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One-Element Extensions of Distributive Groupoids

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In the paper, non-medial distributive groupoids of order 82 are described.

V článku se popisují nemediální distributivní grupoidy řádu 82.

В статье описиваются немедиальные дистрибутивные группоиды порядка 82.

1. In this section, let G be a distributive groupoid containing a subgroupoid H and an element a such that $a \notin H$ and $G = H \cup \{a\}$. Put $A = \{x \in H;\ ax = a\}$, $B = \{x \in H;\ xa = a\}$, $C = \{x \in H;\ ax \neq a\}$, $D = \{x \in H;\ xa \neq a\}$ and $b = aa$.

1.1. Lemma. $A \cap C = B \cap D = \emptyset$ and $A \cup C = B \cup D = H$.

Proof. Obvious.

1.2. Lemma. (i) $CC \subseteq C$, $DD \subseteq D$, $BC \subseteq C$, $DA \subseteq D$, $CA \subseteq C$, $BD \subseteq D$.

(ii) If $a = b$ then $AA \subseteq A$, $BB \subseteq B$, $BA \subseteq A \cap B$.

(iii) If $a \neq b$ then $AA \subseteq C$, $BB \subseteq D$, $BA \subseteq C \cap D$.

Proof. Let $x, u \in A$, $y, v \in B$, $z, w \in C$ and $r, s \in D$. Then $a . xu = ax . au = aa = b$, $yv . a = b$, $a . zw = az . aw \neq a$, $rs . a \neq a$, $a . yx = ay . ax = ay . a = a . ya = aa = b$, $yx . a = ya . xa = a . xa = ax . a = aa = b$, $a . yz = ya . yz = y . az \neq a$, $rx . a = rx . ax = ra . x \neq a$, $a . zx = ax . zx = az . x \neq a$ and $yr . a = yr . ya = y . ra \neq a$.

1.3. Lemma. C (resp. D) is either empty or a right (resp. left) ideal of H.

Proof. Use 1.2(i).

1.4. Lemma. If $D = \emptyset$ then C is either empty or an ideal of H.

Proof. Use 1.3 and 1.2(i).

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1.5. Lemma. Suppose that H is left-ideal-free.

(i) Either $D = H$ or $B = H$.

(ii) If $B = H$ then either $C = H$ or $A = H$.

Proof. Use 1.3.

1.6. Lemma. Suppose that H is both left and right-ideal-free. Then either $A = B = H$ or $A = D = H$ or $C = B = H$ or $C = D = H$.

Proof. Use 1.3.

1.7. Lemma. Suppose that H is both left and right-ideal-free and $A = D = H$. Then H is trivial and G is a two-element semigroup of left zeros.

Proof. We have $ax = a$ and $xa \in H$ for every $x \in H$. Then $xa \,.\, xa = xa = x \,.\, ay = xa \,.\, xy$ for all $x, y \in H$. From this, $(xa, xy) \in q$ where q is defined by $(u, v) \in q$ iff $zu = zv$ for every $z \in H$ (take into account that H is regular). Now, it is clear that $(xy, xz) \in q$ for all $x, y, z \in H$, H/q is a semigroup of left zeros and it is trivial, since it is right-ideal-free. In particular, $q = H \times H$, H is a semigroup of left zeros and H is trivial by similar arguments.

1.8. Lemma. If either $A = H$ or $B = H$ then $a = b$.

Proof. Obvious.

1.9. Lemma. Suppose that $C = H$ (resp. $D = H$) and put $f(x) = ax$ (resp. $g(x) = xa$) for every $x \in H$. Then f (resp. g) is an endomorphism of H and $f(x)\, y = f(y) \,.\, xy$ (resp. $x\, g(y) = xy \,.\, g(x)$) for all $x, y \in H$.

Proof. We have $f(xy) = a \,.\, xy = ax \,.\, ay = f(x) f(y)$ and $f(x)\, y = ax \,.\, y = ay \,.\, xy = f(y) \,.\, xy$.

1.10. Lemma. Suppose that $C = D = H$ and consider the endomorphisms f, g defined in 1.9. Then $x f(y) = g(x) \,.\, xy$ and $g(x)\, y = xy \,.\, f(y)$ for all $x, y \in H$. Moreover, $fg = gf$.

Proof. We have $x f(y) = x \,.\, ay = xa \,.\, xy = g(x) \,.\, xy$, $g(x)\, y = xa \,.\, y = xy \,.\, ay = xy \,.\, f(y)$ and $f\, g(x) = a \,.\, xa = aax \,.\, a = g\, f(x)$.

2. In this section, let H be a distributive groupoid such that H is a right (resp. left) quasigroup.

2.1. Lemma. Let f be an endomorphism of H such that $f(x)\, y = f(y) \,.\, xy$ for all $x, y \in H$. Then there exists an element $a \in H$ with $f(x) = ax$ for every $x \in H$.

Proof. Take an element $b \in H$. Then $f(b) = ab$ for some $a \in H$ and we have $ab = f(b) = f(b) f(b) = f(b) \,.\, ab = f(a)\, b$ which implies $f(a) = a$. Now, $ac \,.\, ac = ac = f(a)\, c = f(c) \,.\, ac$ and $ac = f(c)$ for every $c \in H$.

2.2. Lemma. Let f and g be endomorphisms of H such that $f(x)\,y = f(y)\,.\,xy$ and $x\,f(y) = g(x)\,.\,xy$ for all $x, y \in H$. Then there exists an element $a \in H$ with $f(x) = ax$ and $g(x) = xa$ for each $x \in H$.

Proof. By 2.1, there is an element $a \in H$ such that $f(x) = ax$ for every $x \in H$. Now, $xa\,.\,x = x\,.\,ax = x\,f(x) = g(x)\,x$, and hence $xa = g(x)$.

3. In this section, let H be a distributive idempotent groupoid, $b \in H$, $a \notin H$ and $G = H \cup \{a\}$. Define three groupoids $H[a, 1] = G(+)$, $H[a, b, 2] = G(-)$ and $H[a, b, 3] = G(:)$ by $x + y = x - y = x : y = xy$, $x + a = a + x = a + a = a - a = a$, $x - a = x : a = xb$, $a - x = a : x = bx$, $a : a = b$ for all $x, y \in H$.

3.1. Lemma. The groupoids $H[a, 1]$ and $H[a, b, 2]$ are distributive and idempotent and the groupoid $H[a, b, 3]$ is distributive and not idempotent.

Proof. Let $x, y, z \in G$. If $x, y, z \in H$ then both the distributive laws for these elements are clear. If $x = a$ and $y, z \in H$ then $x + (y + z) = a = (x + y) + (x + z)$, $(y + z) + x = a = (y + x) + (z + x)$, $x - (y - z) = a - yz = b\,.\,yz = by\,.\,bz = (a - y)(a - z) = (a - y) - (a - z) = (x - y) - (x - z)$, $(y - z) - x = (y - x) - (z - x)$, $x : (y : z) = (x : y) : (x : z)$ and $(y : z) : x = (y : x) : (z : x)$. If $y = a$ and $x, z \in H$ then $x + (y + z) = a = (x + y) + (x + z)$, $(y + z) + x = a = (y + x) + (z + x)$, $x - (y - z) = x - bz = x\,.\,bz = xb\,.\,xz = (x - a)\,.\,xz = (x - y) - (x - z)$, $(y - z) - x = bz - x = bz\,.\,x = bx\,.\,zx = (a - x)\,.\,zx = (y - x) - (z - x)$, $x : (y : z) = (x : y) : (x : z)$ and $(y : z) : x = (y : x) : (z : x)$. If $z = a$ and $x, y \in H$ then $x + (y + z) = a = (x + y) + (x + z)$, $(y + z) + x = a = (y + x) + (z + x)$, $x - (y - z) = x - yb = x\,.\,yb = xy\,.\,xb = (x - y) - (x - a) = (x - y) - (x - z)$, $(y - z) - x = (y - x) - (z - x)$, $x : (y : z) = (x : y) : (x : z)$ and $(y : z) : x = (y : x) : (z : x)$. If $x = y = a$ and $z \in H$ then $x + (y + z) = a = (x + y) + (x + z)$, $(y + z) + x = a = (y + x) + (z + x)$, $x - (y - z) = a - (a - z) = (a - a) - (a - z) = (x - y) - (x - z)$, $(y - z) - x = (a - z) - a = bz\,.\,b = b\,.\,zb = a - (z - a) = (y - x) - (z - x)$, $x : (y : z) = a : (a : z) = b\,.\,bz = (a : a) : (a : z) = (x : y) : (x : z)$, $(y : z) : x = (a : z) : a = bz\,.\,b = b\,.\,zb = (a : a) : (z : a) = (y : x) : (z : x)$. If $x = a = z$ and $y \in H$ then $x + (y + z) = a = (x + y) + (x + z)$, $(y + z) + x = (y + x) + (z + x)$, $x - (y - z) = a - (y - a) = b\,.\,yb = by\,.\,b = (x - y) - (x - z)$, $(y - z) - x = (y - a) - a = (y - a) - (a - a) = (y - x) - (z - x)$, $x : (y : z) = a : (y : a) = b\,.\,yb = by\,.\,b = (x : y) : (x : z)$, $(y : z) : x = (y : a) : a = yb\,.\,b = (y : x) : (z : x)$. If $x \in H$ and $y = z = a$ then $x + (y + z) = (x + y) + (x + z)$, $(y + z) + x = (y + x) + (z + x)$, $x - (y - z) = x - a = xb = xb\,.\,xb = (x - a) - (x - a) = (x - y) - (x - z)$, $(y - z) - x = (y - x) - (z - x)$, $x : (y : z) = (x : y) : (x : z)$, $(y : z) : x = (y : x) : (z : x)$. Finally, if $x = y = z = a$ then $x + (y + z) = (x + y) + (x + z)$, $(y + z) + x = (y + x) + (z + x)$, $x - (y - z) = (x - y) -$

$-(x-z)$, $(y-z)-x=(y-x)-(z-x)$, $x:(y:z)=a:b=bb=(a:a):$ $:(a:a)=(x:y):(x:z)$, $(y:z):x=bb=(y:x):(z:x)$.

3.2. Lemma. Suppose that H has no zero element. Then the groupoids $H[a, 1]$ and $H[a, b, 2]$ are not isomorphic.

Proof. Obvious.

3.3. Lemma. Suppose that H is a left (resp. right) quasigroup and $b, c \in H$. Then the groupoids $H[a, b, 2]$ and $H[a, c, 2]$ are isomorphic.

Proof. There is an element $d \in H$ such that $c = bd$. Now, define a permutation f of G by $f(x) = xd$ for every $x \in H$ and $f(a) = a$. Let $x, y \in G$. If $x, y \in H$ then $f(x-y) = f(xy) = xy \,.\, d = xd \,.\, yd = f(x)f(y) = f(x) - f(y)$. If $x = a$ and $y \in H$ then $f(x-y) = f(by) = by \,.\, d = bd \,.\, yd = c \,.\, yd = a - yd = f(a) - f(y) = f(x) - f(y)$. If $x \in H$ and $y = a$ then $f(x-y) = f(xb) = xb \,.\, d = xd \,.\, bd = xd \,.\, c = xc - a = f(x) - f(a) = f(x) - f(y)$. If $x = y = a$ then $f(x-y) = f(a) = a = a - a = f(a) - f(a) = f(x) - f(y)$. We have proved that f is an isomorphism of $H[a, b, 2]$ onto $H[a, c, 2]$.

3.4. Lemma. Suppose that H is a left (resp. right) quasigroup. Then the groupoids $H[a, b, 3]$ and $H[a, c, 3]$ are isomorphic for all $b, c \in H$.

Proof. Similar to that of 3.3.

3.5. Lemma. Suppose that H contains no zero element and no subgroupoid K with $\operatorname{card}(H - K) = 1$. Let P be a distributive idempotent groupoid such that P is not isomorphic to H and let $d \in P$, $c \notin P$. Then the groupoids $H[a, 1]$, $H[a, b, 2]$, $H[a, b, 3]$, $P[c, 1]$, $P[c, d, 2]$ and $P[c, d, 3]$ are pair-wise non-isomorphic.

Proof. Easy.

4. In this section, let H be a non-trivial left-right-ideal-free distributive groupoid such that H is a left (resp. right) quasigroup. Take two elements $a \notin H$ and $b \in H$ and put $H[1] = H[a, 1]$, $H[2] = H[a, b, 2]$ and $H[3] = H[a, b, 3]$.

4.1. Proposition. (i) If H is a subgroupoid of a distributive groupoid G such that $G - H$ is a one-element set then G is isomorphic to exactly one of the groupoids $H[1]$, $H[2]$ and $H[3]$.
(ii) The groupoids $H[1]$, $H[2]$ and $H[3]$ are pair-wise non-isomorphic one-element extensions of H.

Proof. (i) Let $G = H \cup \{a\}$. Consider the subsets A, B, C, D defined in the first section. If $A = B = H$ then G is isomorphic to $H[1]$. If this is not true then $C = D = H$ by 1.6, 1.7 and its dual. In that case, by 1.9, 1.10 and 2.2, there is an element $b \in H$ such that $ax = bx$ and $xa = xb$ for each $x \in H$. If $a = aa$ then G is isomorphic to $H[2]$ by 3.3. If $a \neq aa$ then $aa \, . \, aa = a \, . \, aa = b \, . \, aa = aa \, . \, a = aa \, . \, b$, $b = aa$ and G is isomorphic to $H[3]$ by 3.4.

(ii) This follows from 3.1 and 3.2.

4.2. Proposition. Every two-element distributive groupoid contains a one-element subgroupoid and the number of isomorphism classes of two-element distributive groupoids is equal to 4.

Proof. Easy.

5. For every positive integer $n \geqq 1$, let $a(n)$ (resp. $b(n)$, $c(n)$, $d(n)$) designate the number of isomorphism classes of non-medial (resp. idempotent non-medial, medial, idempotent medial) distributive groupoids of order n.

5.1. Proposition. $a(82) = 18$ and $b(82) = 12$.

Proof. First, let G be a non-medial distributive groupoid of order. 82 by [3], G contains a non-medial subquasigroup Q of order 81. Now, according to 4.1(i), G is isomorphic to one of the groupoids $Q[1]$, $Q[2]$ and $Q[3]$. Conversely, as proved in [4], there exist up to isomorphism just six non-medial distributive quasigroups of order 81 and the result follows from 3.5.

5.2. Remark. According to 5.1, [2], [3] and [4], we have the following table:

n	1	2	3	...	80	81	82
$a(n)$	0	0	0	...	0	6	18
$b(n)$	0	0	0	...	0	6	12
$c(n)$	1	4	19	...	$>10^{79}$	$>10^{80}$	$>10^{81}$
$d(n)$	1	3	13	...	$>10^{79}$	$>10^{80}$	$>10^{81}$

References

[1] BOL, G.: Gewebe und Gruppen, Math. Ann. 114 (1937), 414—431.

[2] JEŽEK, J., KEPKA, T. and NĚMEC, P.: Distributive groupoids, Rozpravy ČSAV 91/3, Praha 1981, 96 pp.

[3] JEŽEK, J. and KEPKA, T.: Non-medial distributive groupoids of small orders (to appear).

[4] KEPKA, T. and NĚMEC, P.: Commutative Moufang loops and distributive groupids of small orders, Czech. Math. J. 31 (1981), 633—669.