Acta Universitatis Carolinae. Mathematica et Physica

E. Suková; Jiří Vinárek

Different types of products and subdirectly irreducible

*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 27 (1986), No. 2, 23--30

Persistent URL: `http://dml.cz/dmlcz/142571`

# Different Types of Products and Subdirectly Irreducible Graphs

E. SUKOVÁ, J. VINÁREK
Department of Mathematics, Charles University, Prague*)

*Received 10 January, 1986*

Subdirect irreducibility of graphs is studied for different cases of products — categorical, cartesian and mixed ones.

V práci se zkoumá subdirektní ireducibilita grafů pro různé typy součinů — kategoriální, kartézské a smíšené.

В работе исследуется подпрямая неприводимость для разных типов произведений графов

## Introduction

Throughout this paper, the topic "graph" denotes an undirected graph without loops and multiple edges.

It may be sometimes useful to construct general graphs from those which are – in some sense – simple, using products and subobjects. Which products and which subobjects? If one does not want to lose useful properties of graphs, it is good to use induced subgraphs as subobjects. This problem was studied for the case of direct (categorical) products of graphs in [2]. In the present paper we attack the problem of constructions of general graphs from simple ones for other two types of products – cartesian and mixed products.

## 1. Conventions and notations

Given graph $G$, we denote $V(G)$ its set of vertices and $E(G)$ its set of edges. In the case of an indexed family of graphs $\{G_i; i \in I\}$ we shall put usually $V(G_i) = V_i$, $E(G_i) = E_i$. If $G$ is (some) product of graphs $G_1, \ldots, G_n$ then $i$-th projection $p_i$ is a mapping defined by $p_i(x_1, \ldots, x_n) = x_i$. A join $G = G_1 \cup G_2$ of graphs is defined by $G = (V_1 \cup V_2, E_1 \cup E_2)$. $K_n$ denotes – as usual-a complete graph with $n$ vertices.

## 2. Products of graphs and subdirect irreducubility

**2.1. Definition.** Let $I = \{1, \ldots, n\}$ be an indexed set, $G_i$ be graphs.

*) 186 00 Praha 8, Sokolovská 83, Czechoslovakia

Then:

a) A *direct product* $D = G_1 \times \ldots \times G_n = \underset{I}{\mathbb{X}} G_i$ is a graph defined by:
$V(D) = V_1 \times \ldots \times V_n$,
$E(D) = \{\{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\};\ \{x_i, y_i\} \in E_i \text{ for any } i \in I\}$

b) A *cartesian product* $C = G_1 \,\square \ldots \square\, G_n = \underset{I}{\square}\, G_i$
is a graph defined by:
$V(C) = V_1 \times \ldots \times V_n$
$$E(C) = \{\{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\}; (\exists j \in I)(\{x_j, k_j\} \in V_j) \wedge \wedge (\forall i \in I)(i \neq j \Rightarrow x_i = y_i)\}.$$

c) A *mixed product* $M = G_1 \otimes \ldots \otimes G_n = \otimes\, G_i$ is a graph defined by:
$V(M) = V_1 \times \ldots \times V_n$,
$E(M) = E(C) \cup E(D)$.
(I.e. $M = C \cup D$).

**2.2. Definition.** A graph $G$ is *subdirectly irreducible* (abbreviated SI) if, whenever $G$ is embedded as an induced subgraph (with an embedding $e$) into a product $\Pi G_i$ of graphs $G_1, \ldots, G_n$ such that all $p_i e$ are onto, then at least one $p_i e$ is an isomorphism. (This definition can be used for $\Pi = \mathbb{X}$, $\square$ or $\otimes$).

$$\begin{tikzcd} G \arrow[r, "e"] \arrow[dr] & \Pi G_i \arrow[d, "p_i"] \\ & G_i \end{tikzcd}$$

Fig. 1.

**2.3. Convention.** We shall use abbreviations DSI (CSI, MSI resp.) for subdirect irreducibility with respect to direct (cartesian, mixed resp.) products.

### 3. Subdirect irreducibility with respect to direct products

DSI was characterized in [2] using categorical methods. In this paper we are going to find subdirectly irreducibles directly.

**3.1. Proposition.** Every complete graph is DSI.

Proof. Let $G = K_n$ be the complete graph, $e: G \to \underset{I}{\mathbb{X}} G_i$ be an induced subgraph such that all $p_i e$ are onto (recall that $p_i$ are the projections). Since $p_i e$ is a homomorphism of graphs and $G$ is complete, $G_i$ is complete for any $i \in I$ as well. Hence, $G_i \cong G$ and $G$ is DSI.

**3.2. Proposition.** Any DSI graph is complete.

Proof. One can check that for $G = (V, E)$ with $E = \bigcap\limits_{i \in I} E_i$, $E \neq E_i$, there is an embedding $G \to \mathop{\mathbb{X}}\limits_{i \in I} (V, E_i)$. Hence, any DSI graph $G$ has at most one edge missing (i.e. it is meet irreducible – see the general categorical theorem from [2]) Suppose that $G = (V, E)$ is a graph with just one edge $\{x, y\}$ missing. Denote $G_1 = (V, E \cup \{x, y\})$, $V_2 = V - \{y\}$, $G_2 = (V_2, (V_2 \times V_2) \cap E)$ Clearly, $G_1$ and $G_2$ are complete graphs and $e: G \to G_1 \times G_2$ defined by

$$e(v) = (v, v) \quad \text{for} \quad v \neq y\,,$$
$$e(y) = (y, x)$$

is an embedding with $p_1 e$, $p_2 e$ homomorphisms onto. Thus, $G$ is not DSI.

**3.3. Theorem.** A graph is DSI iff it is complete.

Proof follows from 3.1. and 3.2.

## 4. Subdirect irreducibility with respect to cartesian products

**4.1. Proposition.** Every complete graph is CSI.

Proof. Suppose a complete $G$ is an induced subgraph of $\mathop{\Box}\limits_{I} G_i$ ($e$ is the embedding) Suppose there exist $x, y, z \in V(G)$, $i, j \in I$, $i \neq j$, such that $p_i e(x) \neq p_i e(y)$, $p_i e(y) = p_i e(z)$, $p_j e(x) = p_j e(y)$, $p_j e(y) \neq p_j e(z)$. Hence, $p_i e(x) \neq p_i e(z)$, $p_j e(x) \neq p_j e(z)$. According to the definition of the cartesian product, there is $\{v(x), v(z)\} \notin E(\Box G_i)$ and $\{x, z\} \notin E(G)$ which contradicts the assumption $G$ is complete. Hence, there exists just one $i \in I$ such for any $x, y \in V(G)$ there is $p_i e(x) \neq p_i e(y)$ and $p_j e(x) = p_j e(y)$ for any $j \neq i$. Since $p_i e$ is onto and $G$ is complete, there is $G \cong G_i$. Thus, $G$ is CSI.

**4.2. Proposition.** Any graph $G$ with just one edge missing is CSI iff $|V(G)| \geqq 4$.

Proof. Denote by $K_n'$ the graph with $n$ vertices and just one edge missing. One can check that $K_2'$ and $K_3'$ are not CSI:

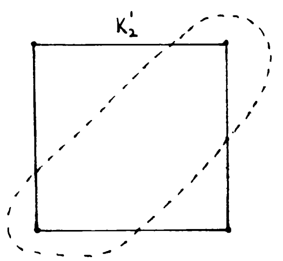


Fig. 2.

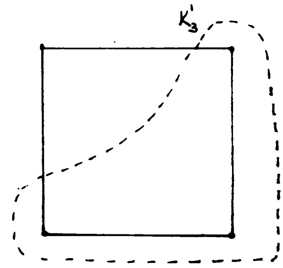


Fig. 3.

Suppose $n \geqq 4$, $G = (V, E)$ where $V = \{x_1, \ldots, x_n\}$, $E = \{\{x_i, x_j\};\ i \neq j,\ (i, j) \neq (1, n)\}$. For $j = 2, \ldots, n-1$ there is $G / \{x_{j-1}, x_j, x_{j+1}\}$ (an induced subgraph) isomorphic to $K_3$. Denote by $e$ an embedding of $G$ in $\square\, G_i$. Hence – according to 4.1. – there exists just one $i(j)$ such that $p_{i(j)}\, e(x_{j-1}) \neq p_{i(j)}\, e(x_j) \neq p_{i(j)}\, e(x_{j+1})$ and $p_i\, e(x_{j-1}) = p_i\, e(x_j) = p_i\, e(x_{j+1})$ for any $i \neq i(j)$. It is evident that $i(2) = \ldots = i(n-1)$. Thus, $G_i$ are trivial for all $i \neq i(2)$. Since $e$ is an embedding, $G_{i(2)} \cong G$ and $G$ is CSI.

**4.3. Proposition.** Let $G = (V, E)$ with $V = \{x_0, \ldots, x_n\}$ $(n \geqq 3)$ be a graph such that $G_1 = G / \{x_0, \ldots, x_{n-1}\}$ and $G_2 = G / \{x_1, \ldots, x_n\}$ are CSI. Then $G$ is CSI, too.

Proof. Suppose that assumptions of Proposition are satisfied and there is an embedding $e: G \to \square\, H_i$ with all $p_i e$ onto. Denote $H_{1i} = H_i / \{p_i\, e(x_0), \ldots, p_i\, e(x_{n-1})\}$, $H_{2i} = H_i / \{p_i\, e(x_1), \ldots, p_i\, e(x_n)\}$. Since $G_1$ is CSI, there exists $k \in I$ such that $H_{1k} \cong G_1$. Using methods of 4.2. one can prove that $|H_{1p}| = 1$ for any $p \neq k$. Similarly, subdirect irreducibility of $G_2$ implies that there exists $q \in I$ such that $H_{2q} \cong G_2$; moreover $|H_{2r}| = 1$ for any $r \neq q$. If $k \neq q$ then $|H_{2k}| = 1$ and $p_k\, e(x_1) = \ldots = p_k\, e(x_{n-1})$. But on the other hand $H_{1k} \cong G_1$ and $p_k\, e(x_1) \neq p_k\, e(x_2) \neq \ldots \neq p_k\, e(x_{n-1})$ which is a contradiction. Hence, $k = q$, $p_k e$ is bijective and $|H_i| = 1$ for any $i \neq k$. Therefore, $G$ is CSI.

**4.4. Remark.** Proposition 4.3. does not hold without assumption $n \geqq 3$. E.g. if $G = (\{0, 1, 2\}, \{\{0, 1\}, \{1, 2\}\})$ then $G / \{0, 1\} \cong G / \{1, 2\} \cong K_2$ is SI but $G \cong K_3'$ is an induced subgraph of $K_2 \times K_2$ and it is not CSI.

**4.5. Theorem.** Let $G = (V, E)$ be a graph with $V(G) = \{x_0, \ldots, x_{n-1}\}$. If $E \supseteq \{\{x_i, x_j\};\ 1 \leqq |i - j| \leqq 2\}$ then $G$ is CSI.

Proof.

(i) If $n \leqq 3$ then $G$ is complete and according to 4.1 it is CSI.

(ii) Suppose that Proposition holds for $n = k$; we are going to prove it for $n = k + 1$. Clearly, $G_1 = G / \{x_0, \ldots, x_k\}$ satisfies assumptions of Proposition for $n = k$ and therefore $G_1$ is CSI. Denote $G_2 = G / \{x_1, \ldots, x_k\}$ and put $y_0 = x_1$, $y_1 = x_2, \ldots, y_{k-1} = x_k$. $G_2$ also satisfies assumptions of Proposition for $n = k$. Hence, $G_2$ is CSI and – according to 4.3 – $G$ is CSI as well.

**4.6. Theorem.** Let $G_1 = (X, R)$, $G_2 = (Y, S)$ be graphs, $X \cap Y = \{a\}$. Then $G = (X \cup Y, R \cup S\}$ is not CSI.

Proof. Define an embedding: $f$: $G \to G_1 \square\, G_2$ as follows:

$$f(x) = (x, a) \quad \text{whenever} \quad x \in X$$
$$f(y) = (a, y) \quad \text{whenever} \quad y \in Y$$

One can check that $f$ is an embedding and $p_1 f$ ($p_2 f$ resp.) are mappings onto $G_1 (G_2)$. Therefore, $G$ is not CSI.

**4.7. Remark.** Propositions 4.1., 4.2 show that DSI $\Rightarrow$ CSI but the converse is not true. Not only complete graphs but also graphs with just one edge missing with at least 4 vertices are CSI, too. Of course, there are also other arbitrarily large CSI graphs which can be constructed using 4.3 and 4.5 (see e.g. Fig. 4). Theorem 4.6 gives an example of a class of graphs which are not CSI. Full characterization of CSI is still open.

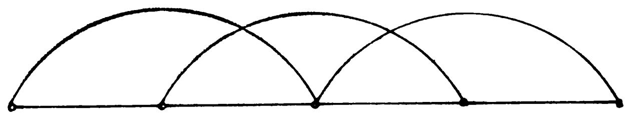

Fig. 4

### 5. Subdirect irreducibility with respect to mixed products

**5.1. Proposition.** A complete graph $K_n$ is MSI if $n \leqq 2$.

Proof.

(i) $K_1$ is trivial, hence MSI.

(ii) Let $e: K_2 \to \otimes G_i$ be an embedding such that all $p_i e$ are onto. Then $|G_i| \leqq 2$ and since $|K_2| = 2$ there exists $j \in I$ such that $|G_j| = 2$. Since $\otimes G_i$ contains $K_2$ as an induced subgraph, there is $G_j \cong K_2$. Therefore, $K_2$ is MSI.

(iii) Suppose $n \geqq 3$. Without loss of generality, one can suppose $V = \{0, 1, \ldots, n-1\}$ Define an integer $m$ by $2^{m-1} \leqq n - 1 < 2^m$.

Now we are going to define an embedding $e$: $K_n \to \bigotimes_{i=0}^{m-1} G_i$ where $G_i \cong K_2$, $V(G_i) = \{0, 1\}$.

Let $i = a_{i,0} + 2a_{i,1} + \ldots + 2^{m-1}a_{i,m-1}(a_{i,j} \in \{0, 1\})$ be the dyadic notation of an integer $i(0 \leqq i \leqq n-1)$. Define $e(i) = (a_{i,0}, \ldots, a_{i,m-1})$. Clearly, $e$ is an embedding onto an induced subgraph of $\otimes G_i$. Since $p_j e(2^j) = 1$, $p_j e(2^j - 1) = 0$, any $p_j e$ maps $K_n$ onto $G_j$.

Hence, $G_n$ is not MSI.

**5.2. Proposition.** An incomplete graph $G$ with just one edge missing (to completness) is MSI iff $|V(G)| \leqq 3$.

Proof.

(i) If $|V| = 2$ then $E = \{\emptyset\}$. Let $e$: $G \to \bigotimes_I G_i$ be an embedding such that all $p_i e$ are onto. If all $G_i$ are complete then $\bigotimes_I G_i$ is also complete which is a contradiction. Hence, there is a $j \in I$ such that $G_j$ is not complete. Since $p_j e$ is onto, $|V(G_j)| = 2$ and $G_j \cong G$.

(ii) Suppose $|V| = 3$, $V = \{x, y, z\}$, $E = \{\{x, y\}, \{y, z\}\}$. Let $e$: $G \to \otimes G_i$ be an embedding such that all $p_i e$ are onto. Hence, $|V(G_i)| \leqq 3$ for any $i \in I$. If $|V(G_i)| \leqq$

$\leqq 2$ for all $i \in I$ then $\otimes G_i$ is either complete or a join of complete graphs and $G$ is not an induced subgraph of $\otimes G_i$. Thus, there exists $j \in I$ such that $|V(G_j)| = 3$ and $G_j$ is not complete. Since $p_j e$ is onto, $G \cong G_j$. Hence, $G$ is MSI.

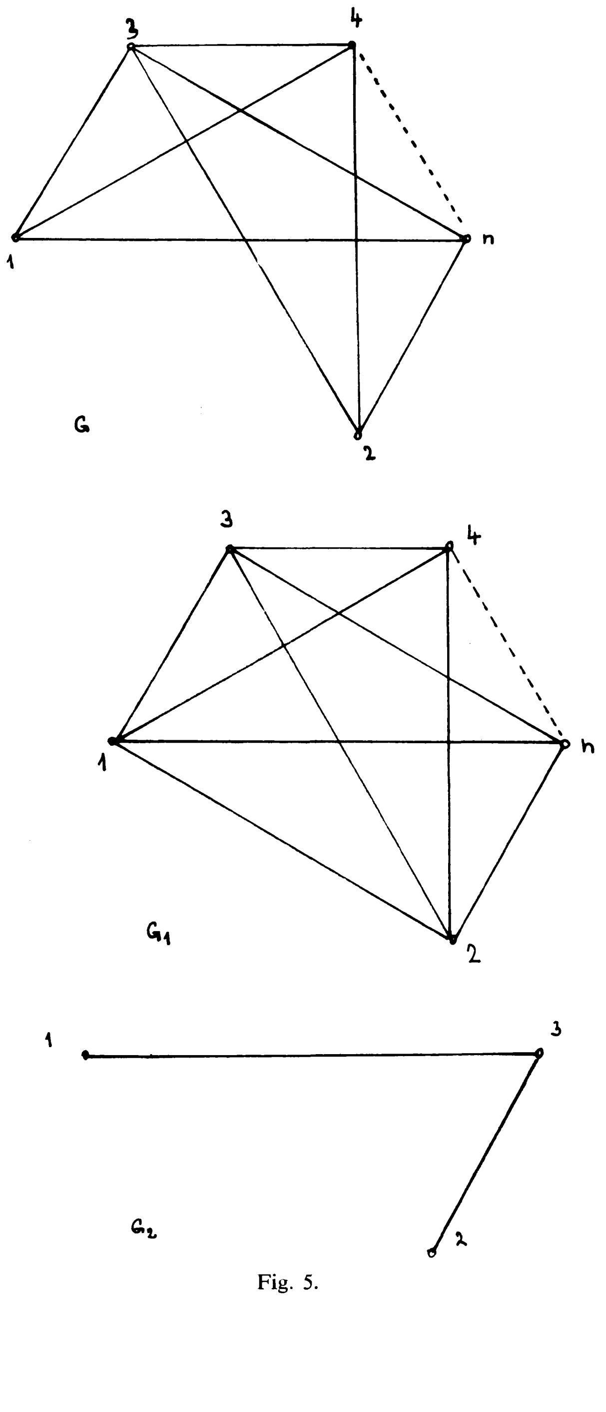


Fig. 5.

(iii) Suppose $G = (V, E)$ where $V = \{1, \dots, n\}, n \geqq 4$,
$E = \{\{i, j\}; i \neq j, \{i, j\} \neq \{1, 2\}\}$.
Define $G_1 = (\{1, \dots, n\}; \{\{i, j\}; i \neq j\})$ (i.e. a complete graph with $n$ vertices), $G_2 = (\{1, 2, 3\}; \{\{1, 3\}, \{2, 3\})$. Further define an embedding $e: G \to G_1 \otimes G_2$ by $e(i) = (i, \min(i, 3))$. We have to prove that the definition of $e$ is correct and that every $p_i e$ is onto.

(a) If $\{i, j\} \in E(G)$ then $\{p_1 e(i), p_1 e(j)\} \in E(G_1)$ and either $\{p_2 e(i), p_2 e(j)\} \in E(G_2)$ or $p_2 e(i) = p_2 e(j)$, Hence $\{e(i), e(j)\} \in E(G_1 \otimes G_2)$.

(b) $\{1, 2\} \in E(G)$, $p_2 e(1) \neq p_2 e(2)$ and $\{p_2 e(1), p_2 e(2)\} \notin E(G_2)$. Hence, $\{e(1), e(2)\} \notin E(G_1 \otimes G_2)$ and $e$ is an embedding.

(c) For any $i = 1, \dots, n$ there is $p_1 e(i) = p_1(i, \min(i, 3)) = i$. Hence, $p_1 e$ is onto.

(d) For $i = 1, 2, 3$ there is $p_2 e(i) = p_2(i, \min(i, 3)) = i$. Hence, $p_2 e$ is onto. Since $p_1 e$, $p_2 e$ are onto, $G \not\leq G_1$, $G \not\leq G_2$, $G$ is not MSI.

**5.3. Proposition.** An incomplete graph with at least two edges missing (to completeness) is not MSI.

Proof. Suppose $G = (V, E)$ such that there are $i, j, r, s \in V$, $i \neq j, r \neq s$, $|\{i, j, r, s\}| \geqq 3$, $\{i, j\} \notin E$, $\{r, s\} \notin E$. Define $G_1 = (V, E \cup \{i, j\})$, $G_2 = (V, E \cup \cup \{r, s\})$. Further define $e: G \to G_1 \otimes G_2$ by $e(x) = (x, x)$.

If $(x, y) \in E$ then $p_1 e(x) \neq p_1 e(y)$, $p_2 e(x) \neq p_2 e(y)$. Moreover, $\{x, y\} \in E$ iff simultaneously $\{p_1 e(x), p_1 e(y)\} \in E(G_1)$ and $\{p_2 e(x), p_2 e(y)\} \in E(G_2)$. Hence, $e$ is an embedding of $G$ onto an induced subgraph of $G_1 \otimes G_2$. Since $p_i e$ are onto for $i = 1, 2$ and $G_1 \not\leq G \not\leq G_2$, $G$ is not MSI.

**5.4. Theorem.** Graph is MSI iff it is isomorphic to one of the following four graphs:

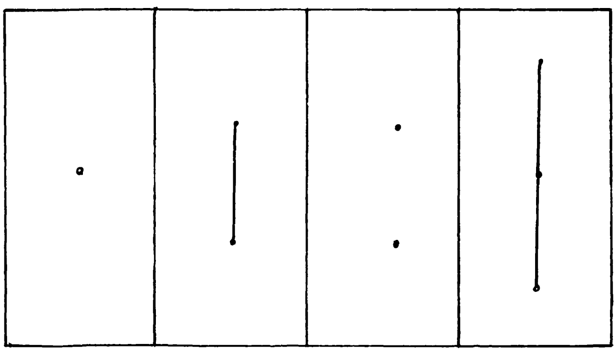

Fig. 6.

Proof follows from Proposition 5.1 – 5.3.

## References

[1] BIRKHOFF, G., Lattice Theory, AMS Coll. Publ. *25* Providence RI, 1967.
[2] PULTR, A., VINÁREK, J., Discr. Math. *20* 1977, 159.
[3] VINÁREK, J., Czechosl. Math. J. *32*, 1982, 116.