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Properties of Operators Related to the Boltzmann Collision Term

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Some recent results on boundedness and compactness of certain operators closely related to the Boltzmann collision term are reviewed.

Je dán přehled některých nových výsledků o omezenosti a kompaktnosti jistých operátorů úzce souvisejících s Boltzmannovým srážkovým členem.

Сведены некоторые новейшие результаты об ограниченности и компактности определенных операторов находящихся в тесной связи со столкновительным членом Больцмана.

Introduction

The Boltzmann collision term Q, characterizing the effect of collisions between two sorts of particles with velocity distribution functions f and g upon the change of f, has the form

$$(1) \qquad Q[f, g](v) = \int_{R^3 \times \langle 0, \pi/2) \times \langle 0, 2\pi)} [f(v')\, g(v'_*) - f(v)\, g(v_*)] \times \times B(\vartheta, \|V\|)\, dv_*\, d\vartheta\, d\varepsilon\ (v \in \mathbb{R}^3),$$

provided only binary elastic collisions are admitted. Here,

$$V = v - v_*,$$

$$v' = v - \frac{2}{1 + \varkappa} (V \,.\, n)\, V,$$

$$v'_* = v_* + \frac{2\varkappa}{1 + \varkappa} (V \,.\, n)\, V,$$

$$n = \begin{pmatrix} \sin\vartheta \cos\varepsilon \\ \sin\vartheta \sin\varepsilon \\ \cos\vartheta \end{pmatrix},$$

and $\varkappa$ is the ratio of the masses of the colliding particles;

$$B\colon \langle 0, \tfrac{1}{2}\pi) \times (0, \infty) \to \langle 0, \infty)$$

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is a measurable function closely related to the differential cross section [1]. In case that the collisions are governed by a repulsive force inversely proportional to some power, say, k-th ($k > 2$), of the distance between the (two) colliding particles, the function B is factorized as follows [1]:

(2) $$B(\vartheta, u) = u^{\gamma}\, b(\vartheta) \quad (0 \leqq \vartheta < \tfrac{1}{2}\pi,\ u > 0)$$

where

$$\gamma = \frac{k-5}{k-1}$$

(hence, $-3 < \gamma < 1$). If an angular cut-off [1], [2] is introduced, the function b becomes integrable over $\langle 0, \frac{1}{2}\pi)$. Note that the well-known hard sphere model of interactions gives the function B of the form (2) with $\gamma = 1$.

Throughout this note we shall deal with a more general version of (1), namely

$$Q[f, g](v) = \int_{R^3 \times \langle 0,\pi/2) \times \langle 0,2\pi)} [f(v')\, g(v'_*) - f(v)\, g(v_*)] \times$$
$$\times\, B(\vartheta, \|V\|) \exp(-\lambda \|v_*\|^2)\, dv_*\, d\vartheta\, d\varepsilon \quad (v \in \mathbb{R}^3)$$

where $\lambda \geqq 0$ is a parameter. A term of this type appears in an equation arising from the Boltzmann kinetic equation by means of a cetrain transformation [3]. Define

$$\nu[g](v) = 2\pi \int_{R^3 \times \langle 0,\pi/2)} g(v_*)\, B(\vartheta, \|V\|) \exp(-\lambda \|v_*\|^2)\, dv_*\, d\vartheta \quad (v \in \mathbb{R}^3)\,.$$

Given functions f and g such that $Q[f, g]$ exists and $\nu[g]$ is finite on $\mathbb{R}^3$, we can decompose Q as follows:

$$Q[f, g] = Q^+[f, g] - f \cdot \nu[g]$$

where

$$Q^+[f, g](v) = \int_{R^3 \times \langle 0,\pi/2) \times \langle 0,2\pi)} f(v')\, g(v'_*)\, B(\vartheta, \|V\|) \times$$
$$\times \exp(-\lambda \|v_*\|^2)\, dv_*\, d\vartheta\, d\varepsilon \quad (v \in \mathbb{R}^3)\,.$$

Letting f and g vary within some function space, we may regard Q^+ as a bilinear operator. If one of the functions f and g is fixed, we consider the following linear operators:

$$Q^{(1)} = Q^+[f, \cdot]\,, \quad Q^{(2)} = Q^+[\cdot, g]\,, \quad Q^{(3)} = f \cdot \nu[\cdot]\,.$$

The purpose of this note is to review criteria on boundedness of the operators Q^+ and $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$, as well as criteria on compactness of the last three operators in quotes. The function B will correspond to an interaction model from a specified class involving, inter alia, the models of inverse-power repulsive interparticle forces with angular cut-off and the hard sphere model. The subject matter presented was treated in full detail in recent author's papers [3], [4], which extend former results due to Carleman [5], Grad [2], [6], Dorfman [7], and Molinet [8].

Notation and agreements

We shall work in spaces of the L_p-type. The space $L_p(\mathbb{R}^3)$ will be denoted just L_p and the norm $\|\cdot\|_p$. For $\sigma \geqq 1$ we consider the space

$$L_{p(\sigma)} = \{h; (1 + \|v\|^2)^{\sigma/2} h(v) \in L_p\}$$

endowed with the norm

$$\|h\|_{p(\sigma)} = \|(1 + \|v\|^2)^{\sigma/2} h(v)\|_p .$$

For easy reference recall that p is conjugated with q, $1 \leqq q \leqq \infty$, if

$$(3) \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{for} \quad 1 < q < \infty\,, \quad p = 1 \quad \text{for} \quad q = \infty, \quad p = \infty \quad \text{for } q = 1\,.$$

Given γ, define $\mathscr{B}_\gamma$ to be the set of measurable functions $B(\vartheta, u)$ satisfying

$$0 \leqq B(\vartheta, u) \leqq \text{const}\,.\, u^\gamma \sin\vartheta \cos^\gamma \vartheta \quad (0 \leqq \vartheta < \tfrac{1}{2}\pi, u > 0)\,,$$

the indicated multiplicative constant depending on B.

Given α, define $\mathscr{M}_\alpha$ to be the set of measurable functions $h(v)$ satisfying

$$|h(v)| \leqq \text{const}\,.\, \exp(-\alpha\|v\|^2) \quad (v \in \mathbb{R}^3)\,,$$

the indicated multiplicative constant depending on h.

In any discussion involving $\varkappa$ it will be tacitly assumed that $\varkappa$ is an arbitrary positive parametre unless its value is explicitly specified.

Results concerning the bilinear operator $Q^+[\cdot,\cdot]$

Theorem (see [4], 2.1). Let

$$\lambda > 0\,;$$

let $B(\vartheta, u)$ be a (nonnegative measurable) function such that

$$B(\vartheta, u) \leqq u^\gamma\, b(\vartheta) \quad (0 \leqq \vartheta < \tfrac{1}{2}\pi, u > 0)$$

with

$$-3 < \gamma \leqq 0\,,$$

and let

$$b \in L_q(0, \tfrac{1}{2}\pi)$$

for some q satisfying

$$1 \leqq q < \frac{3}{|\gamma|} \quad \text{if} \quad \gamma \neq 0\,,$$

$$(4) \quad 1 \leqq q \leqq \infty \quad \text{if} \quad \gamma = 0\,.$$

Defining p by (3), $Q^+[\cdot, \cdot]$ is a bounded bilinear operator **from L_p** $\times L_p$ into L_p.

Corollary (see [4], 2.3). Let

$$\lambda > 0$$

and

$$B \in \mathscr{B}_\gamma, \quad -1 < \gamma \leqq 0 .$$

Let q satisfy $1 \leqq q < 1/|\gamma|$ if $\gamma \neq 0$, or (4). Then, defining p by (3), $Q^+[\cdot, \cdot]$ is a bounded bilinear operator from $L_p \times L_p$ into L_p.

Theorem (see [4], 3.5). Let

$$\lambda > 0$$

and

$$B \in \mathscr{B}_\gamma, \quad -1 < \gamma \leqq 0 .$$

Let q and σ satisfy

$$1 \leqq q \leqq 2 + |\gamma| ,$$

$$1 \leqq \sigma \leqq \frac{|\gamma|}{q} + \min \left\{1, \frac{2}{q}\right\} .$$

Then, defining p by (3), $Q^+[\cdot, \cdot]$ is a bounded bilinear operator from $L_{p(\sigma)} \times$ $\times$ $L_{p(\sigma)}$ into $L_{p(\sigma)}$.

Theorem (see [4], 3.6). Let

$$\lambda > 0 ,$$

$$\varkappa = 1 ,$$

and

$$B \in \mathscr{B}_\gamma, \quad -1 < \gamma \leqq 1 .$$

Then for every $1 \leqq p \leqq \infty$ and every $1 \leqq \sigma (< \infty)$, $Q^+[\cdot, \cdot]$ is a bounded bilinear operator from $L_{p(\sigma)} \times L_{p(\sigma)}$ into $L_{p(\sigma)}$.

Results concerning the linear operators $Q^{(i)}$ $(i = 1, 2, 3)$

Some criteria on boundedness of $Q^{(1)}$ and $Q^{(2)}$ arise immediately from those of $Q^+[\cdot, \cdot]$. If the (fixed) functions f and g are majorized by a Maxwellian, more conclusive results can be established, as seen below.

Theorem (see [3], 5.7). Let

$$\lambda = 0 ,$$

$$B \in \mathscr{B}_\gamma, \quad -1 < \gamma \leqq 0 ,$$

and

$$f, g \in \mathscr{M}_\alpha, \quad \alpha > 0 .$$

Then each of $Q^{(i)}$ $(i = 1, 2, 3)$ is a bounded linear operator from L_1 into itself.

Theorem (see [3], 5.7*)). Let

$$\lambda = 0,$$

$$\varkappa \neq 1,$$

$$B \in \mathscr{B}_\gamma, \quad -1 < \gamma \leqq 0,$$

and

$$g \in \mathscr{M}_\alpha, \quad \alpha > 0.$$

Then for every $1 \leqq p \leqq \infty$, $Q^{(2)}$ is a bounded linear operator from L_p into itself.

Theorem (see [3], 6.9). Let

$$\lambda > 0,$$

$$B \in \mathscr{B}_\gamma, \quad -1 < \gamma \leqq 1,$$

and

$$f, g \in \mathscr{M}_\alpha, \quad \alpha > 0.$$

Then for every $1 \leqq p \leqq \infty$, each of $Q^{(i)}$ $(i = 1, 2, 3)$ is a bounded linear operator from L_p into itself.

Theorem (see [3], 6.16). Let

$$\lambda > 0,$$

$$B \in \mathscr{B}_\gamma, \quad -1 < \gamma \leqq 1,$$

and

$$f, g \in \mathscr{M}_\alpha, \quad \alpha > 0.$$

Then each of $Q^{(i)}$ $(i = 1, 2, 3)$ is a compact operator from L_2 into itself.

Concluding remarks

The results reviewed above may be found interesting for mathematicians and physicists engaged in the kinetic theory. The results originate from author's thesis, which provides mathematical background for investigating transport processes in ionized gases in an external electromagnetic field. A problem of this sort is considered, e.g., in connection with determining the mobility of charged particles in operation media of electrotechnical devices under high-voltage stresses. The thesis was worked out at the Mathematical Institute of the Charles University, Prague.

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*) The clause "provided $\varkappa \neq 1$" should be attached to the last sentence of Theorem 5.7 in [3].

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