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Modules which are Epi-Equivalent to Projective Modules

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The purpose of this paper is to initiate the study of improjective modules. A module X is *im-projective* if given any epimorphism $f: B \to A$ and any homomorphism $f: X \to A$, there exists a homomorphism $h: X \to B$ such that $f h(X) \supseteq g(X)$. The class of improjective modules is shown to be closed under direct sums, and to contain the principal right ideals of a ring R for which $xR = x^2 R$ and the class of modules which are epi-equivalent to projective modules. A module is shown to be projective if and only if it is im-projective and quasi-projective. Completely reducible rings are characterized as rings for which every im-projective module is projective. Finally, we prove that if R is a ring in which each homogeneous component is finitely generated, then every finitely generated projective module is directly finite if and only if every module epi-equivalent to a finitely generated projective module is projective. This result allows us to reformulate a well known open problem on finitely generated directly finite projective modules over a von Neumann regular ring in terms of im-projective modules.

V článku se zavádí a studuje pojem im-projektivního modulu. Modul X se nazývá im-projektivní, jestliže ke každým dvěma homomorfizmům $f: B \to A$, $g: X \to A$, kde f je epi, existuje homomorfizmus $h: X \to B$ takový, že $f h(X) \supseteq g(X)$.

Изучается понятие im-проективного модуля. Модуль X называется im-проективным, если для всякого эпиморфизма $f: B \to A$ и всякого гомоморфизма $g: X \to A$ существует гомоморфизм $h: X \to B$ такой, что $f h(X) \supseteq g(X)$.

In what follows R will denote an associative ring with unity, and modules are unital right R-modules. For modules X and Y, $X \leqq Y$, $X \simeq Y$, X^N, $X \equiv Y$ symbolize X is a submodule of Y, X is R-isomorphic to Y, the direct sum of N copies of X, and X is epi-equivalent to Y (i.e. there are epimorphisms $f: X \to Y$ and $g: Y \to X$), respectively. A module X is called *strictly im-projective* if X is im-projective but not projective. A *homogeneous component* of a module X is the sum of all submodules which are isomorphic to a fixed minimal submodule of X. The right annihilator of a set S in R will be denoted by $\text{ann}_R(S)$. A sequence

$$\ldots \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \ldots$$

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of homomorphisms is said to be (*semi-*)*exact* if and only if the image of the input homomorphism (is contained in) equals the kernel of the output homomorphism at every module other than the ends of the sequence.

I. Characteristics and examples

Theorem 1.1. Let X be a module, then the following are equivalent:

(i) X is im-projective.

(ii) There exists a projective module F and homomorphisms $f\colon F \to X$ and $h\colon X \to F$ such that $f\,h(X) = X$.

(iii) Given any epimorphism $f\colon B \to A$ and any homomorphism $g\colon X \to A$ there exists an epimorphism $k\colon X \to X$ and a homomorphism $h\colon X \to B$ such that $f\,h(s) = g\,k(s)$ for all $s \in X$.

(iv) For every epimorphism $f\colon Y \to X$, $Y = \bar{X} + \operatorname{Ker} f$ where $\bar{X}$ is a homomorphic image of X (hence $\bar{X}$ is epi-equivalent to X).

(v) There exists a projective module F such that $F = M + K$ where M is a homomorphic image of F/K and X is epi-equivalent to M (hence M is epi-equivalent to F/K).

(vi) There exists a projective module F and $g \in \operatorname{End}(F)$ such that $g^2(F) = g(F)$ and X is epi-equivalent to $g(F)$.

(vii) There exists a projective module F and $f, h \in \operatorname{End}(F)$ such that $f^2h = f$ and X is epi-equivalent to $f(F)$.

(viii) There exist elements $\{x_i \mid i \in I\}$ in X and homormorphisms $\{q_i \mid i \in I\}$ in $\operatorname{Hom}(X, R)$ such that for each $y \in X$ there exists $s \in X$, depending on y, such that $y = \sum x_i\, q_i(s)$ and $q_i(s) = 0$ for all but finitely many $i \in I$.

(ix) For every exact sequence $X \xrightarrow{g} A \xrightarrow{d} C$ and any exact sequence $B \xrightarrow{f} A \to 0$ there is an induced semi-exact sequence $X \xrightarrow{h} B \xrightarrow{df} C$ such that $X \xrightarrow{fh} A \xrightarrow{d} C$ is exact.

Proof. The proof can be done in the following sequence: i $\Rightarrow$ ii $\Rightarrow$ iii $\Rightarrow$ iv $\Rightarrow$ v $\Rightarrow$ vi $\Rightarrow$ vii $\Rightarrow$ ii $\Rightarrow$ i, ii $\Leftrightarrow$ vii, i $\Leftrightarrow$ ix. We will indicate only parts of the proof since the remainder is similar.

(ii $\Rightarrow$ iii). There exists a projective module F and homomorphisms $p\colon F \to X$ and $q\colon X \to F$ such that $p\,q(X) = X$. Since F is projective there exists a homomorphism $\bar{h}\colon F \to B$ such that $f\bar{h} = gp$. Therefore $f\bar{h}q = gpq$. Let $h = \bar{h}q$ and $k = pq$. Consequently $fh = gk$.

(ii $\Rightarrow$ viii). In condition (ii) the projective module F can be taken to be free. Let $y \in X$. From (ii) there exists $s \in X$ such that $fh(s) = y$. Then $h(s) \in F$ and $h(s) = \sum t_i r_i$ where $\{t_i \mid i \in I\}$ is a basis for F and $r_i \in R$. Hence $y = fh(s) = \sum f(t_i)\, r_i =$

$= \sum x_i r_i = \sum x_i\, q_i(s)$ where $\{x_i \mid i \in I\}$ is a set of generators for X and $q_i\colon X \to R$ is defined by $q_i(x) = c_i$ for all $x \in X$ where $h(x) = \sum t_i c_i$ and $c_i \in R$.

(ix $\Rightarrow$ i). Consider the exact sequence $X \xrightarrow{g} A \xrightarrow{d} A/\mathrm{Im}\, g$ where d is the natural epimorphism. Let $f\colon B \to A$ be an epimorphism. Then there exists $h\colon X \to B$ such that $X \xrightarrow{fh} A \xrightarrow{d} A/\mathrm{Im}\, g$ is exact. Hence $\mathrm{Im}\, fh = \mathrm{Ker}\, d = \mathrm{Im}\, g$. Consequently, X is im-projective. This completes the proof.

From Theorem 1.1 (ii) we observe that any module epi-equivalent to an im-projective is itself im-projective, and if $X \equiv P$ where P is projective then $X = \bar{P} \oplus K$ with $\bar{P} \simeq P$. Part (iii) shows that every im-projective module is pseudo-projective in the sense of [3] and [4]. Part (viii) is a generalization of the Dual Basis Lemma.

Corollary 1.2. Let xR be a principal right ideal of R, and let X be a cyclic module. Then:

(i) X is im-projective if and only if X is epi-equivalent to xR where $xR = x^2R$.

(ii) $xR = x^2R$ if and only if $R = xR + \mathrm{ann}_R(x)$.

(iii) If xR is a maximal right ideal then xR is im-projective if and only if $\mathrm{ann}_R(x) \nsubseteq xR$.

Thus if R is a p.p. ring (i.e. every principal right ideal is projective), then every cyclic im-projective module is epi-equivalent to a direct summand of R. Also, if R is fully idempotent (i.e. $I = I^2$ for all ideals of R) then every principal right ideal which is an ideal is im-projective.

Example 1.3. Let R be a ring such that $R = A \oplus B$ with $R \simeq A \simeq B$ (e.g. the endomorphism ring of an infinite dimensional vector space over a field) [6]. Since R is not completely reducible there exists a maximal right ideal K which is not a direct summand of R [9, p. 67]. Then $R \oplus R/K \equiv R$. Thus $R \oplus R/K$ is a cyclic strictly im-projective module which is a direct sum of a projective module and a simple, hence quasi-projective, module.

Corollary 1.4. If R is a (semi-) hereditary or a right perfect ring then a (finitely generated) module is im-projective if and only if it is epi-equivalent to a projective module.

Proof. If R is (semi-) hereditary then every (finitely generated) submodule of a projective module is projective. If R is right perfect then every module has a projective cover. The result now follows from Theorem 1.1.

Proposition 1.5. Let M be a module and F be an infinitely generated free module such that M is a homomorphic image of F. Then $M \oplus F$ is epi-equivalent to F, hence $M \oplus F$ is im-projective.

Proof. Since F is infinitely generated, $F \simeq F \oplus F$. There exists an epimorphism $f: F \oplus F \to M \oplus F$. Consequently, $F \equiv M \oplus F$.

Theorem 1.6. A direct sum of im-projective modules is im-projective.

Proof. Let $\{X_i \mid i \in I\}$ be a set of im-projective modules. From Theorem 1.1 for each $i \in I$ there exists a projective module F_i and homomorphisms $f_i: F_i \to X_i$ and $h_i: X_i \to F_i$ such that $f_i h_i(X_i) = X_i$. Therefore there exists homomorphisms $f: \oplus_{i \in I} F_i \to \oplus_{i \in I} X_i$ and $h: \oplus_{i \in I} X_i \to \oplus_{i \in I} F_i$ such that $fh(\oplus_{i \in I} X_i) = \oplus_{i \in I} X_i$. Since $\oplus_{i \in I} F_i$ is projective, then $\oplus_{i \in I} X_i$ is im-projective by Theorem 1.1.

Proposition 1.7. Let G be a non-projective generator then there exists a cardinal number N such that the direct sum of N copies of G is strictly im-projective.

Proof. Let G be a generator. By [*1*, p. 112] $G^n \simeq R \oplus L$ where n is a natural number. There exists an infinite cardinal number N such that there is an epimorphism $f: R^N \to G^n$. But then there is an epimorphism $\bar{f}: (R^N)^N \to (G^n)^N$, and an epimorphism $g: (G^n)^N \to R^N$. However $(R^N)^N \simeq R^N$ and $(G^n)^N \simeq G^N$. Consequently, $R^N \equiv G^N$. Therefore G^N is im-projective.

II. Im-projectivity equivalent to projectivity

Theorem 2.1. Let X be a module. Then X is projective if and only if X is im-projective and for every epimorphism $k \in \mathrm{End}(X)$ there exists $g \in \mathrm{End}(X)$ such that kg is an automorphism.

Proof. Assume X is im-projective and for every epimorphism $k \in \mathrm{End}(X)$ there exists $g \in \mathrm{End}(X)$ such that kg is an automorphism. From Theorem 1.1 (ii), there exists a projective module F and homomorphisms $f: F \to X$ and $h: X \to F$ such that $f\,h(X) = X$. Hence there exists $g \in \mathrm{End}(X)$ such that fhg is an automorphism. Therefore $F = \mathrm{Im}\, hg \oplus \mathrm{Ker}\, f$ and $X \simeq \mathrm{Im}\, hg$. Consequently, X is projective. The converse is obvious.

Corollary 2.2. A module is projective if and only if it is im-projective and quasi-projective.

Corollary 2.3. If X is im-projective and dual continuous, then X is projective.

Proof. The proof follows from Theorem 2.1 and the definition of a dual continuous module [*10*].

Corollary 2.3 has no converse since there are projective modules which are not dual continuous [*10*].

A module is *hopfian* if every onto endomorphism is an automorphism (e.g. any noetherian module is hopfian) [*11*]. For projective modules the hopfian condition is equivalent to the directly finite condition (i.e. a module is directly finite if it is not isomorphic to any proper direct summand of itself [7, p. 49]. Also, if $A \equiv B$ and either A or B is hopfian then $A \simeq B$. From Theorem 2.1 it follows that a hopfian im-projective module is projective.

Proposition 2.4. Let P be a projective module which is epi-equivalent to X. If P or X is directly finite, then P is isomorphic to X.

Proof. Routine.

Theorem 2.5. The following are equivalent:

(i) R is completely reducible.
(ii) Every im-projective module is projective.
(iii) Every module epi-equivalent to a projective module is projective.
(iv) Every module is im-projective.
(v) Every simple module is im-projective.

Proof. (i) implies (ii), (iii), (iv), (v) because a ring is completely reducible if and only if every module is projective. By Theorem 1.1, ii $\Rightarrow$ iii. By Proposition 1.5, iii $\Rightarrow$ i.

(iv & v $\Rightarrow$ i). Every simple module is im-projective and hopfian. Thus every simple module is projective. Hence every maximal right ideal is a direct summand of R. By [9, p. 67] R is completely reducible. This completes the proof.

Example 1.3 shows that not every finitely generated im-projective module is projective. However there is a large class of rings in which finitely generated im-projective modules are projective, such a ring will be called a FIMP ring.

Theorem 2.6. The following types of rings are FIMP:

(i) Commutative rings.
(ii) Right noetherian rings.
(iii) Semiperfect rings.
(iv) Unit regular rings.
(v) Semifirs.
(vi) Rings which have finite Goldie dimension and for which each finitely generated im-projective module is epiequivalent to a projective module (e.g. semihereditary rings with finite Goldie dimension).

Proof. (i and ii). If R is commutative or right noetherian, then every finitely generated module is hopfian [*16*] and [9, p. 23].

(iii). If R is semiperfect, then every finitely generated im-projective module is epi-equivalent to its projective cover. From [*13*], every finitely generated projective module is hopfian. Hence every finitely generated im-projective module is isomorphic to its projective cover.

(iv). If R is unit regular, then it is semihereditary and every finitely generated projective module is directly finite [7, p. 50]. By Corollary 1.4 and Proposition 2.4, R is FIMP.

(v). From [6, p. 218], it follows that a semifir is FIMP.

(vi). If R has finite Goldie dimension then every finitely generated projective module is directly finite. By Proposition 2.4, R is FIMP. This completes the proof.

From [6], a ring has IBN (i.e. invariant basis number) if the rank of any free module is uniquely determined. Any noetherian ring is IBN.

Proposition 2.7. Let R be a ring such that every finitely generated module which is epi-equivalent to a projective module is projective. Then R is IBN. In particular, FIMP rings are IBN.

Proof. Assume every finitely generated module which is epi-equivalent to a projective module is projective, but R is not IBN. Then there exist positive integers n and p such that $R^n \simeq R^{n+p}$. Also, since R is not completely reducible there exists a nonzero $A \leqq R$ such that R/A is not projective. There exist epimorphisms $f\colon R^{n+p} \to \to R^{n+p-1} \oplus R/A$ and $g\colon R^{n+p-1} \oplus R/A \to R^n$. Hence $R^n \equiv R^{n+p-1} \oplus R/A$. Thus $R^{n+p-1} \oplus R/A$ is strictly im-projective and finitely generated. Contradiction! Consequently R is IBN.

Proposition 2.8. In the following types of rings every cyclic im-projective module is projective:

(i) FIMP rings.

(ii) Reduced rings.

(iii) Directly finite p.p. rings.

Proof. The proof follows from Corollary 1.2 and Proposition 2.4.

In [6], Cohn has constructed domains which are not IBN. Hence, by Proposition 2.8 (ii), these are examples of non-FIMP rings in which each cyclic im-projective module is projective. Thus we see that the finitely generated condition in Proposition 2.7 cannot be relaxed to a cyclic condition.

III. Applications

Theorem 3.1. Let P be a projective module which is not directly finite. Then there exists $K \neq 0$ such that $P = \bar{P} \oplus K$ where P is isomorphic to $\bar{P}$, and one and only one of the following conditions is satisfied:

(i) There exists $X \leqq P$ such that P/X is epi-equivalent to P where P/X is strictly im-projective.

(ii) For any decomposition of the form $P = \bar{P} \oplus K$ where $\bar{P}$ is isomorphic to P, then K is a completely reducible module.

Proof. (not i $\Rightarrow$ ii). Suppose $P = \bar{P} \oplus H$ where $P \simeq \bar{P}$. Let $A \leqq H$. Then there exists an epimorphism $f: P/A \to \bar{P}$. Therefore $P/A \equiv P$. Hence P/A is projective. Hence A is a direct summand of P. Thus A is a direct summand of H. Consequently, H is completely reducible.

(i $\Rightarrow$ not ii). Assume $P/X \equiv P$ where P/X is strictly im-projective. Then there exists $H \geqq X$ such that $P/H \simeq P$. Hence $P = \bar{P} \oplus H$ where $\bar{P} \simeq P$. If H is completely reducible, then X is a direct summand of P. Thus P/X is projective. Contradiction! Thus H is not completely reducible. Consequently, part (ii) is not satisfied. This completes the proof.

Corollary 3.2. Let R be a ring in which every homogeneous component is finitely generated. Then every finitely generated projective module is directly finite if and only if every module which is epi-equivalent to a finitely generated projective module is projective. In particular, R is directly finite if and only if every module which is epi-equivalent to a cyclic projective module is projective.

Proof. If every finitely generated projective module is directly finite, then by Proposition 2.4 every module which is epi-equivalent to a finitely generated projective module is projective. Conversely, suppose that every module which is epi-equivalent to a finitely generated projective module is projective. Let P be a finitely generated projective module which is not directly finite. From Theorem 3.1, $P = \bar{P} \oplus K$ where $\bar{P} \simeq P$ and $K \neq 0$ is completely reducible. By an induction argument, for any positive integer n there exists P_n and X_n such that $P = P_n \oplus X_n$, $P_n \simeq P$ and $X_n = \bigoplus_{i=1}^{n} K_i$ with $K_i \simeq K$ for $i = 1, 2, \ldots, n$. But this contradicts the fact that every homogeneous component of R is finitely generated [*1*, pp. 119, 110, 111]. Consequently, every finitely generated projective module is directly finite. In the cyclic case, the proof is similar.

Corollary 3.3. Let R be a semihereditary ring in which every homogeneous component is finitely generated. Then every finitely generated projective module is directly finite if and only if R is a FIMP ring.

Proof. Corollary 3.2 and Corollary 1.4.

Corollary 3.3 allows us to give the following equivalent formulation of problem 1 in [*7*, p. 344]: If R is a simple directly finite regular ring, is R a FIMP ring? Hence an example of a finitely generated strictly im-projective module over a simple directly finite regular ring would provide a negative answer. We observe, from Corollary 3.2, that such an example could not be cyclic.

In [*6*] Cohn gives examples of IBN domains which have a non-hopfian finitely generated free module. Since a domain has zero socle, it follows from Theorem 3.1 that these domains have a strictly im-projective module which is epi-equivalent to a finitely generated free module. Hence these domains are not FIMP although every cyclic im-projective module is projective by Proposition 2.8. Thus Proposition 2.7 has no converse.

Two problems which naturally arose in this work are to determine "nice" characterizations of:

1. Rings for which every (finitely generated) im-projective module is epi-equivalent to a projective module.
2. FIMP rings and rings in which every module which is epi-equivalent to a finitely generated projective module is projective.

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