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On the Maximum Principle in the Linear-Elasticity Theory

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Let us have the system of partial differential equations of the linear elasticity. The solution of this system with the bounded boundary condition (in general) is not bounded (i. e. the displacement vector is unbounded). In the paper it is shown, that the maximum principle does not hold even for the continuous solutions of this system. The maximum principle does not hold even for the smooth solutions of the system with the smooth coefficients.

О принципе максимума в теории линейной упругости. Рассмотрим систему дифференциальных уравнений (в частных производных) теории упругости. Решение этой системы с ограниченным краевым условием не является, вообще говоря, ограниченным (т.е. не ограниченны компоненты вектора трансляции). В статье приводятся примеры, показающие, что принцип максимума не имеет места для непрерывного решения этой системы. Принцип максимума не имеет места ни тогда, когда предполагается, что решение и коэффициенты системы обладают непрерывными производными всех порядков.

O principu maxima v teorii lineární pružnosti. Uvažujme systém parciálních diferenciálních rovnic lineární pružnosti. Řešení tohoto systému s omezenou okrajovou podmínkou obecně není omezené (tj. nejsou omezené složky vektoru posunutí). V článku jsou podány příklady, které ukazují, že princip maxima neplatí ani pro spojitá řešení tohoto systému. Princip maxima neplatí ani pro hladká řešení systému s hladkými koeficienty.

Let $D(\Omega)$ be the class of real functions, each of which is infinitely differentiable and has its support in the domain $\Omega \subset E_3$. Let $W_2^1(\Omega)$, $\mathring{W}_2^1(\Omega)$ be usual Sobolev spaces, and C_{ijkl}, $i, j, k, l = 1, 2, 3$, – the tensor of elastic coefficients in Ω with the following properties:

$$(1) \quad \begin{cases} C_{ijkl}(x) \text{ are measurable, bounded in } \Omega, \\ C_{ijkl} = C_{klij} = C_{jikl}, \; i, j, k, l = 1, 2, 3, \\ C_{ijkl}\, \xi_{ij}\, \xi_{kl} \geq \sigma\, \xi_{ij}\, \xi_{ij} \text{ for any symmetric matrix } (\xi_{ij})_1^3, \end{cases}$$

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where $\sigma > 0$ is a constant. Denote the strain tensor e_{kl} by $e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$, $k, l = 1, 2, 3$ (where $u \in [W_2^1(\Omega)]^3$ is the displacement vector).

Let $u_0 \in [W_2^1(\Omega)]^3$. We say, that the vector function $u \in [W_2^1(\Omega)]^3$ *is the generalized solution of the second problem of mathematical theory of elasticity in Ω with the boundary condition $u(x) = u_0(x)$ on $\partial \Omega$*, if the following two conditions are fulfilled:

(i) $$\int_\Omega C_{ijkl} \frac{\partial v_i}{\partial x_j} e_{kl}\, dx = 0 \qquad \text{for every } v \in [\mathring{W}_2^1(\Omega)]^3$$

(we neglect body forces),

(ii) $$u - u_0 \in [\mathring{W}_2^1(\Omega)]^3.$$

The existence and the uniqueness of a solution follows from the inequality

(2) $$\|v\|_{1,2}^2 \leq C \int_\Omega C_{ijkl} \frac{\partial v_i}{\partial x_j} \frac{\partial v_k}{\partial x_l}\, dx, \quad v \in [\mathring{W}_2^1(\Omega)]^3,$$

where C is a positive constant depending only on σ and on Ω, and from the Lax-Milgram lemma.

Define $$\Omega = \{x \in E_3; \|x\| < 1\}$$

(3) $$C_{ijkl}(x) = \frac{1}{2}(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) + \delta_{ij}\,\delta_{kl} + \frac{3}{\|x\|^2}(\delta_{ij}x_k x_l + \delta_{kl}x_i x_j) + \frac{9}{\|x\|^4} x_i x_j x_k x_l,\ \|x\| \neq 0,\ i, j, k, l = 1, 2, 3,$$

where δ_{ij} is the Kronecker symbol delta,

$$u(x) = x\|x\|^\alpha = (x_1\|x\|^\alpha, x_2\|x\|^\alpha, x_3\|x\|^\alpha), \text{ where } \alpha = \frac{3(1 - \sqrt{17})}{2\sqrt{17}}.$$

Theorem. The displacement vector $u(x) = x\|x\|^\alpha$ is the generalized solution of the second problem of mathematical theory of elasticity (with the coefficients (3)) in Ω with the boundary condition $u(x) = x$ on $\partial\Omega$. The solution u is not bounded in Ω. The proof is contained in [3].

Now we find the sequence of the coefficients $\{C_{ijkl}^n\}_{n=2}^\infty$ and the corresponding solutions $\{u^n\}_{n=2}^\infty$ in Ω: Let $n > 1$ be any natural number. Put

$$\Omega_1^n = \{x \in E_3;\ \|x\| < 1/n\},\ \Omega_2^n = \{x \in E_3;\ 1/n < \|x\| < 1\},\ n = 2, 3, \ldots$$

Let $A = (3\sqrt{17} - 1)/2$. For $x \in \overline{\Omega_1^n}$ we define:

(4) $$C_{ijkl}^n(x) = \frac{A}{4}\left[\frac{1}{2}(\delta_{ik}\,\delta_{lj} + \delta_{il}\,\delta_{jk}) + \delta_{ij}\,\delta_{kl}\right], i, j, k, l = 1, 2, 3.$$

For $x \in \Omega_2^n$ we define $C_{ijkl}^n(x)$ by the formula (3). We see, that the sequence $\{C_{ijkl}^n\}_{n=2}^{\infty}$ is uniformly bounded on Ω and that the relations (1) holds for C_{ijkl}^n with $\sigma = 1$. Now we define the corresponding generalized solutions:

$$u^n(x) = x(1/n)^{\alpha}, \quad x \in \overline{\Omega_1^n}$$
$$u^n(x) = x\|x\|^{\alpha}, \; x \in \Omega_2^n, \; n = 2, 3, \ldots$$

These vector functions are continuous in Ω, $u^n \in [W_2^1(\Omega)]^3$ and $u^n(x) = u_0(x) = x$ for $x \in \partial\Omega$, $n = 2, 3, \ldots$

Theorem. For any natural $n > 1$ the continuous displacement vector $u^n(x)$ is the generalized solution of the second problem of the mathematical theory of elasticity (with the coefficients C_{ijkl}^n) in Ω with the boundary conditions $u^n(x) = x$ on $\partial\Omega$. The sequence $\{u^n\}$ is unbounded in Ω.

Proof. For $\|x\| \neq 0$ put $n(x) = \dfrac{x}{\|x\|}$; $n(\mathbf{0}) = \mathbf{0}$. Then is

$$n_j C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l} = n_i \left(\frac{1}{n}\right)^{\alpha} A, \quad \frac{\partial}{\partial x_j}\left(C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l}\right) = 0, \; x \in \Omega_1^n, \; i = 1, 2, 3,$$

$$n_j C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l} = n_i \|x\|^{\alpha} A, \quad \frac{\partial}{\partial x_j}\left(C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l}\right) = 0, \; x \in \Omega_2^n, \; i = 1, 2, 3.$$

According to Green's theorem is for $v \in [\mathring{W}_2^1(\Omega)]^3$:

$$\int_{\Omega_1^n} C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l}\frac{\partial v_i}{\partial x_j} dx = \int_{\partial\Omega_1^n} v_i n_j C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l} dS = A\left(\frac{1}{n}\right)^{\alpha} \int_{\|x\| = 1/n} n_i v_i dS,$$

$$\int_{\Omega_2^n} C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l}\frac{\partial v_i}{\partial x_j} dx = \int_{\partial\Omega_2^n} v_i(-n_j) C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l} dS = -A\left(\frac{1}{n}\right)^{\alpha} \int_{\|x\| = 1/n} n_i v_i dS.$$

Hence is

$$\int_{\Omega} C_{ijkl}^n \frac{\partial u_k^n}{\partial x_l}\frac{\partial v_i}{\partial x_j} dx = \int_{\Omega^n} \ldots + \int_{\Omega_2^n} \ldots = 0.$$

The rest of the proof is obvious.

Now we define

$$\tilde{C}_{ijkl}^n(x) = \frac{1}{2}(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) + \delta_{ij}\delta_{kl} + (1 - \varphi(nx)) \,.$$

$$\cdot \left[\frac{3}{\|x\|^2}(\delta_{ij} x_k x_l + \delta_{kl} x_i x_j) + \frac{9}{\|x\|^4} x_i x_j x_k x_l\right], \; i, j, k, l = 1, 2, 3, \; n = 1, 2, \ldots,$$

$$\text{where } \varphi \in D(\Omega), \; 0 \leq \varphi \leq 1 \text{ on } \Omega, \; \varphi(x) = 1 \text{ for } \|x\| \leq \frac{1}{2}\,.$$

Then the assumptions (1) are fulfilled for $\sigma = 1$, the sequence $\{\tilde{C}_{ijkl}^n\}_{n=1}^{\infty}$ is uniformly bounded, and $\tilde{C}_{ijkl}^n$ are infinitely differentiable on $\bar{\Omega}$, $n = 1, 2, \ldots$.

Hence the corresponding solutions $\tilde{u}^n$ with the boundary conditions $\tilde{u}^n(x) = x$ on $\partial\Omega$ are also infinitely differentiable on Ω. For $u(x) = x\|x\|^a$ is $\tilde{u}^n - u \in [\mathring{W}_2^1(\Omega)]^3$. Then is (due to our first theorem and to (2)):

$$\|u - \tilde{u}^n\|_{1,2}^2 \leq C \int_\Omega \tilde{C}_{ijkl}^n \frac{\partial(u - \tilde{u}^n)_i}{\partial x_j} \frac{\partial(u - \tilde{u}^n)_k}{\partial x_l} dx =$$

$$= C \int_\Omega \tilde{C}_{ijkl}^n \frac{\partial u_i}{\partial x_j} \frac{\partial(u - \tilde{u}^n)_k}{\partial x_l} dx = C \int_\Omega (\tilde{C}_{ijkl}^n - C_{ijkl}) \frac{\partial u_i}{\partial x_j} \frac{\partial(u - \tilde{u}^n)_k}{\partial x_l} dx \leq$$

$$\leq C_1 \left(\int_{\|x\|<1/n} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial xj} dx \right)^{1/2} \cdot \left(\int_{\|x\|<1/n} \frac{\partial(u - \tilde{u}^n)_k}{\partial x_l} \frac{\partial(u - u^n)_k}{\partial x_l} dx \right)^{1/2} \leq$$

$$\leq C_2 \left(\int_{\|x\|<1/n} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx \right)^{1/2} \cdot \|u - \tilde{u}^n\|_{1,2},$$

and consequently

$$\|u - \tilde{u}^n\|_{1,2} \leq C_2 \left(\int_{\|x\|<1/n} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx \right)^{1/2} \to 0, \text{ when } n \to \infty.$$

Hence the sequence $\{\tilde{u}^n\}_{n=1}^\infty$ is unbounded in Ω.

We found the sequence of the uniformly bounded and "uniformly" elliptic smooth coefficients and the unbounded sequence of the smooth solutions of the corresponding systems in the sphere with the uniform bounded (smooth) boundary condition. Hence the maximum principle for the smooth solutions of the system with the smooth coefficients does not hold.

References

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