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Cancellative Relations and Matrices

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In the paper, a connection between cancellative relations and certain matrices is investigated.

V článku se studuje souvislost mezi relacemi s krácením a jistými maticemi.

В статье изучается связь между отношениями со сокращением и некоторыми матрицами.

1. Introduction

Throughout this paper, let N^* denote the set of non-negative integers, N that of positive integers and Z the ring of integers.

Let $n \in N$ and let r be an n-ary relation defined on a set M, i.e. $r \subseteq M^{(n)}$. We denote by $Q_i(r)$, $1 \leqq i \leqq n$, the set of $a \in M$ such that $(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \in$ $\in r$ for some $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in M$ and we put $q_i(r) = \operatorname{card}(Q_i(r))$. The relation r is said to be cancellative if $b = c$ whenever $1 \leqq i \leqq n$, $b_1, \ldots, b_{i-1}$, $b_{i+1}, \ldots, b_n, b, c \in M$ and $(b_1, \ldots, b_{i-1}, b, b_{i+1}, \ldots, b_n), (b_1, \ldots, b_{i-1}, c, b_{i+1}, \ldots, b_n) \in r$.

Let $Q(r) = Q_1(r) \cup \ldots \cup Q_n(r)$ and let S be a semigroup with unit element 1_S. A mapping $\varphi\colon Q(r) \to S$ is said to be a homomorphism if $\varphi(a_1) \ldots \varphi(a_n) = 1_S$ for every $(a_1, \ldots, a_n) \in r$. The homomorphism φ is said to be trivial if $\operatorname{card}(\varphi(Q_i(r))) =$ $= 1$ for each $1 \leqq i \leqq n$.

Let $G(+)$ be an Abelian group and let $1 \leqq i \leqq n$. A mapping $\varphi\colon Q(r) \to G(+)$ is said to be an i-homomorphism if $\varphi(a_1) + \ldots + \varphi(a_{i-1}) + \varphi(a_{i+1}) + \ldots + \varphi(a_n) =$ $= \varphi(a_i)$ for every $(a_1, \ldots, a_n) \in r$.

2. Matrices of special type

Throughout this paper, let R be a non-trivial associative and commutative ring with unit element 1_R. We denote by $P = P(R)$ the prime subring of R, i.e. the subring generated by 1_R. Then P is isomorphic to Z_σ for uniquely determined $\sigma \in N^*$, $\sigma \neq 1$. Notice that σ is a prime number, provided $\sigma \neq 0$ and R is an integral domain.

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For $m, n \in N$, let $\mathfrak{M}(R, m, n)$ designate the set of matrices of type (m, n) over R. Then $\mathfrak{M}(R, m, n)$ is a free R-module of rank equal to mn. If A $(B, \ldots)$ is a matrix then a_{ij} $(b_{ij}, \ldots)$ will be the elements of A $(B, \ldots)$, a_i $(b_i, \ldots)$ the rows and ${}_ja$ $({}_jb, \ldots)$ the columns.

Consider the following conditions for a matrix $A \in \mathfrak{M}(R, m, n)$:

(1) $a_{ij} \in \{0, 1\}$ for all $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$.

(2) For every $1 \leqq j \leqq n$ there exists at least one $1 \leqq i \leqq m$ with $a_{ij} = 1$.

Let $\alpha \in N$, $2 \leqq \alpha \leqq n + 2$.

$(3, \alpha)$ If $1 \leqq i < k \leqq m$ then $a_{ij} = 1 = a_{kj}$ for at most $\alpha - 2$ indices j, $1 \leqq j \leqq n$.

Let $\alpha \in N$, $\alpha \leqq n$ and let $\beta \in N^{(\alpha)}$ be such that $\Sigma\beta = \beta_1 + \ldots + \beta_\alpha = n$.

$(4, \alpha, \beta)$ For every $1 \leqq i \leqq m$ there are exactly α indices $t_1, \ldots, t_\alpha$ with $1 \leqq t_1 \leqq \leqq \beta_1 < t_2 \leqq \beta_1 + \beta_2 < \ldots \leqq \beta_1 + \ldots + \beta_{\alpha-1} < t_\alpha \leqq \beta_1 + \ldots + \beta_\alpha = n$ and $a_{ij} = 1$ iff $j \in \{t_1, \ldots, t_\alpha\}$.

2.1. Lemma. (i) If A satisfies (1) then $A \in \mathfrak{M}(P, m, n)$.

(ii) If A satisfies (1) and (2) then A satisfies $(4, \alpha, \beta)$ for at most one ordered pair (α, β) with $1 \leqq \alpha \leqq n$, $\beta \in N^{(\alpha)}$ and $n = \Sigma\beta$.

Proof. (i) This is obvious.

(ii) Suppose that A satisfies $(4, \alpha, \beta)$ and $(4, \alpha', \beta')$. Let $\beta_1 < \beta'_1$ and $\beta_1 < j \leqq \beta'_1$. Then it is easy to see that the column ${}_ja$ is a zero column, a contradiction with (2). Hence $\beta_1 = \beta'_1$. Similarly, $\beta_2 = \beta'_2$, etc., and therefore $\alpha = \alpha'$.

2.2. Lemma. Let $3 \leqq \alpha \leqq n$, $(n - \alpha + 2)/2 < m \in N$, $\beta \in N^{(\alpha)}$, $\Sigma\beta = n$, be such that there exists a matrix $A \in \mathfrak{M}(R, m, n)$ satisfying (1), $(3, \alpha)$ and $(4, \alpha, \beta)$. Then:

(i) $\operatorname{card}(\{1 \leqq i \leqq \alpha \mid \beta_i = 1\}) \leqq \alpha - 3$.

(ii) $\alpha + 3 \leqq n$ and $3 \leqq m$.

Proof. Easy.

3. Annihilators

Let $A \in \mathfrak{M}(R, m, n)$. We denote by $\operatorname{Mod}(A, R)$ (resp. $\operatorname{Mod}(R, A)$) the submodule generated by the rows (resp. columns) of A in the free R-module $R^{(n)}$ (resp. $R^{(m)}$). Further, let $\operatorname{Ann}(A, R) = \{w \in R^{(n)} \mid Aw = 0\}$ and $\operatorname{Ann}(R, A) = \{w \in R^{(m)} \mid wA = = 0\}$. Then $\operatorname{Ann}(A, R)$ is a submodule of $R^{(n)}$ and $\operatorname{Ann}(R, A)$ is a submodule of $R^{(m)}$.

Now, suppose that $A \in \mathfrak{M}(P, m, n)$. Then there exist invertible matrices $C \in \in \mathfrak{M}(P, m, m)$ and $D \in \mathfrak{M}(P, n, n)$ such that the matrix $B = CAD$ is diagonal and b_{ii} divides $b_{i+1, i+1}$ in P for all $1 \leqq i \leqq k = \min(m, n)$.

3.1. Lemma. $Rb_{11} \supseteq Rb_{22} \supseteq \ldots \supseteq Rb_{kk}$ and both the R-modules $\operatorname{Mod}(A, R)$ and $\operatorname{Mod}(R, A)$ are isomorphic to the outer direct sum $Rb_{11} \oplus \ldots \oplus Rb_{kk}$ of principal ideals of R.

Proof. Obvious.

3.2. Lemma. $(0 : b_{11})_R \subseteq (0 : b_{22})_R \subseteq \ldots \subseteq (0 : b_{kk})_R$, the R-module $\operatorname{Ann}(A, R)$ is isomorphic to the outer direct sum $(0 : b_{11})_R \oplus \ldots \oplus (0 : b_{kk})_R \oplus R^{(n-k)}$ and the R-module $\operatorname{Ann}(R, A)$ is isomorphic to the outer direct sum $(0 : b_{11})_R \oplus \ldots \ldots \oplus (0 : b_{kk})_R \oplus R^{(m-k)}$ of annihilator ideals of R.

Proof. Obvious.

Although the invertible matrices C and D and the diagonal matrix B are not determined uniquely by A, it follows from the next lemma that the number of non-zero elements of B is determined uniquely by A. We denote this number by $\operatorname{rank}(A)$. So, $0 \leqq \operatorname{rank}(A) \leqq \min(m, n)$.

3.3. Lemma. Let $1 \in N$ and let $x_1, \ldots, x_l, y_1, \ldots, y_l \in R$ be such that $Rx_1 \supseteq \ldots \ldots \supseteq Rx_l$, $Ry_1 \supseteq \ldots \supseteq Ry_l$ and the R-modules $X = Rx_1 \oplus \ldots \oplus Rx_l$ and $Y = = Ry_1 \oplus \ldots \oplus Ry_l$ are isomorphic. Then $Rx_1 \cong Ry_1, \ldots, Rx_l \cong Ry_l$.

Proof. We shall proceed by the induction on l. For $l = 1$ there is nothing to prove. Assume $l \geqq 2$ and let φ be an isomorphism of X onto Y. Clearly, $\varphi(x_1) = \lambda y_1$ and $\varphi^{-1}(y_1) = \varrho x_1$ for some $\lambda, \varrho \in R$. Consequently, $\varphi(Rx_1) = Ry_1$ and the R-modules $Rx_2 \oplus \ldots \oplus Rx_l \cong X/Rx_1$ and $Ry_2 \oplus \ldots \oplus Ry_l \cong Y/Ry_1$ are isomorphic.

3.4. Corollary. Suppose that either $\sigma = 0$ or σ is a prime. Then $\operatorname{Ann}(A, R)$ is a free R-module of rank equal to $n - \operatorname{rank}(A)$ and $\operatorname{Ann}(R, A)$ is a free R-module of rank equal to $m - \operatorname{rank}(A)$.

Let $1 \leqq \alpha \leqq n$ and let $\beta \in N^{(\alpha)}$ be such that $\Sigma\beta = n$. We denote by $R(\alpha, \beta)$ the set of all $w \in R^{(n)}$ such that $w_1 = \ldots = w_{\beta_1}, w_{\beta_1+1} = \ldots = w_{\beta_1+\beta_2}, \ldots, w_{\beta_1+\ldots+\beta_{\alpha-1}+1} = \ldots \ldots = w_n$ and $w_{\beta_1} + w_{\beta_1+\beta_2} + \ldots + w_{\beta_1+\ldots+\beta_{\alpha-1}+1} + w_n = 0$. Obviously, $R(\alpha, \beta)$ is a submodule of $R^{(n)}$ and $R(\alpha, \beta)$ is a free R-module of rank $\alpha - 1$.

A matric $A \in \mathfrak{M}(R, m, n)$ will be called (R, α, β)-flat if $\operatorname{Ann}(A, R) = R(\alpha, \beta)$.

3.5. Lemma. Let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$. Then:

(i) $R(\alpha, \beta) \subseteq \operatorname{Ann}(A, R)$.

(ii) $\operatorname{rank}(A) \leqq n - \alpha + 1$.

(iii) $\operatorname{rank}(A) = n - \alpha + 1$ and $m \geqq n - \alpha + 1$, provided A is (R, α, β)-flat.

Proof. (i) This is an easy consequence of $(4, \alpha, \beta)$.

(ii) By (i), $P(\alpha, \beta) \subseteq \operatorname{Ann}(A, P)$ and $P(\alpha, \beta)$ is isomorphic to $P^{(\alpha-1)}$. By 3.3, $\operatorname{Ann}(A, P)$ is isomorphic to $P^{(n-l)} \oplus M$ where $l = \operatorname{rank}(A)$ and $M = = (0 : b_{11})_P \oplus \ldots \oplus (0 : b_{ll})_P$. In particular, $bM = 0$, $b = b_{ll}$. If $\sigma = 0$ then $M = 0$ and clearly $\alpha - 1 \leqq n - l$. Suppose that $\sigma \neq 0$. Then the module $(Pb)^{(n-l)}$ contains a submodule isomorphic to $(Pb)^{(\alpha-1)}$, and hence $r^{\alpha-1} \leqq r^{n-l}$, $r = \operatorname{card}(Pb) \geqq 2$, r finite, so that $\alpha - 1 \leqq n - l$.

(iii) Put $l = \operatorname{rank}(A)$. By (ii), $l \leqq n - \alpha + 1$. By 3.3, $\operatorname{Ann}(A, R)$ possesses a free direct summand of rank $n - l$. But $\operatorname{Ann}(A, R) = R(\alpha, \beta)$ is a free module of rank $\alpha - 1$. Consequently, $n - l \leqq \alpha - 1$ and $n - \alpha + 1 \leqq l$.

3.6. Lemma. Let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$. Suppose that A is (R, α, β)-flat. Then $l = \operatorname{rank}(A) = n - \alpha + 1$ and $(0 : b_{11})_R = \dots$ $\dots = (0 : b_{ll})_R = 0$.

Proof. First, $l = n - \alpha + 1$ by 3.5(iii). Further, by 3.3, $\operatorname{Ann}(A, R)$ is isomorphic to $R^{(\alpha-1)} \oplus M$, $M = (0 : b_{11})_R \oplus \dots \oplus (0 : b_{ll})_R$. On the other hand, $\operatorname{Ann}(A, R) =$ $= R(\alpha, \beta)$ is isomorphic to $R^{(\alpha-1)}$, and therefore $R^{(\alpha-1)}$ is isomorphic to $R^{(\alpha-1)} \oplus M$. We are going to show that $M = 0$. Assume on the contrary that $M \neq 0$. The module M as a direct summand of $R^{(\alpha-1)}$ is finitely generated. Hence $IM \neq M$ for a maximal ideal I of R. Put $S = R/I$, so that S is a field. The S-modules $S^{(\alpha-1)}$ and $S^{(\alpha-1)} \oplus$ $\oplus M/IM$ are isomorphic, hence $M/IM = 0$, $IM = M$, a contradiction.

3.7. Proposition. Let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$. The following conditions are equivalent:
(i) A is (P, α, β)-flat.
(ii) $l = \operatorname{rank}(A) = n - \alpha + 1$ and $(0 : b_{ll})_P = 0$ (then $(0 : b_{11})_P = \dots = (0 : b_{ll})_P =$ $= 0$).

Proof. (i) implies (ii) by 3.6.
(ii) implies (i). First, let $\sigma = 0$ and let Q be a quotient field of P. Then both $Q(\alpha, \beta)$ and $\operatorname{Ann}(A, Q)$ are isomorphic to $Q^{(\alpha-1)}$. Consequently, $Q(\alpha, \beta) = \operatorname{Ann}(A, Q)$ and it is easy to see that A is (P, α, β)-flat. Now, let $\sigma \neq 0$. Then the equality $\operatorname{card}(P(\alpha, \beta)) = \operatorname{card}(\operatorname{Ann}(A, P))$ yields the equality $P(\alpha, \beta) = \operatorname{Ann}(A, P)$.

3.8. Corollary. Suppose that either $\sigma = 0$ or σ is a prime. Let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$. Then A is (P, α, β)-flat iff $\operatorname{rank}(A) = n - \alpha + 1$.

3.9. Proposition. Suppose that R is a domain. Let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$. Then A is (R, α, β)-flat iff $\operatorname{rank}(A) = n - \alpha + 1$.

Proof. Similar to that of 3.7.

4. Auxiliary results

Let $\alpha, n, m \in N$, $\alpha \leqq n$, $\beta \in N^{(\alpha)}$, $\Sigma\beta = n$. Let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$.

4.1. Lemma. Suppose that A is (R, α, β)-flat. Then A satisfies (2).

Proof. Easy.

4.2. Lemma. Let $a_i = a_k$ for some $1 \leqq i < k \leqq m$ and let B be the matrix obtained from A by omitting the k-th row. Then:
(i) $B \in \mathfrak{M}(R, m - 1, n)$ satisfies (1) and $(4, \alpha, \beta)$.
(ii) B satisfies (2) iff A does so.
(iii) $\operatorname{Ann}(A, R) = \operatorname{Ann}(B, R)$.
(iv) B is (R, α, β)-flat iff A is so.

Prcof. Easy.

Now, let $1 \leqq i < k \leqq m$ be such that $a_i \neq a_k$ and $a_{ij_1} = a_{kj_1} = \ldots = a_{ij_{\alpha-1}} = a_{kj_{\alpha-1}} = 1$ for some $1 \leqq j_1 < \ldots < j_{\alpha-1} \leqq n$ (in particular, A does not satisfy (3)). Then there exist $1 \leqq r, s \leqq n$ such that $r \neq s$, $r, s \neq j_1, \ldots, j_{\alpha-1}$ and $a_{ir} = a_{ks} = 1$. Moreover, both r and s are contained in just one from the intervals $[1, \beta_1]$, $[\beta_1 + 1, \beta_1 + \beta_2]$, $\ldots$, $[\beta_1 + \ldots + \beta_{\alpha-1} + 1, n]$. Let $r, s \in [\beta_0 + \ldots + \beta_l + 1, \beta_1 + \ldots + \beta_{l+1}]$, where $\beta_0 = 0$, $0 \leqq l \leqq \alpha - 1$ Put $t = \min(r, s)$ and $u = \max(r, s)$. We shall define a matrix $C \in \mathfrak{M}(R, m - 1, n - 1)$ as follows: $c_{xy} = a_{xy}$ and $c_{zy} = a_{z+1,y}$ for all $1 \leqq x < k \leqq z \leqq m - 1$ and all y such that either $1 \leqq y < t$ or $t < y < u$; $c_{xy} = a_{x,y+1}$ for all $1 \leqq x < k$ and all $u \leqq y \leqq n - 1$; $c_{zy} = a_{x+1,y+1}$ for all $k \leqq x \leqq m - 1$ and $u \leqq y \leqq n - 1$; $c_{xt} + a_{xr} + a_{xs}$ for all $1 \leqq x < k$; $c_{xt} = a_{x+1,r} + a_{x+1,s}$ for all $k \leqq x \leqq m - 1$.

4.3. Lemma. (i) $C \in \mathfrak{M}(R, m - 1, n - 1)$ satisfies (1) and C satisfies (2) iff A does so.
(ii) C satisfies $(4, \alpha, \gamma)$ for $\gamma = (\beta_1, \ldots, \beta_1, \beta_{l+1} - 1, \beta_{l+2}, \ldots, \beta_\alpha)$.
(iii) If $z \in \operatorname{Ann}(A, R)$ then $z_r = z_s$.
(iv) C is (R, α, γ)-flat iff A is (R, α, β)-flat.

Proof. Let $w \in R^{(n-1)}$ and $x \in R^{(n)}$ be such that $x_1 = w_1, \ldots, x_{u-1} = w_{u-1}$, $x_u = w_t$, $x_{u+1} = w_u, \ldots, x_n = w_{n-1}$. Let $Cw = v$ and $Ax = y$. Then $v_1 = y_1, \ldots, v_{k-1} = y_{k-1}$, $v_k = y_{k+1}, \ldots, v_{n-1} = y_n$ and $v_i = y_k = y_i$.

In the next lemma, let $\varrho_1, \ldots, \varrho_m \in R$ and $1 \leqq i \leqq m$ be such that $\varrho_1 a_1 + \ldots + \varrho_m a_m = 0$ and $(0 : \varrho_i)_R = 0$. Denote by D the matrix obtained from A by omitting the i-th row.

4.4. Lemma. (i) $D \in \mathfrak{M}(R, m - 1, n)$ satisfies (1) and $(4, \alpha, \beta)$ and D satisfies (2) iff A does so.
(ii) $\operatorname{Ann}(D, R) = \operatorname{Ann}(A, R)$.
(iii) D is (R, α, β)-flat iff A is so.

Proof. Easy.

5. Determinants

Let $\alpha, n, m \in N$, $2 \leqq \alpha \leqq n$ and $m = n - \alpha + 1$. Let $1 \leqq r_1 < \ldots < r_{\alpha-1} \leqq n$. For a matrix $A \in \mathfrak{M}(R, m, n)$ we denote by $A(r_1, \ldots, r_{\alpha-1})$ the square matrix of type (m, n) obtained from A by omitting the $r_1, \ldots, r_{\alpha-1}$-th columns. Further, define a square matrix $C = A[r_1, \ldots, r_{\alpha-1}]$ of type (n, n) as follows: $c_i = a_i$ for all $1 \leqq i \leqq m$; $c_{m+j,k} = 0$ for all $1 \leqq j \leqq \alpha - 1$ and $1 \leqq k \leqq n$, $k \neq r_j$; $c_{m+j,r_j} = 1$ for all $1 \leqq j \leqq \alpha - 1$.

5.1. Lemma. $\det C = (-1)^l \det B$ for $l = r_1 + \ldots + r_{\alpha-1} + (\alpha - 1) n - (\alpha - 1)(\alpha - 2)/2$.

Proof. Easy.

5.2. Lemma. (i) $w \in \operatorname{Ann}(C, R)$ iff $w \in \operatorname{Ann}(A, R)$ and $w_{r_1} = \ldots = w_{r_{\alpha-1}} = 0$.

Proof. Obvious.

Now, let $\beta \in N^{(\alpha)}$, $\Sigma\beta = n$. Denote by $J(\beta)$ the set of all ordered $\alpha - 1$-tuples $s = (s_1, \ldots, s_{\alpha-1})$ such that $1 \leqq s_1 < \ldots < s_{\alpha-1} \leqq n$ and each from the intervals $[1, \beta_1]$, $[\beta_1 + 1, \beta_1 + \beta_2], \ldots, [\beta_1 + \ldots + \beta_{\alpha-1} + 1, n]$ contains at most one from the numbers $s_1, \ldots, s_{\alpha-1}$. Then exactly $\alpha - 1$ intervals contain each just one of those numbers and one interval contains none.

Let $r = (r_1, \ldots, r_{\alpha-1}) \in J(\beta)$, let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$ and let $B = A(r)$ and $C = A[r]$.

5.3. Lemma. $\operatorname{Ann}(C, R) \cap R(\alpha, \beta) = 0$.

Proof. This is an easy consequence of 5.2.

5.4. Lemma. Suppose that A is (R, α, β)-flat. Then:
(i) $\operatorname{Ann}(C, R) = 0$.
(ii) $\operatorname{rank}(C) = n$.
(iii) $(0 : \det C)_R = 0$.
(iv) If $\sigma \neq 0$ then C is invertible in $\mathfrak{M}(P, n, n)$.
(v) C is invertible in $\mathfrak{M}(S, n, n)$ where S is the classical quotient ring of R.

Proof (i) We have $R(\alpha, \beta) = \operatorname{Ann}(A, R)$, $\operatorname{Ann}(C, R) \subseteq \operatorname{Ann}(A, R)$ and $\operatorname{Ann}(C, R) \cap R(\alpha, \beta) = 0$. Thus $\operatorname{Ann}(C, R) = 0$.
(ii) This is an easy consequence of 3.2.
(iii) This is clear from (i).
(iv) Put $c = \det C$. Then $(0 : c)_R = 0$, so that c is invertible in P and C in $\mathfrak{M}(P, n, n)$.
(v) Similar to (iii).

5.5. Lemma. If $\operatorname{Ann}(C, R) = 0$ then A is (R, α, β)-flat.

Proof. Let $w \in \operatorname{Ann}(A, R)$. We shall restrict ourselves to the case $r_{\alpha-1} \leqq \beta_1 + \ldots + \beta_{\alpha-1}$, the other cases being similar. Consider $x \in R^{(n)}$ such that $x_1 = \ldots = x_{\beta_1} = w_{r_1}$, $x_{\beta_1+1} = \ldots = x_{\beta_1+\beta_2} = w_{r_2}, \ldots, x_{\beta_1+\ldots+\beta_{\alpha-2}+1} = \ldots$

$\ldots = x_{\beta_1+\ldots+\beta_{\alpha-1}} = w_{r_{\alpha-1}}$ and $x_{\beta_1+\ldots+\beta_{\alpha-1}+1} = \ldots = x_n = -w_{r_1} - \ldots - w_{r_{\alpha-1}}$. Then $x \in R(\alpha, \beta)$. Moreover, $w - x \in \operatorname{Ann}(C, R)$ by 5.2(i). Hence $w = x$ and $w \in R(\alpha, \beta)$.

5.6. Proposition. The following conditions are equivalent:
(i) A is (R, α, β)-flat.
(ii) $\operatorname{Ann}(C, R) = 0$.
(iii) $(0 : \det C)_R = 0$.

Proof. (i) implies (ii) and (iii) by 5.4, (ii) implies (i) by 5.5 and the implication (iii) implies (ii) is easy.

5.7. Proposition. Suppose that either $\sigma = 0$ or σ is a prime. The following conditions are equivalent:
(i) A is (P, α, β)-flat.
(ii) $\operatorname{rank}(A) = n - \alpha + 1$.
(iii) $\operatorname{rank}(C) = n$.
(iv) $\det C \neq 0$.
(v) $\det B \neq 0$.

Proof Use 3.8 and the fact that P is an integral domain.
Let $r, s \in J(\beta)$.

5.8. Lemma. Let I be a prime ideal of R. Then $\det A[r] \in I$ iff $\det A[s] \in I$.

Proof. Denote by g the natural homomorphism of R onto $S = R/I$ and suppose that $\det A[r] \in I$. We have $g(A[r]) = g(A)\,[r]$, $0 = g(\det A[r]) = \det g(A)\,[r]$ and $\operatorname{rank}(g(A)) < n - \alpha + 1$ by 5.7 and 3.5(ii). However, now we can proceed conversely and we have $\det A[s] \in I$.

6. Some matrices

Let $4 \leqq m \in N$, m even, $k = m/2$ and $n = m + 2$. Let $B \in \mathfrak{M}(Z, k, k)$ be a matrix satisfying (1), (2) and $(4, 1, k)$ such that $b_{ii} = 0$ for each $1 \leqq i \leqq k$. Notice that each column as well as each row of B contains just one non-zero element. Consider the following matrix $A \in \mathfrak{M}(Z, m, n)$:

$$\left(\begin{array}{c|c|c|c} I_k & I_k & \begin{matrix}1\\ \vdots\\ 1\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline I_k & B & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \end{array}\right),$$

where $I_k \in \mathfrak{M}(R, k, k)$ is the unit matrix.

6.1. Lemma. A satisfies (1), (2), (3, 3) and (4, 3, β) for $\beta = (k, k, 2)$.

Proof. Obvious.

Now, put $C = A(m, n)$ (clearly $(m, n) \in J(A)$). The matrix C has the following form:

$$\left(\begin{array}{c|c|c} I_k & \begin{matrix} I_{k-1} \\ 0 \ldots 0 \end{matrix} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \hline I_k & D & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right),$$

where D is the matrix obtained from B by omitting the last column. It is easy to see that $\det C = \pm \det E$, where E is the following matrix:

$$\left(\begin{array}{c|c|c} I_k & \begin{matrix} I_{k-1} \\ 0 \ldots 0 \end{matrix} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ \hline 0 & D - L & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \end{array}\right).$$

Here, $L = \begin{pmatrix} I_{k-1} \\ 0 \ldots 0 \end{pmatrix} \in \mathfrak{M}(Z, k, k-1)$.

Similarly, $\det C = \pm \det F$, where F is as follows:

$$\left(\begin{array}{c|c|c} I_k & 0 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & D - L & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \end{array}\right).$$

Denote by G the matrix $\left(D - L, \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix}\right) \in \mathfrak{M}(Z, k, k)$.

6.2. Lemma. $\det C = \pm \det G$.

Proof. The result follows easily from the equality $\det C = \pm \det F$.

Let H be the matrix obtained from B by omitting the last row and the last column. Put $K = H - I_{k-1}$.

6.3. Lemma. $\det C = \pm k \det K$.

Proof. If we add in G to the last row the remaining rows, we get the following matrix:

$$\left(\begin{array}{c|c} K & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ \hline 0 \dots 0 & -k \end{array}\right).$$

6.4. Lemma. $\det K \in \{0, 1, -1\}$.

Proof. By induction on $l \in N$, we shall prove such an assertion: Let $V \in \mathfrak{M}(Z, l\ l)$ be a matrix satisfying (1) such that each row as well as each column of V contains at most one non-zero element. Then $\det W \in \{0, 1, -1\}$, where $W = V - I_l$. First, observe that if $v_{ii} = 1$ for some $1 \leqq i \leqq l$ then W contains a zero row, so that $\det W = 0$. Hence assume that $v_{ii} = 0$ for each i. Further, if the matrix V contains a zero row (resp. column), then we can apply the induction hypothesis. So, let each row as well as each column of V contain exactly one non-zero element. Then the sum of all rows of W is equal to zero, so that $\det W = 0$.

There is a permutation μ of the set $\{1, 2, \dots, k\}$ such that $b_{ij} = 1$ iff $j = \mu(i)$.

6.5. Lemma. Suppose that $k \geqq 3$. Then $\det K \neq 0$ iff μ is a cycle.

Proof. Denote by f the natural homomorphism of Z onto $Z/2Z = S$. Clearly, $\det K \neq 0$ iff $\det f(K) \neq 0$. However, $\det f(K) \neq 0$ iff the rows of $f(K)$ are linearly independent over S. It is easy to see that if T is a proper non empty subset of $\{1, 2, \dots, k\}$ then $\sum_{i \in T} f(k_i) = 0$ iff $\mu(T) = T$. The rest is clear.

6.6. Corollary. Let $m \geqq 6$. The matrix A is $(Z, 3, \beta)$-flat iff μ is a cycle.

7. The numbers $\delta(R, I, \alpha)$

Let $1 \leqq \alpha \leqq n$, $\beta \in N^{(\alpha)}$, $\Sigma\beta = n$, $3 \leqq n$, $m = n - \alpha + 1$. Let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and $(4, \alpha, \beta)$. We denote by $\lambda(A)$ the set of all non-zero (proper) prime ideals I of R such that $\det A[r] \in I$ for some (and then for all – see 5.8) $r \in J(\beta)$.

7.1. Lemma. The following conditions are equivalent for a non-zero prime ideal I of R:

(i) $I \in \lambda(A)$.

(ii) $f_I(A)$ is not (S, α, β)-flat where f_I: $R \to R/I = S$ is the natural projection.

(iii) $\operatorname{rank}(f_I(A)) < m$.

Proof. The conditions (ii) and (iii) are equivalent by 3.9 and the conditions (i) and (iii) are equivalent by 5.7.

Let I be a non-zero prime ideal of R and let $2 \leqq \alpha \in N$. We let $\delta(R, I, \alpha)$ designate the minimum of all $m \in N$ such that there exist $\beta \in N^{(\alpha)}$ with $n = \Sigma\beta = m + \alpha - 1$ and a matrix $A \in \mathfrak{M}(R, m, n)$ satisfying (1), (2), (3, α) and (4, α, β) such that $I \in \lambda(A)$ and A is (R, α, β)-flat. We put $\delta(R, I, \alpha) = \infty$ if there exists no matrix with these properties. For a prime $\pi \in N$, let $\delta(\pi, \alpha) = \delta(Z, Z\pi, \alpha)$.

7.2. Lemma. Suppose that R is a domain .Let $m, n \in N$, $\beta \in N^{(\alpha)}$, $\Sigma\beta = n$ and let $A \in \mathfrak{M}(R, m, n)$ be a matrix satisfying (1) and (4, α, β) such that A is (R, α, β)-flat and $f_I(A)$ is not $(R/I, \alpha, \beta)$-flat. Then $\delta(R, I, \alpha) \leqq m$.

Proof. First, A satisfies (2) by 4.1. Further, with respect to 4.2 and 4.3, we can assume that A satisfies (3, α). By 3.5, rank $(A) = n - \alpha + 1 \leqq m$. On the other hand, with regard to 4.4, we can assume that the rows of A are linearly independent over R. Then they are linearly independent over the quotient field of R, so that $m \leqq$ rank (A) and $m = n - \alpha + 1$.

7.3. Proposition. Let $2 \leqq \alpha$, $\pi \in N$ and let π be a prime. Then $\log_\alpha \pi < \delta(\pi, \alpha)$.

Proof. Let $\beta \in N^{(\alpha)}$, $\Sigma\beta = n = m + \alpha - 1$ and let $A \in \mathfrak{M}(Z, m, n)$ be a matrix satisfying (1), (2), (3, α) and (4, α, β) such that $I = Z\pi \in \lambda(A)$ and A is (Z, α, β)-flat. Take $r \in J(\alpha)$ and put $B = A(r)$. Then B is a square matrix of type (m, m) and each row of B contains at most α non-zero elements. Moreover, at least one row contains at most $\alpha - 1$ non-zero elements. Hence $|\det B| < \alpha^m$. Finally, $\det f_I(B) = f_I(\det B) = 0$, so that π divides $\det B$ and $\pi < \alpha^m$. From, this $\log_\alpha \pi < m$.

7.4. Proposition. Let $3 \leqq \pi$ be a prime. Then $\delta(\pi, 3) \leqq 2\pi$.

Proof. The result is an easy consequence of 6.6.

8. Cancellative relations and matrices

Let $2 \leqq \alpha \in N$ and let r be a non-empty α-ary relation defined on a finite set M. Put $q_i = q_i(r)$, $n = \Sigma q_i$, $\beta = (q_1, \ldots, q_\alpha)$, $m = \text{card}(r)$, $Q_i(r) = \{z_{i1}, \ldots, z_{iq_i}\}$, $Q = Q(r)$. Further, let ϱ be a linear ordering of $r = \{x_1, \ldots, x_m\}$, $(x_l, x_k) \in \varrho$ iff $k \leqq l$. We shall define a matrix $E = E(r, \varrho) \in \mathfrak{M}(R, m, n)$ as follows: $e_{ij} \in \{0, 1\}$ for all $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$; if $1 \leqq i \leqq m$ and $x_i = (a_1, \ldots, a_\alpha)$ then, for $1 \leqq j \leqq n$, $e_{ij} = 1$ iff there exists $0 \leqq k \leqq \alpha - 1$ with $q_1 + \ldots + q_k < j \leqq q_1 + \ldots \ldots + q_{k+1}$ and $a_{k+1} = z_{k+1,l}$, $l = j - q_1 - \ldots - q_k$.

8.1. Proposition. The matrix $E(r, \varrho)$ satisfies the conditions (1), (2), (3, α) and (4, α, β).

Proof. Obvious.

Let $\varphi: Q \to R$ be a mapping. Define $w \in R^{(\alpha)}$ by $w_1 = \varphi(z_{11}), \ldots, w_{q_1} = \varphi(z_{1q_1})$, $w_{q_1+1} = \varphi(z_{21}), \ldots, w_{q_1+q_2} = \varphi(z_{2q_2}), \ldots, w_n = \varphi(z_{\alpha q_\alpha})$.

8.2. Lemma. φ is a homomorphism of r into the additive group $R(+)$ iff $Ew = 0$. In this case, φ is trivial iff $w \in R(\alpha, \beta)$.

Proof. Obvious.

8.3. Corollary. Every homomorphism of r into $R(+)$ is trivial iff $E(r, \varrho)$ is (R, α, β)-flat.

Let $\varphi: Q \to R$ be a mapping and let $1 \leqq i \leqq \alpha$. Define

$$\begin{aligned} &v \in R^{(\alpha)} \text{ by } v_1 = \varphi(z_{11}), \ldots, v_{q_1+\cdots+q_{i-1}} = \varphi(z_{i-1,q_{i-1}}), \\ &v_{q_1+\cdots+q_{i-1}+1} = -\varphi(z_{i1}), \ldots, v_{q_1+\cdots+q_i} = -\varphi(z_{iq_i}), \\ &v_{q_1+\cdots+q_i+1} = \varphi(z_{i+1,1}), \ldots, v_n = \varphi(z_{\alpha q_\alpha}). \end{aligned}$$

8.4. Lemma. φ is an i-homomorphism of r into $R(+)$ iff $Ev = 0$. In this case, φ is trivial iff $v \in R(\alpha, \beta)$.

8.5. Corollary. Every i-homomorphism of r into $R(+)$ is trivial iff $E(r, \varrho)$ is (R, α, β)-flat.