\cos h_{\mathrm{II}}^{(m)} W}{K_{\mathrm{II}zz}} + \frac{p_{\mathrm{I}} \sin h_{\mathrm{II}}^{(m)} W}{h_{\mathrm{II}}^{(m)} n_{\mathrm{I}}^2} \right) + \left(\frac{p_{\mathrm{I}} \cos h_{\mathrm{II}}^{(m)} W}{n_{\mathrm{I}}^2} - \frac{h_{\mathrm{II}}^{(m)} \sin h_{\mathrm{II}}^{(m)} W}{K_{\mathrm{II}zz}} \right) \Big] -$$

$$- (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(e)^2}) (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(m)^2}) \times$$

$$\times \Big[\frac{p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{III}}^2} \left(\frac{\cos h_{\mathrm{II}}^{(e)} W}{K_{\mathrm{II}zz}} + \frac{p_{\mathrm{I}} \sin h_{\mathrm{II}}^{(e)} W}{h_{\mathrm{II}}^{(e)} n_{\mathrm{I}}^2} \right) +$$

$$+ \left(\frac{p_{\mathrm{I}} \cos h_{\mathrm{II}}^{(e)} W}{n_{\mathrm{I}}^2} - \frac{h_{\mathrm{II}}^{(e)} \sin h_{\mathrm{II}}^{(e)} W}{K_{\mathrm{II}zz}} \right) \Big] \Big\} .$$

We divide (20a) by (20b) assuming that both sides of equation (20b) are non-zero and obtain desired equation of guided modes in gyrotropic waveguide

$$(21) \quad (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(m)^2}) (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(e)^2}) \Big\{ \Big[\left(p_{\mathrm{I}} + \frac{p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{III}}^2} \right) \cos h_{\mathrm{II}}^{(m)} W +$$

$$+ \left(\frac{p_{\mathrm{I}} p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{III}}^2 h_{\mathrm{II}}^{(m)}} - h_{\mathrm{II}}^{(m)} \right) \sin h_{\mathrm{II}}^{(m)} W \Big] - \Big[\left(p_{\mathrm{I}} + \frac{p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{III}}^2} \right) \cos h_{\mathrm{II}}^{(e)} W +$$

$$+ \left(\frac{p_{\mathrm{I}} p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{III}}^2 h_{\mathrm{II}}^{(e)}} - h_{\mathrm{II}}^{(e)} \right) \sin h_{\mathrm{II}}^{(e)} W \Big] \Big\} (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(m)^2}) (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(e)^2}) \times$$

$$\times \Big\{ \Big[\left(\frac{p_{\mathrm{I}}}{n_{\mathrm{I}}^2} + \frac{p_{\mathrm{III}}}{K_{\mathrm{II}zz}} \right) \cos h_{\mathrm{II}}^{(m)} W + \left(\frac{p_{\mathrm{I}} p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{I}}^2 h_{\mathrm{II}}^{(m)}} - \frac{h_{\mathrm{II}}^{(m)}}{K_{\mathrm{II}zz}} \right) \sin h_{\mathrm{II}}^{(m)} W \Big] -$$

$$- \Big[\left(\frac{p_{\mathrm{I}}}{n_{\mathrm{I}}^2} + \frac{p_{\mathrm{III}}}{k_{\mathrm{II}zz}} \right) \cos h_{\mathrm{II}}^{(e)} W + \left(\frac{p_{\mathrm{I}} p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{I}}^2 h_{\mathrm{II}}^{(e)}} - \frac{h_{\mathrm{II}}^{(e)}}{K_{\mathrm{II}zz}} \right) \sin h_{\mathrm{II}}^{(e)} W \Big] \Big\} =$$

$$= \Big\{ (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(m)^2}) (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(e)^2}) \Big[\left(\frac{p_{\mathrm{I}}}{n_{\mathrm{I}}^2} + \frac{p_{\mathrm{III}}}{n_{\mathrm{III}}^2} \right) \cos h_{\mathrm{II}}^{(m)} W +$$

$$+ \left(\frac{p_{\mathrm{I}} p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{I}}^2 n_{\mathrm{III}}^2 h_{\mathrm{II}}^{(m)}} - \frac{h_{\mathrm{II}}^{(m)}}{K_{\mathrm{II}zz}} \right) \sin h_{\mathrm{II}}^{(m)} W \Big] - (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(e)^2}) \times$$

$$\times (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(m)^2}) \Big[\left(\frac{p_{\mathrm{I}}}{n_{\mathrm{I}}^2} + \frac{p_{\mathrm{III}}}{n_{\mathrm{III}}^2} \right) \cos h_{\mathrm{II}}^{(e)} W + \left(\frac{p_{\mathrm{I}} p_{\mathrm{III}} K_{\mathrm{II}zz}}{n_{\mathrm{I}}^2 n_{\mathrm{III}}^2 h_{\mathrm{II}}^{(e)}} - \frac{h_{\mathrm{II}}^{(e)}}{K_{\mathrm{II}zz}} \right) \times$$

$$\times \sin h_{\mathrm{II}}^{(e)} W \Big] \Big\} \Big\{ (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(e)^2}) (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(m)^2}) \Big[(p_{\mathrm{I}} + p_{\mathrm{III}}) \cos h_{\mathrm{II}}^{(m)} W +$$

$$+ \left(\frac{p_{\mathrm{I}} p_{\mathrm{III}}}{h_{\mathrm{II}}^{(m)}} - h_{\mathrm{II}}^{(m)} \right) \sin h_{\mathrm{II}}^{(m)} W \Big] - (\beta^2 - k_0^2 K_{\mathrm{II}yy} + h_{\mathrm{II}}^{(m)^2}) (k_0^2 K_{\mathrm{II}zz} - h_{\mathrm{II}}^{(e)^2}) \times$$

$$\times \left[(p_{\rm I} + p_{\rm III}) \cos h_{\rm II}^{(e)} W + \left(\frac{p_{\rm I} p_{\rm III}}{h_{\rm II}^{(e)}} - h_{\rm II}^{(e)} \right) \sin h_{\rm II}^{(e)} W \right] \Bigg\} .$$

This is the desired complicated dispersion relation as reported in [2]. The reader is referred to this work for its physical discussion as well as correspondence with previously reported special case.

References

[1] Yamamoto S., Koyomada Y., Makimoto, T., J. Appl. Phys. 43 1972, 5090.
[2] Matyáš, M., jr., Čápek V., J. Opt. Soc. Am. 5 1988, 1901.