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## Decomposition Properties in Module Categories

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The characterization of rings  $R$  by properties of the category of left  $R$ -modules can be extended to the investigation of  $R$ -modules  $M$  by the study of the category  $\sigma[M]$  subgenerated by  $M$ . Here we consider modules  $M$  of finite length, for which every object in  $\sigma[M]$  is direct sum of (homogeneously) uniserial modules. For  $M = R$  our main results give (old and new) characterizations of a left artinian ring to be serial or a left and right principal ideal ring.

Charakterizace okruhů vlastnostmi jejich modulů je rozšířena na vyšetřování modulů studiem kategorií těchto moduly generovaných.

Характеризация колец свойствами модулей расширяется на исследование модулей изучением категорий порожденных этими модулями.

*Contents:* 1. Preliminaries. 2. QF-3 modules and their duals. 3. Modules of serial representation type. 4. Modules flat or FP-injective over their endomorphism rings. 5. Modules of homoserial representation type.

### 1. Preliminaries

Let  $R$  denote an associative ring with identity and  $R\text{-MOD}$  ( $\text{MOD-}R$ ) the category of unitary left (right) modules over  $R$ . Morphisms will be written on the opposite side of the scalars. For modules  $M, N$  in  $R\text{-MOD}$  we say that  $N$  is *subgenerated* by  $M$ , if it is a submodule of an  $M$ -generated module. By  $\sigma[M]$  we denote the full subcategory of  $R\text{-MOD}$  whose objects are all  $R$ -modules subgenerated by  $M$ . A finitely generated, projective generator in  $\sigma[M]$  is called *progenerator* in  $\sigma[M]$ . If there is a progenerator  $P$  in  $\sigma[M]$ , then the category  $\sigma[M]$  is equivalent to  $\text{End}({}_R P)\text{-MOD}$ . In case  $M$  is a module of finite length, then every finitely generated module in  $\sigma[M]$  has finite length, and there is only a finite number of non-isomorphic simple modules in  $\sigma[M]$  (factors of submodules of  $M$ ).

(1.1) *Lemma.* Let  $M$  be an  $R$ -module and assume that there is an artinian progenerator in  $\sigma[M]$ . Then for every  $N \in \sigma[M]$  there is an artinian progenerator in  $\sigma[N]$ .

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*Proof.* Let  $P$  be an artinian progenerator in  $\sigma[M]$ . For  $N \in \sigma[M]$  take the reject  $K := \text{Rej}(P, N) = \bigcap \{ \text{Ker } f / f \in \text{Hom}_R(P, N) \}$ . Since  $N$  is  $P$ -generated it is also  $P/K$ -generated and hence  $N \in \sigma[P/K]$ . On the other hand,  $P/K$  is finitely co-generated by  $N$  and therefore  $P/K \in \sigma[N]$ . Since  $K$  is a fully invariant submodule of  $P$ , the factor module  $P/K$  is a progenerator in  $\sigma[P/K] = \sigma[N]$ .

The next result can be obtained by standard arguments known from  $R\text{-MOD}$ :

(1.2) *Lemma.* Let  $M$  be an  $R$ -module of finite length and  $E_1, \dots, E_k$  the non-isomorphic simple modules in  $\sigma[M]$ . Assume their  $M$ -injective envelopes  $\hat{E}_1, \dots, \hat{E}_k$  to be finitely generated and set  $W = \bigoplus_{i \leq k} \hat{E}_i$ ,  $S = \text{End}({}_R W)$ . Then

- (1)  $S$  is a right artinian ring;
- (2)  ${}_R W$  is an injective cogenerator of finite length in  $\sigma[M]$ ;
- (3)  $W_S$  is an injective cogenerator in  $\text{MOD-}S$ ;
- (4) the functors  $\text{Hom}_R(-, {}_R W)$  and  $\text{Hom}(-, W_S)$  define a duality between the finitely generated modules in  $\sigma[M]$  and  $\text{MOD-}S$ ;
- (5) if there is a finitely generated cogenerator in  $\text{MOD-}S$ , then there is a progenerator in  $\sigma[M]$ .

(1.3) An  $R$ -module  $M$  of finite length is said to be of *finite representation type*, if there are only a finite number of nonisomorphic indecomposable modules in  $\sigma[M]$  (see [20]). In this case the  $M$ -injective envelopes of the simple modules in  $\sigma[M]$  are finitely generated, and the endomorphism ring of the minimal cogenerator in  $\sigma[M]$  is also of finite representation type. Hence it follows from (1.2) that there is a progenerator in  $\sigma[M]$ . In addition, since every indecomposable module in  $\sigma[M]$  is finitely presented, every module in  $\sigma[M]$  is direct sum of finitely generated modules, i.e.  $M$  is *pure semisimple* (see Héaulme [11]).

## 2. QF-3 modules and their duals

Any  $R$ -module which cogenerates the  $R$ -module  $M$  is called an  *$M$ -cogenerator*. An  $M$ -cogenerator is said to be *minimal*, if it is a direct summand of every  $M$ -cogenerator in  $\sigma[M]$ . A module  $M$  is called *QF-3 module*, if there is a minimal  $M$ -cogenerator in  $\sigma[M]$ . Obviously, a ring  $R$  is *left QF-3 ring*, if and only if  ${}_R R$  is QF-3 module. The following properties of QF-3 modules are immediately evident:

(2.1) *Proposition.* Let  $M$  be a finitely generated QF-3 module with minimal  $M$ -cogenerator  $Q$  in  $\sigma[M]$ . Then

- (1)  $Q$  is a finitely generated,  $M$ -injective submodule of  $M$ ;
- (2)  $Q = \bigoplus_{i \leq r} \hat{E}_i$ , where  $E_1, \dots, E_r$  are the non-isomorphic simple submodules of  $M$  ( $\hat{E}_i = M$ -injective envelope of  $E_i$ );
- (3)  $Q$  cogenerates  $\hat{M}$  and hence  $M$  also cogenerates  $\hat{M}$ ;
- (4)  $\text{End}({}_R Q)$  is a semiperfect ring.

The next result gives characterizations of QF-3 modules and provides examples for modules of this type:

(2.2) *Proposition.* For a finitely generated  $R$ -module  $M$  the following statements are equivalent:

- (a)  $M$  is QF-3 module;
- (b) there are finitely many non-isomorphic simple submodules  $E_1, \dots, E_r$  of  $M$  such that  $Q = \bigoplus_{i \leq r} \hat{E}_i$  is an  $M$ -cogenerator and  $Q \subset M^{(\mathbb{N})}$ .

If  $M$  is  $M$ -projective, then to (a) is equivalent:

- (c) There are finitely many simple submodules  $E_1, \dots, E_r$  in  $M$  such that  $Q = \bigoplus_{i \leq r} \hat{E}_i$  is an  $M$ -projective  $M$ -cogenerator.

If  $M$  is a finitely cogenerated  $R$ -module, then to (a) is equivalent:

- (d)  $M$  cogenerates  $\hat{M}$ .

If  $M$  is  $M$ -projective and finitely cogenerated, then to (a) is equivalent:

- (e)  $\hat{M}$  is projective in  $\sigma[M]$ .

The *proof* is easily obtained by standard arguments known for QF-3 rings (see Tachikawa [16]) taking into account that every projective module in  $\sigma[M]$  is a submodule of a direct sum of copies of  $M$ .

A very general definition of *Quasi-Frobenius modules* is given in Hauger-Zimmermann [10]. For noetherian modules this definition reduces to: A noetherian module  $M$  is called *QF module*, if it is an injective cogenerator in  $\sigma[M]$  (see Wisbauer [19]). Several characterizations of noetherian QF modules are given in [10]. Obviously, these modules are also QF-3 modules.

Dualizing the notion of QF-3 modules we make the following definitions: A module which generates the module  $M$  is an  *$M$ -generator*. An  $M$ -generator is called *minimal*, if it is a direct summand of every  $M$ -generator in  $\sigma[M]$ . The modules dual to QF-3 modules are those modules  $M$  which have a *minimal  $M$ -generator*. Recall that an  $R$ -module  $M$  is  *$\sigma$ -semiperfect*, if every factor module of  $M$  has a projective cover in  $\sigma[M]$  (Wisbauer [18]) and that  $M/\text{Rad}(M)$  is semisimple in this case.

(2.3) *Proposition.* Let  $M$  be a finitely generated  $\sigma$ -semiperfect module,  $E_1, \dots, E_k$  the (non-isomorphic) simple factor modules of  $M$  and  $P(M), P(E_i)$  their projective covers in  $\sigma[M]$ . If  $M$  has a minimal  $M$ -generator  $P$ , then

- (1)  $P$  is a finitely generated,  $M$ -projective factor module of  $M$ ;
- (2)  $P \simeq \bigoplus_{i \leq k} P(E_i)$ ,  $P$  generates  $P(M)$  and hence  $M$  also generates  $P(M)$ .
- (3)  $\text{End}({}_R P)$  is a semiperfect ring.

The proof is dual to that of (2.2). Modules which generate their projective cover in  $R\text{-MOD}$  are called *pseudo-projective modules* in Bican [2]. It is evident that every semiperfect ring has a minimal  $R$ -generator (as left and right module). Also, every finitely generated, self-projective and  $\sigma$ -semiperfect module  $M$  has a minimal  $M$ -generator.

The following statement generalizes Proposition 10 in Azumaya [1]:

(2.4) *Proposition.* Let the  $R$ -module  $M$  be  $M$ -projective and a direct sum of indecomposable modules of finite length with only finitely many summands non-isomorphic. Suppose that

- (i) every indecomposable direct summand of  $M$  has a simple socle,
- (ii) the  $M$ -injective envelope of every simple submodule of  $M$  has a simple top.

Then  $M$  is QF-3 module.

*Proof.* We may assume that  $M$  has finite length and have to show that  $M$  contains the  $M$ -injective envelope of every simple submodule: Clearly, by (i) a simple submodule  $L \subset M$  is the socle of one of the indecomposable summands –  $M_1$  – of  $M$ .  $L$  is essential in  $M_1$  and hence also in the  $M$ -injective envelope  $\hat{M}_1$ . By (ii)  $\hat{M}_1$  has a simple top  $T = \hat{M}_1/\text{Rad}(\hat{M}_1)$  and – since  $\hat{M}_1$  is  $M$ -generated – there is an indecomposable summand  $M_2$  of  $M$  and an epimorphism  $f: M_2 \rightarrow T$ . The diagram

$$\begin{array}{ccccccc}
 & & & & M_2 & & \\
 & & & & f \downarrow & & \\
 O & \rightarrow & \text{Rad}(\hat{M}_1) & \rightarrow & \hat{M}_1 & \rightarrow & T \rightarrow O
 \end{array}$$

can be completed by  $g: M_2 \rightarrow \hat{M}_1$  to a commutative diagram. Since  $(M_2)g \not\subset \text{Rad}(\hat{M}_1)$ ,  $g$  is an epimorphism by (ii).  $M_1$  being projective, the submodule  $(M_1)g^{-1} \subset M_2$  decomposes into  $\text{Ker } g \oplus M_1$ . However, by (i)  $(M_1)g^{-1}$  is indecomposable and hence  $\text{Ker } g = 0$  and  $\hat{M}_1 \simeq M_2 \subset M$ .

Dualizing the above arguments we obtain:

(2.5) *Proposition.* Let the  $R$ -module  $M$  be  $M$ -injective and direct sum of indecomposable modules of finite length with only finitely many summands non-isomorphic. Assume that every simple factor module of  $M$  has a projective cover in  $\sigma[M]$  and that

- (i) every indecomposable direct summand of  $M$  has a simple top,
- (ii) the projective cover of every simple factor module of  $M$  has a simple socle.

Then  $M$  has a minimal  $M$ -generator.

### 3. Modules of serial representation type

An  $R$ -module is said to be *uniserial*, if its submodules are linearly ordered by inclusion. Direct sums of uniserial modules are called *serial*. We say that an  $R$ -module  $M$  is of *serial representation type*, if every module in  $\sigma[M]$  is serial.

From the fact that every finitely  $M$ -generated uniserial module is a factor module of  $M$  we readily deduce:

(3.1) *Lemma.* Let  $M$  be a uniserial  $R$ -module.

- (1) If  $M$  is finitely generated and  $M$ -projective, then every finitely generated left ideal in  $\text{End}({}_R M)$  is principal.
- (2) If  $M$  is finitely cogenerated and  $M$ -injective, then every finitely generated right ideal in  $\text{End}({}_R M)$  is principal.

The following simple observation is useful for our investigations:

(3.2) *Lemma.* Let  $M$  be a direct sum of uniserial modules of finite length  $\{U_i\}_{i \in I}$  with index set  $I$ . Assume that

- (i) there are only finitely many non-isomorphic modules in  $\{U_i\}$ ,
- (ii) the  $M$ -injective envelopes of uniserial modules in  $\sigma[M]$  are uniserial.

Then  $M$  contains an  $M$ -injective summand.

*Proof.* Set  $M_0 = U_1 \oplus \dots \oplus U_k$  with all the non-isomorphic modules  $U_1, \dots, U_k$  in  $\{U_i\}$ . Assume  $U_1$  to be a module with maximal length among the  $U_i$ 's. Then the  $M$ -injective envelope  $\hat{U}_1$  is uniserial and  $M_0$ -generated. Therefore  $\hat{U}_1 = \text{Trace}(M_0, \hat{U}_1) = \sum_{i \leq k} \text{Trace}(U_i, \hat{U}_1) = \text{Trace}(U_r, \hat{U}_1)$  for some  $r \leq k$ . This implies that  $\hat{U}_1$  is a factor module of  $U_r$  and  $\text{length}(\hat{U}_1) \leq \text{length}(U_r)$ . Now  $U_1 \neq \hat{U}_1$  would mean  $\text{length}(\hat{U}_1) > \text{length}(U_1) \geq \text{length}(U_r)$ , a contradiction. Hence  $U_1 = \hat{U}_1$ , i.e.  $U_1$  is  $M$ -injective.

(3.3) *Proposition.* Let  $M$  be an  $R$ -module of finite length and of serial representation type. Then

- (1) the  $M$ -injective envelope of every simple module in  $\sigma[M]$  is a factor module of  $M$ ;
- (2) the length of every indecomposable module in  $\sigma[M]$  is bounded by the length of  $M$ ;
- (3) every module  $N \in \sigma[M]$  has an  $N$ -injective summand;
- (4) every indecomposable module in  $\sigma[M]$  is self-injective;
- (5) every self-projective module in  $\sigma[M]$  is QF-3 module;
- (6) every self-injective module  $N \in \sigma[M]$  has an ( $N$ -projective) minimal  $N$ -generator;
- (7) every module  $N \in \sigma[M]$  has a  $\sigma[M]$ -projective factor module;
- (8) every indecomposable  $N \in \sigma[M]$  is  $\sigma[N]$ -projective;
- (9) the endomorphism ring of every indecomposable module in  $\sigma[M]$  is left and right principal ideal ring.

*Proof.* (1) The  $M$ -injective envelope of a simple module is  $M$ -generated and uniserial; since  $M$  has finite length it is a factor module of  $M$ .

(2) follows easily from (1) and implies that  $M$  is of bounded representation type and hence of finite representation type (see [20]). By (1.3) there is a progenerator in  $\sigma[M]$ .

(3) is implied by (2) and Lemma (3.2); (3) implies (4). In view of (2) property (5) is shown by (2.4), and dually we get (6) from (1.1) and (2.5).

(7) Every module  $N \in \sigma[M]$  generates its  $N$ -injective envelope, which has an  $N$ -projective factor module by (6); hence  $N$  also has an  $N$ -projective factor module.

(8) is implied by (7) and (9) deduces from (3.1).

Having in mind that every finitely generated submodule of an  $M$ -generated module  $L$  is contained in a finitely  $M$ -generated submodule of  $L$ , we can show with familiar arguments:

3.4) *Lemma.* Suppose that the  $R$ -module  $M$  has finite length and every finitely generated indecomposable module in  $\sigma[M]$  has a simple top. Then the  $M$ -injective envelope of every simple module in  $\sigma[M]$  is uniserial of finite length.

We are now able to show:

(3.5) *Theorem.* For an  $R$ -module  $M$  of finite length the following statements are equivalent:

- (a)  $M$  is of serial representation type;
- (b) every finitely generated module  $N \in \sigma[M]$ 
  - (i) is serial, or
  - (ii) has an  $N$ -injective and an  $N$ -projective direct summand, or
  - (iii) has an  $N$ -projective factor module;
- (c) every indecomposable module in  $\sigma[M]$  is uniserial;
- (d) every finitely generated indecomposable module in  $\sigma[M]$ 
  - (i) is uniserial, or
  - (ii) self-injective and self-projective, or
  - (iii) has a simple socle and a simple top;
- (e) there is a progenerator  $P$  in  $\sigma[M]$  and for every fully invariant submodule  $K \subset P$ 
  - (i) the factor module  $P/K$  is QF-3 module, or
  - (ii) the factor module  $P/K$  has a  $P/K$ -injective submodule;
- (f) there is a progenerator  $P$  in  $\sigma[M]$  and  $\text{End}({}_R P)$  is an artinian serial ring;
- (g) there is a finitely generated injective cogenerator  $Q$  in  $\sigma[M]$  and for every fully invariant submodule  $L \subset Q$ 
  - (i)  $L$  has a minimal  $L$ -generator, or
  - (ii)  $L$  has an  $L$ -projective factor module.

*Proof.* The implications (a)  $\Rightarrow$  (b.i)  $\Rightarrow$  (d.i) and (a)  $\Rightarrow$  (c)  $\Rightarrow$  (d.i) are trivial. Also (b.ii)  $\Rightarrow$  (d.ii)  $\Rightarrow$  (d.iii) are immediately evident.

(a)  $\Rightarrow$  (b.ii) and (a)  $\Rightarrow$  (e) were shown in Proposition (3.3).

(d.iii)  $\Rightarrow$  (d.i) follows from (3.4) since under the given conditions every indecomposable module in  $\sigma[M]$  is an essential extension of a simple module.

(d.i)  $\Rightarrow$  (a) The  $M$ -injective envelopes of the simple modules are uniserial of finite length. This implies that every indecomposable module is uniserial with length smaller than the length of  $M$ . Hence  $M$  is of finite representation type (see [20]) and pure semisimple by (1.3).

(e.ii)  $\Rightarrow$  (d.ii) Let  $N$  be a finitely generated indecomposable module in  $\sigma[M]$  and set  $K := \text{Reject}(P, N) = \bigcap \{ \text{Ker } f / f \in \text{Hom}_R(P, N) \}$ . Then  $\sigma[P/K] = \sigma[N]$  (see proof of (1.1)) and  $P/K \subset N^r$ ,  $r \in \mathbf{N}$ .  $P/K$  contains the  $P/K$ -injective envelope  $\bar{E}$

of a simple submodule  $E \subset P/K$  which – as a direct summand – is  $P/K$ -projective and isomorphic to a submodule of  $N$ , hence  $N \simeq \tilde{E}$ .

(a)  $\Rightarrow$  (b.iii)  $\Rightarrow$  (g.ii) and (g.i)  $\Rightarrow$  (g.ii) are readily obtained from (3.3) and (3.4).

(g.ii)  $\Rightarrow$  (d.ii) is shown by an argument dual to the proof of (e.ii)  $\Rightarrow$  (d.ii): For a finitely generated indecomposable module  $N \in \sigma[M]$  consider the trace of  $N$  in  $Q$ .

(a)  $\Leftrightarrow$  (f) is clear by the equivalence of  $\sigma[M]$  and  $\text{End}({}_R P)\text{-MOD}$  and the properties of artinian serial rings.

*Remarks.* (1) Let  $\{U_i\}_{i \in I}$  be a representing family of the finitely generated modules in  $\sigma[M]$ ,  $U = \bigoplus_{i \in I} U_i$ , and consider the *functor ring* of  $\sigma[M]$  (Héaulme [11]):

$T = \widehat{\text{Hom}}_R(U, U) = \{f \in \text{Hom}_R(U, U) \mid (U_i)f = 0 \text{ almost everywhere}\}$ . Using results in [11] and transferring arguments from Fuller-Hullinger [8] to our setting one can show that a module  $M$  of finite length is of serial representation type if and only if  $T$  is a QF-2 ring.

(2) Generalizing decomposition theorems of abelian groups Singh introduced two conditions on a module which he numbered by (I) and (II) (Singh [14]). Evidently, condition (I) posed on the module  $M^{(\mathbf{N})}$  is equivalent to the property that every finitely generated module in  $\sigma[M]$  is serial. There should be a contribution by Surjeet Singh on these modules in this volume.

(3) For  $M = R$  the preceding theorem gives new and old characterizations of a left artinian ring to be serial. For example, for rings (a)  $\Leftrightarrow$  (d.ii) was shown in Fuller [7] and (a)  $\Leftrightarrow$  (e.ii) is due to Azumaya [1]. It was shown in Skornyakov [13] that a ring whose left modules are serial is necessarily left artinian.

#### 4. Modules flat or FP-injective over their endomorphism rings

For an  $R$ -module  $M$  set  $S = \text{End}({}_R M)$  and consider the homomorphisms  $f: M^k \rightarrow M^r$ ,  $k, r \in \mathbf{N}$ . Recall that  $M$  is *flat* over  $S$  if and only if  $\text{Ker } f$  is  $M$ -generated for all  $k, r \in \mathbf{N}$  (or  $k \in \mathbf{N}$ ,  $r = 1$ ). Dually,  $M$  is *FP-injective (coflat)* over  $S$ , if the cokernel of  $f$  is cogenerated by  $M$  for any  $k, r \in \mathbf{N}$  (resp.  $k = 1$ ,  $r \in \mathbf{N}$ ). Evidently, if  $M$  is a generator (cogenerator) in  $\sigma[M]$ , then it is flat (FP-injective) over  $S$ . Results in Camillo-Fuller [5] for  $R\text{-MOD}$  can be generalized to

(4.1) *Proposition.* For an  $R$ -module  $M$  the following properties are equivalent:

- (a) every  $N \in \sigma[M]$  is flat over  $\text{End}({}_R N)$ ;
- (b) every self-injective  $N \in \sigma[M]$  is flat over  $\text{End}({}_R N)$ ;
- (c) every  $N \in \sigma[M]$  is generator in  $\sigma[N]$ .

*Proof.* (a)  $\Leftrightarrow$  (c) can be immediately deduced from [5]. Since every  $N$ -injective module in  $\sigma[N]$  is  $N$ -generated, we see with similar arguments that (b) implies that every submodule of an  $N$ -injective module in  $\sigma[N]$  is  $N$ -generated, hence (b)  $\Rightarrow$  (c).



Dually we obtain:

(4.2) *Proposition.* For an  $R$ -module  $M$  the following properties are equivalent:

- (a) every  $N \in \sigma[M]$  is FP-injective over  $\text{End}({}_R N)$ ;
- (b) every  $N \in \sigma[M]$  is coflat over  $\text{End}({}_R N)$ ;
- (c) every  $N \in \sigma[M]$  is cogenerator in  $\sigma[N]$ .

If there is a progenerator of finite length in  $\sigma[M]$  then (a) is equivalent to:

- (d) every self-projective module  $N \in \sigma[M]$  is FP-injective (coflat) over  $\text{End}({}_R N)$ .

*Proof.* The equivalence of (a), (b) and (c) is seen dually to (a)  $\Leftrightarrow$  (c) in (4.1). The progenerator of finite length in  $\sigma[M]$  guarantees the existence of a progenerator in  $\sigma[N]$  (see (1.1)); dually to (b)  $\Rightarrow$  (c) in (4.1) we can show that by (d) the factors of projective modules in  $\sigma[N]$  are cogenerated by  $N$ , i.e. (d)  $\Rightarrow$  (c).

Modules with these properties will occur in the next paragraph.

## 5. Modules of homo-serial representation type

A uniserial module  $N$  is called *homogeneously uniserial* or *homo-uniserial*, if all the finitely generated submodules of  $N$  have isomorphic tops. Direct sums of homo-uniserial modules are named *homo-serial*. We say that an  $R$ -module  $M$  is of *homo-serial representation type*, if every module in  $\sigma[M]$  is homo-serial. If  $M$  is homo-uniserial, then there is only one simple module in  $\sigma[M]$  and we observe:

(5.1) *Lemma.* Let  $M$  be a homo-uniserial  $R$ -module. Then

- (1) If  $M$  is finitely generated and self-projective, then  $M$  is a progenerator in  $\sigma[M]$ .
- (2) If  $M$  is finitely cogenerated and self-injective, then  $M$  is an (injective) cogenerator in  $\sigma[M]$ .

These facts can be used to sharpen the results on modules of serial representation type to the homo-serial case:

(5.2) *Proposition.* Let  $M$  be an  $R$ -module of finite length and of homo-serial representation type. Then

- (1) every indecomposable module  $N \in \sigma[M]$  is an injective cogenerator and projective generator in  $\sigma[N]$ ;
- (2) every self-projective module  $N \in \sigma[M]$  is an injective cogenerator in  $\sigma[N]$  (QF module);
- (3) every self-injective module  $L \in \sigma[M]$  is a projective generator in  $\sigma[L]$  (QF module);
- (4) every module  $N \in \sigma[M]$  is generator and cogenerator in  $\sigma[N]$ ;
- (5) for two indecomposable modules  $N_1, N_2$  in  $\sigma[M]$  with  $M$ -injective envelopes  $\hat{N}_1, \hat{N}_2$  we have  $\hat{N}_1 \simeq \hat{N}_2$  or  $\text{Hom}_R(\hat{N}_1, \hat{N}_2) = 0$ ;

- (6) if  $N$  is a finite direct sum of non-isomorphic indecomposable  $M$ -injective modules, then  $\text{End}({}_R N)$  is a left and right principal ideal ring.

*Proof.* (1), (2) and (3) follow from (3.3), (2.4), (2.5) and (5.1).

- (4) For any module  $N \in \sigma[M]$  denote by  $P$  the direct sum of the projective covers of the (finitely many) simple modules in  $\sigma[N]$ . By [2] and [3]  $P$  is injective cogenerator and projective generator in  $\sigma[P] = \sigma[N]$ . Since  $P$  is a direct summand of  $N^k$ ,  $k \in \mathbf{N}$ ,  $N$  is also generator and cogenerator in  $\sigma[N]$ .
- (5) The modules  $\hat{N}_1, \hat{N}_2$  are isomorphic if and only if they have isomorphic (simple) socles. Otherwise  $\text{Hom}_R(\hat{N}_1, \hat{N}_2) = 0$ .
- (6) Let  $N$  be direct sum of homo-uniserial, self-projective and self-injective modules  $N_1, \dots, N_k$ . By Lemma (3.1) all the  $\text{End}({}_R N_i)$  are left and right principal ideal rings. It follows from (5) that  $\text{End}({}_R N)$  is a ring product of the  $\text{End}({}_R N_i)$  and hence also a left and right principal ideal ring.

Let us split the characterization of modules of homo-serial representation type in two parts: In the first theorem we list equivalent conditions all of which imply the existence of a progenerator in  $\sigma[M]$ . In the second theorem we give properties which are equivalent if we assume that there is a progenerator in  $\sigma[M]$ :

(5.3) *Theorem.* For an  $R$ -module  $M$  of finite length the next statements are equivalent:

- (a)  $M$  is of homo-serial representation type;  
 (b) every finitely generated module in  $\sigma[M]$  is homo-serial;  
 (c) every (finitely generated) indecomposable module  $N \in \sigma[M]$  is  
 (i) homo-uniserial, or  
 (ii) self-projective and cogenerator in  $\sigma[N]$ ;  
 (d) every self-injective module  $N \in \sigma[M]$  is projective in  $\sigma[N]$ ;  
 (e) there is a finitely generated injective cogenerator  $Q$  in  $\sigma[M]$  and  
 (i) every fully invariant submodule  $L \subset Q$  is a projective generator in  $\sigma[L]$ , or  
 (ii) every fully invariant submodule  $L \subset Q$  is  $L$ -projective.

If  $M$  satisfies these conditions, then there is a progenerator in  $\sigma[M]$ .

*Proof.* The equivalence of (a), (b) and (c.i) is easily obtained from (3.5). We have already seen that (a) implies (c.ii), (d) and (e).

(c.ii)  $\Rightarrow$  (c.i) Let  $N$  be a finitely generated indecomposable module in  $\sigma[M]$  and  $E$  a simple module in  $\sigma[N]$ .  $N$  being a cogenerator in  $\sigma[N]$  it contains the  $N$ -injective envelope of  $E$  as direct summand. This implies that  $N$  is self-injective with simple socle  $E$ . So every finitely generated indecomposable module in  $\sigma[M]$  is self-injective and self-projective and hence uniserial by (3.5). Since  $E$  is the only simple module in  $\sigma[N]$ ,  $N$  has to be homo-uniserial.

(d)  $\Rightarrow$  (c.ii) Let  $N \in \sigma[M]$  be finitely generated and indecomposable and denote by  $\tilde{N}$  the  $N$ -injective envelope of  $N$  in  $\sigma[N]$ . By assumption  $\tilde{N}$  is projective in  $\sigma[\tilde{N}] = \sigma[N]$  and so  $\tilde{N}$  is contained in a direct sum of copies of  $N$ . Therefore the  $N$ -

injective envelope  $\tilde{E}$  of a simple submodule  $E \subset \tilde{N}$  is contained in  $N$  as a direct summand. Hence  $N = \tilde{E}$  is self-injective and self-projective. For every  $K \in \sigma[N]$  with  $N$ -injective envelope  $\tilde{K}$  the sum  $N \oplus \tilde{K}$  is self-injective and therefore projective in  $\sigma[N \oplus \tilde{K}] = \sigma[N]$ . This implies that  $\tilde{K}$  is projective in  $\sigma[N]$  and consequently cogenerated by  $N$ .

(e.ii)  $\Rightarrow$  (c.ii) For a finitely generated indecomposable module  $N \in \sigma[M]$  the module  $L = \text{Trace}(N, Q)$  is a fully invariant submodule of  $Q$ ,  $\sigma[L] = \sigma[N]$ , and  $N$  is self-injective and self-projective.  $L$  is an injective cogenerator in  $\sigma[L]$  and – as projective module in  $\sigma[N]$  – contained in a direct sum of copies of  $N$ . Therefore  $N$  is also a cogenerator in  $\sigma[N]$ .

It was already seen in (3.5) that the conditions of the theorem imply the existence of a progenerator in  $\sigma[M]$ .

We finally come to the second part of the investigation of modules of homo-serial representation type:

(5.4) *Theorem.* Let  $M$  be an  $R$ -module of finite length and suppose that there is a progenerator  $P$  in  $\sigma[M]$ . Then the following statements are equivalent:

- (a)  $M$  is of homo-serial representation type;
- (b) every (self-injective)  $N \in \sigma[M]$  is flat over  $\text{End}({}_R N)$ ;
- (c) every (self-injective)  $N \in \sigma[M]$  is generator in  $\sigma[N]$ ;
- (d) every (self-projective)  $N \in \sigma[M]$  is FP-injective (coflat) over  $\text{End}({}_R N)$ ;
- (e) every (self-projective)  $N \in \sigma[M]$  is cogenerator in  $\sigma[N]$ ;
- (f) every (finitely generated) indecomposable  $N \in \sigma[M]$  is generator and cogenerator in  $\sigma[N]$ ;
- (g) every self-projective module in  $\sigma[M]$  is self-injective;
- (h) for every fully invariant submodule  $K \subset P$  we have
  - (i)  $P/K$  is QF module, or
  - (ii)  $P/K$  is cogenerator in  $\sigma[P/K]$ , or
  - (iii)  $P/K$  is self-injective;
- (j) there is a finitely generated injective cogenerator in  $\sigma[M]$  and every fully invariant submodule  $L \subset Q$  is generator in  $\sigma[L]$ ;
- (k) there is a progenerator in  $\sigma[M]$  whose endomorphism ring is a left and right principal ideal ring.

*Proof.* We have already seen that (a) implies all the other assertions of the theorem. (b)  $\Leftrightarrow$  (c) and (d)  $\Leftrightarrow$  (e) follows from § 4.

(c)  $\Rightarrow$  (j) Let  $N$  be a finitely generated indecomposable module in  $\sigma[M]$ . By (c)  $N$  generates the  $N$ -projective cover of a simple module in  $\sigma[N]$  and hence is isomorphic to it and has a simple top. By Lemma (3.4) the injective envelopes of simple modules in  $\sigma[M]$  are finitely generated and hence we get (j).

(j)  $\Rightarrow$  (f) Let  $N \in \sigma[M]$  be finitely generated and indecomposable and set  $L = \text{Trace}(N, Q)$ . Then  $\sigma[L] = \sigma[N]$  and since  $L$  is  $N$ -generated by definition and generator in  $\sigma[N]$  by (j),  $N$  is also a generator in  $\sigma[N]$  and therefore  $N$ -projective.

This implies that  $N$  is isomorphic to a direct summand of a direct sum of copies of the  $L$ -injective module  $L$  and hence  $N$ -injective. Since there is only one simple module in  $\sigma[N]$ ,  $N$  is a cogenerator in  $\sigma[N]$ .

(f)  $\Rightarrow$  (a) It follows easily from (f) that every finitely generated indecomposable module in  $\sigma[M]$  is self-injective and self-projective and we are done by (5.3).

(h.ii) and (h.iii) both imply that  $M$  is of serial representation type by (3.5). For a finitely generated indecomposable  $N \in \sigma[M]$  set  $K = \text{Reject}(P, N)$  and have  $\sigma[P/K] = \sigma[N]$ . If  $P/K$  is cogenerator in  $\sigma[N]$ , then the same is true for  $N$  and  $N$  is homo-uniserial. If  $P/K$  is injective in  $\sigma[N]$ , then it is generated by  $N$  and  $N$  is a generator in  $\sigma[N]$  and homo-uniserial. So in each case we have (a) by (5.3).

(e)  $\Rightarrow$  (h.ii) and (g)  $\Rightarrow$  (h.iii) are trivial.

(k)  $\Rightarrow$  (a) Let  $P_0$  be a progenerator in  $\sigma[M]$  and  $\text{End}({}_R P_0)$  a left and right principal ideal ring. Then  $\sigma[M]$  is equivalent to  $\text{End}({}_R P_0)\text{-MOD}$  and the (left artinian) ring  $\text{End}({}_R P_0)$  is of homo-serial representation type.

*Remarks.* (1) The notion of a *semiinjective module* as studied in Tuganbaev [17] can also be used to describe modules of homo-serial representation type (compare Theorem (4.7) in [17]).

(2) For  $M = R$  the last two theorems give us a long list of characterizations for a left artinian ring to be left and right principal. Observing that some of the conditions imply that  $R$  is left artinian we obtain — mostly with new proofs — results of Bobylev [3], Byrd [4], Camillo-Fuller [5], Damiano [6], Goursaud [9], Roux [12], Snider [15] and others.

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