Acta Universitatis Carolinae. Mathematica et Physica

J. Blank; P. Exner

Remarks on tensor products and their applications in quantum theory I. General considerations

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 17 (1976), No. 2, 75--98

Persistent URL: http://dml.cz/dmlcz/142389

1976 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 17, NO. 2, Pag. 75–98

Remarks on Tensor Products and Their Applications in Quantum Theory
I. General Considerations

J. BLANK, P. EXNER

Department of Theoretical Nuclear Physics, Faculty of Mathematics and Physics, Charles University, Prague)*

Received 5 June 1975

The notion of tensor product of Hilbert spaces and operators on them is widely used in quantum theory, though it is often understood only intuitively. We collect and examine here the basic definitions and properties concerning the tensor products. Typical examples are also discussed.

О тензорных произведениях и их применениях в квантовой теории — I. Общие соображения. — Тензорные произведения гильбертовых пространств и операторов широко применяются в квантовой теории несмотря на то, что они часто понимаются только интуитивно. В работе разобраны основные определения и дан обзор важнейших свойств тензорных произведений. Рассмотрены также типичные примеры.

Poznámky k tensorovým součinům a jejich použití v kvantové teorii — I. Obecné úvahy. — Pojmů tensorového součinu Hilbertových prostorů a operátorů na nich se široce užívá v kvantové teorii, ačkoli jsou často chápány pouze intuitivně. V práci jsou shromážděny a probrány základní definice a vlastnosti týkající se tensorových součinů. Diskutují se zde též některé typické příklady.

I. Introduction

The concept of tensor product of Hilbert spaces and operators on them is of fundamental importance in quantum theory: it is sufficient to mention the description of particles with spin, of the many-particle systems or the Fock spaces on which quantum field theory is built.

However, in many standard textbooks of quantum theory these things are treated in a way which is far from being satisfactory, not only from the mathematical point of view, but from the point of view of the physical theory itself. In this "elementary" treatment vectors of the Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ are written in the form $/\rangle_1 /\rangle_2$ where $/\rangle_i$ is a vector of $\mathscr{H}_i$. Now, according to the superposition principle, any linear combination of vectors of this type must again belong to $\mathscr{H}$; however, if $/\psi\rangle_1 \neq /\tilde{\psi}\rangle_1$ and $/\varphi\rangle_2 \neq /\tilde{\varphi}\rangle_2$, we cannot write $/\psi\rangle_1/\varphi\rangle_2 +$

*) 110 00 Praha 1, Myslíkova 7, Czechoslovakia

$+ |\tilde{\psi}\rangle_1 |\tilde{\varphi}\rangle_2$ in the form $|\rangle_1 |\rangle_2$. If the $\mathscr{H}_i$'s are spaces of square-integrable functions $\psi(\mathbf{r}_i)$, i.e. $\mathscr{H}_i = L^2(\mathbf{R}^3)$, we guess that $\mathscr{H}$ is the space of all square-integrable functions $\Psi(\mathbf{r}_1, \mathbf{r}_2)$, i.e. $\mathscr{H} = L^2(\mathbf{R}^6)$, and contains not only the products $\psi(\mathbf{r}_1)\,\varphi(\mathbf{r}_2)$. Nevertheless, in more complicated cases the appropriate guess is not so obvious, and thus a rigorous general definition is a prerequisite for the theory.

From the point of view of mathematics the problems concerning tensor products are of no special difficulty. Unfortunately, this topic does not belong to standard contents of textbooks on functional analysis or mathematical physics and, moreover, there is no unique way of treating it; there are differences even in the basic definitions and it is often non-trivial to see their mutual connections.

These circumstances, which undoubtely hinder physicists from a better understanding and a fully qualified use of tensor products, led us to the idea of writing this paper. We tried to define here the basic concepts in a way which should be understandable to physicists who are familiar with elements of the theory of Hilbert spaces and linear operators, to give further a review of different properties of tensor products which are needed in practical applications and finally to illustrate these applications on several typical examples.

Section 2 is devoted to tensor products of Hilbert spaces. We use essentially the Jauch's definition [1] who treats tensor product as a mapping satisfying certain conditions. The questions of existence and uniqueness are examined and it is shown, that other occuring definitions can be regarded as different realizations of this mapping. Several examples of frequently used tensor products are included in order to illustrate the general statements. The basic notions concerning tensor products of linear operators (in general unbounded) are included in Section 3. The last section contains a rather detailed summary of properties of bounded tensor-product operators, special attention being paid to those which are important for applications in quantum theory: Hermitian, unitary, projections and trace classes.

In the second part of this paper we shall examine the question of deriving spectral properties of a tensor product operator from those of its constituent operators and discuss typical examples showing general application of the tensor product formalism in quantum theory.

Since some notions and symbols used in the text might not be commonly known, we give now a brief list of them:

$\mathbf{R}$ ...real line

$\mathbf{C}$ ...complex plane

$\mathsf{A} \times \mathsf{B}$ ...Cartesian product of the sets A, B

$\mathbf{R}^n (n = 1, 2, \ldots)$ ...$\mathbf{R} \times \mathbf{R} \times \ldots \times \mathbf{R}$ (similarly for $\mathbf{C}^n$)

$f\colon \mathsf{M} \to \mathsf{N}$ ...a mapping from the set M to N; f is called injective if it is one-one, surjective if it maps M onto N and bijective if it is both injective and surjective

$\mathscr{H}, \mathscr{H}_r (r = 1, 2, \ldots)$ ...Hilbert spaces (the corresponding scalar products and norms are labelled in the same way)

M_λ ...linear envelope of the set M, i.e. the minimal linear manifold containing M

$\mathsf{M}^\perp$ ...orthogonal complement of M

complete set...a set $\mathsf{M} \subset \mathscr{H}$ is complete if $\mathsf{M}^\perp$ contains only zero vector ($\mathsf{M}^\perp = \{0\}$), which is equivalent to $\bar{\mathsf{M}}_\lambda = \mathscr{H}$

subspace ...closed linear manifold $\mathscr{G}$ in $\mathscr{H}$, i.e. $\mathscr{G} = \bar{\mathscr{G}}_\lambda$

$\mathscr{G}_1 \dot{+} \mathscr{G}_2$, $\sum_{i=1}^{n} \dot{+} \mathscr{G}_i$ ...orthogonal sum of subspaces $\mathscr{G}_i \subset \mathscr{H}$, i.e. direct sum of mutually orthogonal subspaces (general direct sum is denoted by $\oplus$)

$A \upharpoonright \mathsf{D}$ ...restriction of operator A to a subset D of its domain

■ ...end of a proof

2. Tensor product of Hilbert spaces

Definition 1: *Tensor product* $\mathscr{H}_1 \otimes \mathscr{H}_2$ of Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ is the class of all pairs $\mathscr{H}, \varphi$, where $\mathscr{H}$ is a Hilbert space and φ a mapping from $\mathscr{H}_1 \times \mathscr{H}_2$ to $\mathscr{H}$ such that:

(φ1) φ is bilinear

(φ2) $(\varphi(x_1, x_2), \varphi(y_1, y_2)) = (x_1, y_1)_1 (x_2, y_2)_2$ for all $[x_1, x_2]$, $[y_1, y_2] \in \mathscr{H}_1 \times \mathscr{H}_2$

(φ3) $\mathscr{H}$ is spanned by the set $\Phi = \varphi(\mathscr{H}_1 \times \mathscr{H}_2)$, i.e. $\bar{\Phi}_\lambda = \mathscr{H}$.

Each such pair is called *realization* of $\mathscr{H}_1 \otimes \mathscr{H}_2$ and will be denoted by $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ *).

Let us stress that the realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$ is not only a Hilbert space — it must be also specified how elements of $\mathscr{H}_1 \times \mathscr{H}_2$ are related to those of $\mathscr{H}$, which is expressed by φ.

Two questions naturally arise:

(i) Does for any $\mathscr{H}_1$, $\mathscr{H}_2$ at least one realization exist?

(ii) What is the connection between different realizations of $\mathscr{H}_1 \otimes \mathscr{H}_2$ (if two or more of them exist)?

In order to answer the first question consider the following construction (cf. ref. [3]): Let F be the set of all complex bilinear forms $f[.,\ ..]$ on $\mathscr{H}_1 \times \mathscr{H}_2$, the equality of $f_1, f_2 \in \mathsf{F}$ being defined "pointwise", i.e. $f_1 = f_2$ means $f_1[x_1, x_2] = f_2[x_1, x_2]$ for all $[x_1, x_2] \in \mathscr{H}_1 \times \mathscr{H}_2$. The set F becomes a complex vector space if one puts

$$(f_1 + f_2)\,[x_1, x_2] = f_1[x_1, x_2] + f_2[x_1, x_2]$$
$$(\alpha f)\,[x_1, x_2] = \bar{\alpha} f[x_1, x_2] \qquad (\alpha \in \mathbf{C}) .$$

*) Cf. ref. [1]. One can also omit (φ3) and understand by $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ the pair $\bar{\Phi}_\lambda$, φ (see ref. [2]), this slight difference being of little importance. Other definitions and their connections to the above one will be discussed below.

The zero vector of F will be denoted by 0_F, i.e. $0_F[x_1, x_2] = 0$ for all $[x_1, x_2]$. Define now a mapping $\varphi : \mathscr{H}_1 \times \mathscr{H}_2 \to F$, $[y_1, y_2] \to \varphi_{y_1,y_2}$, by

$$\varphi_{y_1,y_2}[., ..] = (., y_1)_1 (.., y_2)_2 . \tag{2.1}$$

This mapping is obviously bilinear and it is not injective (one-one), i.e. the pair $[y_1, y_2]$ is not uniquely determined by the form φ_{y_1,y_2}. However, φ is "almost injective", as the following lemma states:

Lemma 2.1:

(a) If $\varphi_{y_1,y_2} = 0_F$, then at least one of y_1, y_2 is zero.

(b) Let $\varphi_{y_1,y_2} \neq 0_F$ and $\varphi_{y_1,y_2} = \varphi_{z_1,z_2}$,

then $z_1 = \alpha y_1$, $z_2 = \frac{1}{\alpha} y_2$ for some $\alpha \in \mathbf{C}$, $\alpha \neq 0$.

Proof: (a) Substituting $x_1 = y_1$, $x_2 = y_2$ into $\varphi_{y_1,y_2}[x_1, x_2] = 0$, we get from (2.1) $\|y_1\|_1^2 \|y_2\|_2^2 = 0$, i.e. $y_1 = 0$ or $y_2 = 0$.

(b) According to (a) both y_1 and y_2 are non-zero, i.e. $\varphi_{y_1,y_2}[y_1, y_2] \neq 0$. Since

$$\varphi_{y_1,y_2}[x_1, x_2] = \varphi_{z_1,z_2}[x_1, x_2] \tag{*}$$

for all $[x_1, x_2] \in \mathscr{H}_1 \times \mathscr{H}_2$, we have $(y_1, z_1)_1 (y_2, z_2)_2 \neq 0$. Substitution $x_1 = y_1/\|y_1\|_1^2$ into (*) gives $(x_2, y_2)_2 = (x_2, \alpha_2 z_2)_2$ where $\alpha_2 = \frac{1}{\|y_1\|_1^2} \overline{(y_1, z_1)} \neq 0$. Now, x_2 is any vector from $\mathscr{H}_2$ and thus $y_2 = \alpha_2 z_2$. Similarly $y_1 = \alpha_1 z_1$ with $\alpha_1 = \frac{1}{\|y_2\|_2^2} \overline{(y_2, z_2)} \neq 0$ and, since $z_2 = \frac{1}{\alpha_2} y_2$, we get $\alpha_1 = \frac{1}{\alpha_2}$. ■

Let us denote by Φ the φ-image of $\mathscr{H}_1 \times \mathscr{H}_2$ (see condition (φ3)) and introduce the following function Ψ on $\Phi \times \Phi$:

$$\Psi(\varphi_{x_1,x_2}, \varphi_{y_1,y_2}) = (x_1, y_1)_1 (x_2, y_2)_2 . \tag{2.2}$$

This definition makes sense: according to the above lemma and elementary properties of scalar product

$$\Psi(\varphi_{x_1,x_2}, \varphi_{y_1,y_2}) = \Psi(\varphi_{x_1',x_2'}, \varphi_{y_1',y_2'}) ,$$

if $\varphi_{x_1',x_2'} = \varphi_{x_1,x_2}$ and $\varphi_{y_1',y_2'} = \varphi_{y_1,y_1}$. Comparing (2.2) with (2.1) we find

$$\Psi(\varphi_{x_1,x_2}, \varphi_{y_1,y_2}) = \varphi_{y_1,y_2}[x_1, x_2] = \overline{\varphi_{x_1,x_2}[y_1, y_2]} , \tag{2.3a}$$

$$\Psi(\varphi_{x_1,x_2}, \varphi_{y_1,y_2}) = \overline{\Psi(\varphi_{y_1,y_2}, \varphi_{x_1,x_2})} . \tag{2.3b}$$

In view of obvious relations

$$\Psi(0_F, \varphi_{y_1,y_2}) = \Psi(\varphi_{x_1,x_2}, 0_F) = 0$$

one can extend Ψ by linearity from $\Phi \times \Phi$ to $\Phi_\lambda \times \Phi_\lambda$ putting

$$\Psi(f, g) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \bar{\beta}_j (y_1^{(i)}, z_1^{(j)})_1 (y_2^{(i)}, z_2^{(j)})_2 \tag{2.4}$$

if

$$f = \sum_{i=1}^{m} \alpha_i \varphi_{y_1^{(i)}, y_2^{(i)}}, \quad g = \sum_{j=1}^{n} \beta_j \varphi_{z_1^{(j)}, z_2^{(j)}}.$$

Again the value $\Psi(f, g)$ does not change if f and/or g are expressed by means of another linear combinations. This can be easily seen if one takes into account that

$$f[z_1^{(j)}, z_2^{(j)}] = \sum_{i=1}^{m} \bar{\alpha}_i \varphi_{y_1^{(i)}, y_2^{(i)}} [z_1^{(j)}, z_2^{(j)}] = \sum_{i=1}^{m} \overline{\alpha_i (y_1^{(i)}, z_1^{(j)})_1 (y_2^{(i)}, z_2^{(j)})_2}$$

which implies

$$\Psi(f, g) = \sum_{j=1}^{n} \overline{\beta_j f[z_1^{(j)}, z_2^{(j)}]} \,. \tag{2.4a}$$

In the same way one gets

$$\Psi(f, g) = \sum_{i=1}^{m} \alpha_i \, g[y_1^{(i)}, y_2^{(i)}] \,. \tag{2.4b}$$

Thus Ψ satisfies

(i) $\Psi(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \Psi(f_1, g) + \alpha_2 \Psi(f_2, g)$ for all $f_1, f_2, g \in \Phi_\lambda$, $\alpha_1, \alpha_2 \in \mathbf{C}$,

(ii) $\Psi(f, g) = \overline{\Psi(g, f)}$.

We further show that Ψ is a scalar product on Φ_λ:

Proof: Because of (i) and (ii) it is sufficient to prove that $\Psi(f, f) \geq 0$ for all $f \in \Phi_\lambda$ and $\Psi(f,f) = 0$ only if $f = 0_F$. Let $f = \sum_{i=1}^{m} \alpha_i \varphi_{y_1^{(i)}, y_2^{(i)}}$ and assume that $\varphi_{y_1^{(i)}, y_2^{(i)}} \neq 0_F$ at least for one i (otherwise $f = 0_F$ and $\Psi(f, f) = 0$ due to (i)). Denote by $\mathscr{G}_r$ $(r = 1, 2)$ the subspace of $\mathscr{H}_r$ spanned by the vectors $y_r^{(i)}$ $(i = 1, 2, \ldots, m)$; its dimension $m_r = \dim \mathscr{G}_r$ satisfies $1 \leq m_r \leq m$. Let $\{e_r^{(k)}\}_{k=1}^{m_r}$ be an orthonormal basis for $\mathscr{G}_r$ and denote by $\eta_k^{(i,r)}$ the components of $y_r^{(i)}$ in this basis; then

$$\varphi_{y_1^{(i)}, y_2^{(i)}} = \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} \overline{\eta_k^{(i,1)}} \overline{\eta_l^{(i,2)}} \varphi_{e_1^{(k)}, e_2^{(l)}}$$

and

$$f = \sum_{k=1}^{m_1} \sum_{k=1}^{m_2} \gamma_{kl} \, \varphi_{e_1^{(k)}, e_2^{(l)}}$$

where

$$\gamma_{kl} = \sum_{i=1}^{m} \alpha_i \overline{\eta_k^{(i,1)} \eta_l^{(i,2)}} \,.$$

Substituting this expression for f into (2.4), we get

$$\Psi(f, f) = \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} |\gamma_{kl}|^2 \,.$$

Thus $\Psi(f, f) \geq 0$, and if $\Psi(f, f) = 0$ then all the $\gamma_{kl} = 0$, i.e. $f = 0_F$. ■

Now we complete Φ_λ under this scalar product by the standard procedure (see e.g. ref. [4]) and obtain thus a Hilbert space $\mathscr{H}$ such that Φ_λ is dense in it. Hence $\mathscr{H}$ and the bilinear mapping φ satisfy condition (φ3), and they satisfy also (φ2) due to (2.2). Summarizing, we can give the following answer to our first question:

Theorem 1: To any pair of Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$ there exists at least one realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Let us mention now, when the question of existence is positively answered, the following important property of realizations of tensor products:

Lemma 2.1a:

Let $\mathscr{H}, \varphi$ be a realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$.

(a) If $\varphi(x_1, x_2) = 0$, then at least one of x_1, x_2 is zero

(b) If $\varphi(x_1, x_2) \neq 0$ and $\varphi(x_1, x_2) = \varphi(y_1, y_2)$, then

$$y_1 = \alpha x_1, \; y_2 = \frac{1}{\alpha} x_2 \text{ for some } \alpha \in \mathbf{C}, \; \alpha \neq 0.$$

Proof is an obvious modification of that of Lemma 2.1. ■

Let us now pass to the second question. Firstly we prove another useful lemma:

Lemma 2.2: Let $\mathsf{M}_r \, (r = 1, 2)$ be a complete set in $\mathscr{H}_r$, and let a realization $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ be given. Then $\varphi(\mathsf{M}_1 \times \mathsf{M}_2)$ is complete in $\mathscr{H}$.

Proof: Let us take any $y_r \in \mathscr{H}_r$. According to our assumption $y_r \in \overline{(\mathsf{M}_r)_\lambda}$, i.e. there exists a sequence $\{y_r^{(n)}\}_{n=1}^\infty \subset (\mathsf{M}_r)_\lambda$ such that $y_r^{(n)} \to y_r$. Then $\|y_r^{(n)}\|_r \to \|y_r\|_r$ and $(y_r^{(n)}, x)_r \to (y_r, x)_r$ for any $x \in \mathscr{H}_r$. Because of (φ2) we have

$$\|\varphi(y_1, y_2) - \varphi(y_1^{(n)}, y_2^{(n)})\|^2 =$$
$$= \|y_1\|_1^2 \|y_2\|_2^2 + \|y_1^{(n)}\|_1^2 \|y_2^{(n)}\|_2^2 - 2 \operatorname{Re}(y_1, y_1^{(n)})_1 (y_2, y_2^{(n)})_2 \to 0 ,$$

i.e.

$$(2.5) \qquad \Phi = \varphi(\mathscr{H}_1 \times \mathscr{H}_2) \subset \overline{\varphi((\mathsf{M}_1)_\lambda \times (\mathsf{M}_2)_\lambda)} \subset \overline{(\varphi(\mathsf{M}_1 \times \mathsf{M}_2))_\lambda}$$

(the second inclusion is easily verified using (φ1)). Further $\overline{(\varphi(\mathsf{M}_1 \times \mathsf{M}_2))_\lambda}$ is a linear manifold; then (2.5) implies $\Phi_\lambda \subset \overline{(\varphi(\mathsf{M}_1 \times \mathsf{M}_2))_\lambda}$ and also $\overline{\Phi}_\lambda \subset \overline{(\varphi(\mathsf{M}_1 \times \mathsf{M}_2))_\lambda}$. On the other hand, from $\varphi(\mathsf{M}_1 \times \mathsf{M}_2) \subset \Phi$ we get $\overline{(\varphi(\mathsf{M}_1 \times \mathsf{M}_2))_\lambda} \subset \overline{\Phi}_\lambda$; hence $\overline{(\varphi(\mathsf{M}_1 \times \mathsf{M}_2))_\lambda} = \overline{\Phi}_\lambda = \mathscr{H}$. ■

This lemma has some important consequences:

Corollary 1: Let $\mathscr{H}_r$ be separable and $\mathscr{E}_r = \{e_r^{(i)}\}_{i=1}^{\dim \mathscr{H}_r}$ be an orthonormal basis in $\mathscr{H}_r$. Then the (arbitrarily) ordered set $\varphi(\mathscr{E}_1 \times \mathscr{E}_2) = \{\varphi(e_1^{(i)}, e_2^{(j)}) \mid 1 \leq i \leq \dim \mathscr{H}_1, 1 \leq j \leq \dim \mathscr{H}_2\}$ is an orthonormal basis in $\mathscr{H}$. In other words: If $\mathscr{H}_r$ are separable, then for any realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$ the space $\mathscr{H}$ is separable and $\dim \mathscr{H} = \dim \mathscr{H}_1 \, . \, \dim \mathscr{H}_2$ *).

In practical applications, when we deal with a concrete realization, it may turn out difficult to verify (φ3). The above lemma makes it possible to replace this condition by a more convenient one:

*) This statement holds for non-separable Hilbert spaces as well; it is only necessary to use the general definition, according to which dim $\mathscr{H}$ is the cardinality of any complete orthonormal set in $\mathscr{H}$ (see e.g. ref. [5]).

Corollary 2: Let $\mathscr{H}_r$ $(r = 1, 2)$ be separable and $\mathscr{E}_r$ be an orthonormal basis in $\mathscr{H}_r$. If a mapping $\varphi : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}$ satisfies $(\varphi 1)$ and $(\varphi 2)$ and $\varphi(\mathscr{E}_1 \times \mathscr{E}_2)$ is an orthonormal basis in $\mathscr{H}$, then $\overline{(\varphi(\mathscr{H}_1 \times \mathscr{H}_2))_\lambda} = \mathscr{H}$ *).

The following theorem shows that a realization can be constructed for any $\mathscr{H}$ which has an "appropriate" dimension:

Theorem 2: Let $\mathscr{H}_1, \mathscr{H}_2$ and $\mathscr{H}$ be given. Then a realization $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ exists if and only if

$$\dim \mathscr{H}_1 \,.\, \dim \mathscr{H}_2 = \dim \mathscr{H} \,. \tag{2.6}$$

Proof: According to Corollary 1 of Lemma 2.2 the condition (2.6) is necessary. Suppose now that (2.6) holds and denote by $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H}_0,\varphi_0}$ the "constructive" realization from Theorem 1. Then $\dim \mathscr{H}_0 = \dim \mathscr{H}_1 \,.\, \dim \mathscr{H}_2 = \dim \mathscr{H}$, so that $\mathscr{H}_0$ is isomorphic to $\mathscr{H}$. Hence there exists a unitary (isometric) operator U that maps $\mathscr{H}_0$ onto $\mathscr{H}$. The compound mapping $\varphi(.,\, ..) = U(\varphi_0(.,\, ..))$ maps $\mathscr{H}_1 \times \mathscr{H}_2$ to $\mathscr{H}$ and, due to the properties of U and φ_0, it obviously satisfies $(\varphi 1)$, $(\varphi 2)$. Let $y \in (\varphi(\mathscr{H}_1 \times \mathscr{H}_2))^\perp$, i.e. $(\varphi(x_1, x_2), y) = 0$ for all $[x_1, x_2] \in \mathscr{H}_1 \times \mathscr{H}_2$; writing $y = Ux$, $x \in \mathscr{H}_0$, we get $(\varphi_0(x_1, x_2), x)_0 = 0$ for all $[x_1, x_2] \in \mathscr{H}_1 \times \mathscr{H}_2$. Since $\varphi_0(\mathscr{H}_1 \times \mathscr{H}_2)$ is complete in $\mathscr{H}_0$, one has $x = 0$ and $y = Ux = 0$. Hence $(\varphi(\mathscr{H}_1 \times \mathscr{H}_2))^\perp = \{0\}$, which is equivalent to $(\varphi 3)$. ■

Thus we see that for any $\mathscr{H}_1$, $\mathscr{H}_2$ there exists an infinite variety of realizations of $\mathscr{H}_1 \otimes \mathscr{H}_2$. We now show that these realizations are closely connected to each other. For this purpose we introduce the notion of *isomorphic realizations*: $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ is isomorphic to $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H}',\varphi'}$ if a unitary operator U from $\mathscr{H}$ onto $\mathscr{H}'$ exists such that

$$U(\varphi(x_1, x_2)) = \varphi'(x_1, x_2) \tag{2.7}$$

for all

$$[x_1, x_2] \in \mathscr{H}_1 \times \mathscr{H}_2 \,.$$

Notice that this definition requires more than a mere isomorphism of $\mathscr{H}$ and $\mathscr{H}'$ (this is trivial due to Corollary 1 of Lemma 2.2). The reason is clear from the definition of tensor product and the remark following it.

Theorem 3.: For given $\mathscr{H}_1$, $\mathscr{H}_2$ all realizations $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ are mutually isomorphic.

Proof: We take any two realizations and denote as usually: $\Phi = \varphi(\mathscr{H}_1 \times \mathscr{H}_2)$, $\Phi' = \varphi'(\mathscr{H}_1 \times \mathscr{H}_2)$. Because of Lemma 2.1a one can regard relation (2.7) as the definition of a mapping g from Φ onto Φ' which, according to $(\varphi 2)$, preserves scalar product. Since g maps the zero of $\mathscr{H}$ to the zero of $\mathscr{H}'$, we can extend g by

*) This statement can again be generalized for non-separable $\mathscr{H}_r$'s if one replaces everywhere "orthonormal basis" by "complete orthonormal set".

linearity to a linear operator $V_0 : \Phi_\lambda \to \Phi'_\lambda$. Clearly, V_0 maps Φ_λ onto Φ'_λ and

$$(V_0 x, V_0 y)' = (x, y) \quad \text{for all} \quad x, y \in \Phi_\lambda .$$

In particular for $x = y$ one gets that V_0 is bounded. Extending V_0 continuously by the standard procedure (see ref. [2]) to $\overline{\Phi}_\lambda = \mathscr{H}$ and using the above mentioned properties of it, we obtain a bounded operator V which maps $\mathscr{H}$ onto $\overline{\Phi}'_\lambda = \mathscr{H}'$ and preserves scalar product. Hence V is a unitary operator which has all the properties required by the definition of isomorphic realizations. ■

Remark: According to this theorem, the tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ can be represented by any of its realizations, and the realizations can be understood as elements of the equivalence class $\mathscr{H}_1 \otimes \mathscr{H}_2$. It is a usual practice in such cases to interchange the both notions, wherever it is convenient*).

For example, one often says "$\mathscr{H}$ is the tensor product of $\mathscr{H}_1, \mathscr{H}_2$" and writes $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ if $\mathscr{H}, \varphi$ is a currently used realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$, the mapping φ being implicitely "determined" by the "natural" relation of $\mathscr{H}$ to $\mathscr{H}_1$ and $\mathscr{H}_2$ (see Examples 2.1 and 2.2 below). Also from the point of view of applications, namely those in quantum theory, it is convenient to understand by the tensor product one of its realizations (in general unspecified); in the following sections we shall do so. On the other hand, we must be careful with this interchanging of notions when different tensor products are involved. If $\mathscr{H}, \varphi$ and $\tilde{\mathscr{H}}, \tilde{\varphi}$ are given realizations of $\mathscr{H}_1 \otimes \mathscr{H}_2$, $\tilde{\mathscr{H}}_1 \otimes \tilde{\mathscr{H}}_2$, respectively, then different relations between $\mathscr{H}$ and $\tilde{\mathscr{H}}$ may exist but we cannot automatically write these relations between $\mathscr{H}_1 \otimes \mathscr{H}_2$ and $\tilde{\mathscr{H}}_1 \otimes \tilde{\mathscr{H}}_2$ without having a meaningful definition of them.

Thus our second question is fully answered. Let us now prove one property of tensor products that we shall use in Section 4:

Lemma 2.3: Let $\mathscr{H}, \varphi$ be a realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$ and $\mathscr{G}', \mathscr{G}''$ be orthogonal subspaces of $\mathscr{H}_1$. If $\mathscr{G} = \mathscr{G}' \dotplus \mathscr{G}''$, then

$$\overline{(\varphi(\mathscr{G} \times \mathscr{H}_2))_\lambda} = \overline{(\varphi(\mathscr{G}' \times \mathscr{H}_2))_\lambda} \dotplus \overline{(\varphi(\mathscr{G}'' \times \mathscr{H}_2))_\lambda} . \tag{2.8}$$

Proof of this lemma is based on the following statement which is a simple consequence of the definition of the orthogonal sum (see e.g. ref. [2]): If linear manifolds $\mathsf{L}_1, \mathsf{L}_2$ are orthogonal, $\mathsf{L}_1 \perp \mathsf{L}_2$, then their direct sum satisfies

$$\overline{\mathsf{L}_1 \oplus \mathsf{L}_2} = \overline{\mathsf{L}_1} \dotplus \overline{\mathsf{L}_2} . \tag{2.9}$$

Now, the sets $\varphi(\mathscr{G}' \times \mathscr{H}_2) \subset \mathscr{H}$, $\varphi(\mathscr{G}'' \times \mathscr{H}_2) \subset \mathscr{H}$ are orthogonal because of (φ2) and hence $(\varphi(\mathscr{G}' \times \mathscr{H}_2))_\lambda \perp (\varphi(\mathscr{G}'' \times \mathscr{H}_2))_\lambda$.
Since $\mathscr{G}' \subset \mathscr{G}$, $\mathscr{G}'' \subset \mathscr{G}$, it obviously holds

$$(\varphi(\mathscr{G}' \times \mathscr{H}_2))_\lambda \oplus (\varphi(\mathscr{G}'' \times \mathscr{H}_2))_\lambda \subset (\varphi(\mathscr{G} \times \mathscr{H}_2))_\lambda . \tag{2.10}$$

Let us consider an arbitrary $z = (\varphi(\mathscr{G} \times \mathscr{H}_2))_\lambda$, i.e. $z = \sum_{i=1}^{n} \alpha_i \varphi(x_1^{(i)}, x_2^{(i)}), x_1^{(i)} \in \mathscr{G}$,

*) This is the same situation as e.g. in $L^2(a, b)$ and analogous functional spaces, elements of which are usually called "functions", while in fact they are classes of equivalent functions.

$x_2^{(i)} \in \mathscr{H}_2$. Each vector $x_1^{(i)}$ can be uniquely written in the form $x_1^{(i)} = x_1'^{(i)} + x_1''^{(i)}$, where $x_1'^{(i)} \in \mathscr{G}'$ and $x_1''^{(i)} \in \mathscr{G}''$. Using further bilinearity of φ, one can write $z = z' + z''$ where $z' \in (\varphi(\mathscr{G}' \times \mathscr{H}_2))_\lambda$, $z'' \in (\varphi(\mathscr{G}'' \times \mathscr{H}_2))_\lambda$, and therefore equality holds in (2.10). One then obtains (2.8) by putting $\mathsf{L}_1 = (\varphi(\mathscr{G}' \times \mathscr{H}_2))_\lambda$, $\mathsf{L}_2 = (\varphi(\mathscr{G}'' \times \mathscr{H}_2))_\lambda$ in (2.9). ■

Before finishing this general part let us notice that the definition of tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ and of its realizations can be easily generalized for the case of any finite number of the $\mathscr{H}_r$'s by virtue of a multilinear mapping $\varphi : \mathscr{H}_1 \times \mathscr{H}_2 \times \ldots \times \mathscr{H}_n \to \mathscr{H}$ with the properties analogous to (φ2), (φ3); the above theorems can be reformulated for $\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \ldots \otimes \mathscr{H}_n$ in an obvious way. We shall not introduce tensor product for infinite systems of Hilbert spaces.

We shall now discuss several examples of tensor products which occur frequently in applications.

Example 2.1: Consider the realization $(L^2(\mathbf{R}) \otimes L^2(\mathbf{R}))_{L^2(\mathbf{R}^2),\varphi}$ where φ expresses the following "natural" relation of $L^2(\mathbf{R}^2)$ to $L^2(\mathbf{R})$:

$$(\varphi(x_1, x_2))\,(t_1, t_2) = x_1(t_1)\, x_2(t_2) \qquad (x_1, x_2 \in L^2(\mathbf{R}))\,.$$

The Fubini theorem implies that φ is a (bilinear) mapping from $L^2(\mathbf{R}) \times L^2(\mathbf{R})$ to $L^2(\mathbf{R}^2)$ such that (φ2) is satisfied. If $\mathscr{E} = \{e_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\mathbf{R})$, then the (arbitrarily) ordered set $\{\varphi(e_j, e_k)\}_{j,k=1}^{\infty}$ is an orthonormal basis in $L^2(\mathbf{R}^2)$ (see ref. [2]). Hence, due to Corollary 2 of Lemma 2.2, $L^2(\mathbf{R}^2)$, φ is a realization of $L^2(\mathbf{R}) \otimes L^2(\mathbf{R})$, and it is this realization that one currently understands by the tensor product of $L^2(\mathbf{R}) \otimes L^2(\mathbf{R})$ (cf. Section 1). In the sense of the remark to Theorem 3 we conventionally write

$$L^2(\mathbf{R}) \otimes L^2(\mathbf{R}) = L^2(\mathbf{R}^2)\,. \tag{2.11}$$

This result can be obviously generalized as follows

$$L^2(\mathbf{R}^m) \otimes L^2(\mathbf{R}^m) = L^2(\mathbf{R}^{m+n}) \tag{2.11a}$$

for any natural m, n. One obtains further generalization for the spaces $L^2(\mathsf{M}_r, \mathrm{d}\mu_r)$ of μ_r-square integrable functions on a μ_r-measurable set M_r (where μ_r, $r = 1, 2$, is a general measure) in the following way (see ref. [3]): let μ_{12} be product measure of μ_1, μ_2; then

$$L^2(\mathsf{M}_1, \mathrm{d}\mu_1) \otimes L^2(\mathsf{M}_2, \mathrm{d}\mu_2) = L^2(\mathsf{M}_1 \times \mathsf{M}_2, \mathrm{d}\mu_{12})\,. \tag{2.11b}$$

Example 2.2 (vector-valued functions): Let $\mathscr{H}_0$ be separable; consider the vector space of all mappings $f : \mathbf{R}^n \to \mathscr{H}_0$ (with the "pointwise" defined addition and multiplication by complex numbers) such that

$$\int_{\mathbf{R}^n} \|f(t_1, t_2, \ldots, t_n)\|_0^2 \,\mathrm{d}t_1\, \mathrm{d}t_2, \ldots \mathrm{d}t_n < \infty\,.$$

One can define scalar product on this vector space by

$$(f, g) = \int_{\mathbf{R}^n} (f(t_1, t_2, \ldots, t_n), g(t_1, t_2, \ldots, t_n))_0 \,\mathrm{d}t_1\, \mathrm{d}t_2 \ldots \mathrm{d}t_n \tag{2.12}$$

and obtain thus a Hilbert space which is called the *space of vector-valued functions* and denoted by $L^2(\mathbf{R}^n; \mathscr{H}_0)$ (see refs. [2], [3] for further details). For example, $L^2(\mathbf{R}^3; \mathbf{C}^n)$ is the space formed by n-tuples of square integrable functions $\psi_i(\mathbf{r})$ *):

$$\Psi(\mathbf{r}) = \{\psi_1(\mathbf{r}), \psi_2(\mathbf{r}), \ldots, \psi_n(\mathbf{r})\}, \text{ such that}$$

$$\sum_{i=1}^{n} \int_{\mathbf{R}^3} |\psi_i(\mathbf{r})|^2 \, d^3r < \infty ;$$

then

$$(\Psi, \Phi) = \sum_{i=1}^{n} \int_{\mathbf{R}^3} \psi_i(\mathbf{r}) \, \bar{\varphi}_i(\mathbf{r}) \, d^3r .$$

Thus $L^2(\mathbf{R}^3; \mathbf{C}^2)$ is the space used in the elementary quantum-mechanical description of the electron spin, $L^2(\mathbf{R}^3; \mathbf{C}^4)$ is used in Dirac's theory etc.

We show now that $L^2(\mathbf{R}^n; \mathscr{H}_0)$ is a realization of $L^2(\mathbf{R}^n) \otimes \mathscr{H}_0$. **) The corresponding (bilinear) mapping is "naturally" defined by

$$(\varphi(x_1, x_2))\,(t_1, t_2, \ldots, t_n) = x_1(t_1, t_2, \ldots, t_n) \,.\, x_2$$

where $x_1 \in L^2(\mathbf{R}^n)$, $x_2 \in \mathscr{H}_0$, i.e. the value of $\varphi(x_1, x_2)$ at a point $\{t_1, t_2, \ldots, t_n\} \in$ $\in \mathbf{R}^n$ is obtained by multiplying $x_2 \in \mathscr{H}_0$ by the complex number $x_1(t_1, t_2, \ldots, t_n)$. Using (2.12) one easily finds

$$(\varphi(x_1, x_2), \varphi(y_1, y_2)) =$$

$$= (x_2, y_2)_0 \;.\; \int_{\mathbf{R}^n} x_1(t_1, t_2, \ldots, t_n) \, \overline{y_1(t_1, t_2, \ldots, t_n)} \, dt_1 \, dt_2 \ldots dt_n$$

and hence (φ2) is satisfied. Finally, if $\mathscr{E}_1 = \{e_j\}_{j=1}^{\infty}$ and $\mathscr{E}_2 = \{f_k\}_{k=1}^{\dim \mathscr{H}_0}$ are orthonormal bases in $L^2(\mathbf{R}^n)$ and $\mathscr{H}_0$ respectively, then the (arbitrarily) ordered set $\{\varphi(e_j, f_k) \mid j, k = 1, 2, \ldots\}$ is an orthonormal basis in $L^2(\mathbf{R}^n; \mathscr{H}_0)$ (ref. [2]). Thus we can write conventionally

$$L^2(\mathbf{R}^n) \otimes \mathscr{H}_0 = L^2(\mathbf{R}^n; \mathscr{H}_0) . \tag{2.13}$$

Example 2.3 (Fock spaces): Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{H}^n$, φ_n be a realization of the n-fold tensor product $\mathscr{H} \otimes \mathscr{H} \otimes \ldots \otimes \mathscr{H}$ $(n = 2, 3, \ldots)$. The above mentioned "current realizations" are usually used and thus $\mathscr{H}^n$, φ_n need not be explicitely specified. We denote further $\mathscr{H}^0 = \mathbf{C}$, $\mathscr{H}^1 = \mathscr{H}$ and define the *Fock space* $\mathscr{F}(\mathscr{H})$ over $\mathscr{H}$:

$$\mathscr{F}(\mathscr{H}) = \sum_{n=0}^{\infty} \oplus \mathscr{H}^n \text{ ***)} . \tag{2.14}$$

As a typical example we shall consider $\mathscr{F}(L^2(\mathbf{R}^3))$. Denoting again the points of

*) The points of $\mathbf{R}^3$ are denoted by $\mathbf{r}$ – "radius-vectors".

**) In this form, namely as tensor products of a space referring to orbital quantum numbers and of that referring to spin quantum numbers, the spaces $L^2(\mathbf{R}^3; \mathbf{C}^n)$ usually occur in quantum theory — cf. refs. [6], [7], [8].

***) See ref. [2] for the definition of the infinite direct sum of Hilbert spaces.

$\mathbf{R}^3$ by $\mathbf{r}$ and putting $\mathscr{H}^n = L^2(\mathbf{R}^{3n})$ (see Example 2.1), we find that elements of $\mathscr{F}(L^2(\mathbf{R}^3))$ are sequences

$$\Psi = \{\psi_0, \psi_1(\mathbf{r}_1), \psi_2(\mathbf{r}_1, \mathbf{r}_2), \ldots\}$$

where $\psi_0 \in \mathbf{C}$, $\psi_n \in L^2(\mathbf{R}^{3n})$ $(n = 1, 2, \ldots)$, such that

$$|\psi_0|^2 + \sum_{n=1}^{\infty} \int_{\mathbf{R}^{3n}} |\psi_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)|^2 \, d^3r_1 \, d^3r_2 \ldots d^3 r_n < \infty .$$

Two subspaces of $\mathscr{F}(\mathscr{H})$ are of special interest for quantum theory: the *symmetric* (*Bose-Fock*) *space* $\mathscr{F}_s(\mathscr{H})$ and the *antisymmetric* (*Fermi-Fock*) *space* $\mathscr{F}_a(\mathscr{H})$. Construction of these subspaces is discussed in detail in refs. [2], [3], and we shall remind here only its main features.

Let $\mathscr{S}_n$ be the permutation group of n elements $(n \geq 2)$ and denote by $\pi(\sigma)$ the parity of permutation $\sigma \in \mathscr{S}_n$, i.e. $\pi(\sigma) = \pm 1$. To each $\sigma \in \mathscr{S}_n$ a unitary operator $U_n(\sigma)$ on $\mathscr{H}^n$ is assigned in such a way that the family $\{U_n(\sigma) \mid \sigma \in \mathscr{S}_n\}$ is a unitary representation of $\mathscr{S}_n$. This fact implies that the operators

$$S_n = \frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} U_n(\sigma)$$

$$A_n = \frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} \pi(\sigma) \, U_n(\sigma)$$

are projections. The spaces $\mathscr{F}_s(\mathscr{H})$ and $\mathscr{F}_a(\mathscr{H})$ are then defined by

(2.15) $$\mathscr{F}_s(\mathscr{H}) = \sum_{n=0}^{\infty} \oplus \, S_n \mathscr{H}^n$$

(2.16) $$\mathscr{F}^a(\mathscr{H}) = \sum_{n=0}^{\infty} \oplus \, A_n \mathscr{H}^n$$

where $S_r = A_r = I_r$ for $r = 0,1$ (I_r is the unit operator on $\mathscr{H}^r$). For $\mathscr{H}^n = L^2(\mathbf{R}^{3n})$ one obtains from the definition of $U_n(\sigma)$:

$$(U_n(\sigma) \, \psi_n) \, (\mathbf{r}_{\sigma(1)}, \mathbf{r}_{\sigma(2)}, \ldots, \mathbf{r}_{\sigma(n)}) = \psi_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$$

and this relation implies that the space $S_n L^2(\mathbf{R}^{3n})$ consists of functions $\psi_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$ which are invariant under any permutation of $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}$, while the functions belonging to $A_n L^2(\mathbf{R}^{3n})$ change sign according to the parity of the permutation involved.

The importance of $\mathscr{F}_s(\mathscr{H})$ and $\mathscr{F}_a(\mathscr{H})$ for quantum theory is obvious: they serve for the description of boson and fermion field, respectively.

Concluding this section, we shall establish connections between Definition 1 and other occuring definitions of the tensor product. These definitions describe in fact constructions of some special realization, say $\mathscr{H}_d$, φ_d, of $\mathscr{H}_1 \otimes \mathscr{H}_2$. One of these constructions, in which a subset of the space of all complex bilinear forms on $\mathscr{H}_1 \times \mathscr{H}_2$ is used as $\mathscr{H}_d$ (see e.g. ref. [3]), has been applied in the proof of

Theorem 1. The construction described e.g. in ref. [9] starts with the Hilbert space $\mathscr{H}_d$ of antilinear Hilbert-Schmidt operators from $\mathscr{H}_1$ to $\mathscr{H}_2$; then a mapping φ_d from $\mathscr{H}_1 \times \mathscr{H}_2$ to $\mathscr{H}_d$ is constructed that is shown to satisfy $(\varphi 1) - (\varphi 3)$ *). Another construction (see e.g.ref. [10]) consists in the direct extension of $\mathscr{H}_1 \times \mathscr{H}_2$ to a Hilbert space by formal summing.

These definitions have one common disadvantage: the realizations of $\mathscr{H}_1 \otimes \mathscr{H}_2$ which are of practical importance (e.g. (2.11), (2.13) etc.) are usually not identical with $\mathscr{H}_d$, φ_d. It is then necessary to introduce a rather vague notion of "natural isomorphism" [3]: for example $L^2(\mathbf{R}^2)$ is said to be "naturally isomorphic" to $L^2(\mathbf{R}) \otimes L^2(\mathbf{R})$, which in the language used here means that $(L^2(\mathbf{R}) \otimes L^2(\mathbf{R}))_{\mathscr{H}_d, \varphi_d}$ and the realization from Example 2.1 are isomorphic realizations.

On the other hand, we regard as an advantage of Definition 1 that it corresponds to the general algebraical definition of the tensor product of vector spaces (see e.g.ref. [11]; the "category-theoretical fashion" of the definition is, of course, more elegant, but it would need too many preparatory notions to be presented here). Naturally, the latter definition does not contain conditions analogous to $(\varphi 2)$ and $(\varphi 3)$ which are related to the topological structure of involved Hilbert spaces. However, it represents a general starting point for defining tensor products of some other important spaces (e.g. Banach, locally convex etc.): the corresponding definitions may be obtained by adding to it certain "topological" conditions.

3. Tensor product of operators — general case

Throughout this section we shall understand by T_r $(r = 1,2)$ a densely defined linear operator (in general unbounded) on Hilbert space $\mathscr{H}_r$; the domain of T_r will be denoted by D_r, i.e. $\overline{\mathsf{D}}_r = \mathscr{H}_r$. We shall construct from the T_r's an operator on $\mathscr{H}_1 \otimes \mathscr{H}_2$ — this operator will be called *tensor* (*or Kronecker*) *product* of T_1, T_2 and denoted by $T_1 \otimes T_2$. It is clear from the preceding section that the result of such a construction will depend on the choice of realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$. In the sense of the remark to Theorem 3 we shall in the following always understand by $\mathscr{H}_1 \otimes \mathscr{H}_2$ one of its realizations $\mathscr{H}$, φ (without specifying $\mathscr{H}$ and φ explicitely). The connection between operators $T_1 \otimes T_2$ obtained for different realizations of $\mathscr{H}_1 \otimes \mathscr{H}_2$ will be discussed below.

Let us consider a mapping $\mathscr{T}_{12}$ from $\varphi(\mathsf{D}_1 \times \mathsf{D}_2)$ to $\mathscr{H}$ defined by

(3.1) $$\mathscr{T}_{12}(\varphi(x_1, x_2)) = \varphi(T_1 x_1, T_2, x_2) \text{ for all } [x_1, x_2] \in \mathsf{D}_1 \times \mathsf{D}_2 .$$

Using the linearity of T_r's and Lemma 2.1a one easily verifies that $\mathscr{T}_{12}$ is well defined, i.e. $\mathscr{T}_{12}(\varphi(x_1, x_2)) = \mathscr{T}_{12}(\varphi(y_1, y_2))$ if $\varphi(x_1, x_2) = \varphi(y_1, y_2)$. The domains D_r are linear manifolds; therefore $\varphi(\mathsf{D}_1 \otimes \mathsf{D}_2)$ contains the zero vector of $\mathscr{H}$ and clearly $\mathscr{T}_{12}\, 0 = 0$.

*) A similar construction has also been used by Jauch [1] for proving existence of a realization for each $\mathscr{H}_1 \otimes \mathscr{H}_2$.

We thus arrive at the following definition:

Definition 2: Tensor (Kronecker) product of linear operators $T_r\,(r = 1,2)$ with domains D_r is the linear operator which is obtained by linear extension of mapping $\mathscr{T}_{12}$ to $(\varphi(\mathsf{D}_1 \times \mathsf{D}_2))_\lambda$, i.e. the domain $\mathsf{D}_{1\otimes 2}$ of $T_1 \otimes T_2$ is $(\varphi(\mathsf{D}_1 \times \mathsf{D}_2))_\lambda$ and the action of $T_1 \otimes T_2$ on any $x \in \mathsf{D}_{1\otimes 2}$, $x = \sum_{i=1}^{n} \alpha_i \varphi(x_1^{(i)}, x_2^{(i)})$, $x_r^{(i)} \in \mathsf{D}_r$, is given by

$$(3.2)\qquad (T_1 \otimes T_2)\, x = \sum_{i=1}^{n} \alpha_i\, \varphi(T_1 x_1^{(i)}, T_2\, x_2^{(i)})\,.$$

The basic properties of $T_1 \otimes T_2$ are summarized in the following theorem*):

Theorem 4: Let $T_r\,(r = 1, 2)$ be densely defined, i.e. $\bar{\mathsf{D}}_r = \mathscr{H}_r$. Then

(a) $T_1 \otimes T_2$ is densely defined,

(b) the adjoint of $T_1 \otimes T_2$ exists and

$$(3.3)\qquad (T_1 \otimes T_2)^+ \supset T_1^+ \otimes T_2^+ \;{}^{**)}\,,$$

(c) if T_1, T_2 are symmetric so is $T_1 \otimes T_2$,

(d) if T_1, T_2 are closable so is $T_1 \otimes T_2$.

Proof: (a) This statement immediately follows from Lemma 2.2.

(b) Existence of $(T_1 \otimes T_2)^+$ follows from (a). Let $y \in \mathsf{D}(T_1^+ \otimes T_2^+)$, i.e. $y = \sum_{j=1}^{n} \beta_j \varphi(y_1^{(j)}, y_2^{(j)})$ where $y_r^{(j)} \in \mathsf{D}(T_r^+)$ and let us take an arbitrary vector $x \in \mathsf{D}(T_1 \otimes T_2)$, $x = \sum_{i=1}^{m} \alpha_i \varphi(x_1^{(i)}, x_2^{(i)})$, $x_r^{(i)} \in \mathsf{D}_r$. According to definition of T_r^+, it holds $(T_r x_r^{(i)}, y_r^{(j)})_r = (x_r^{(i)}, T_r^+ y_r^{(j)})_r$ for $r = 1, 2$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Using these relations and property (φ2) from Definition 1, we find

$$(3.4)\qquad \begin{aligned} ((T_1 \otimes T_2)\, x, y) &= \Big(\sum_{i=1}^{m} \alpha_i \varphi(T_1 x_1^{(i)}, T_2 x_2^{(i)}), \sum_{j=1}^{n} \beta_j \varphi(y_1^{(j)}, y_2^{(j)})\Big) = \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \bar{\beta}_j (T_1 x_1^{(i)}, y_1^{(j)})_1\, (T_2 x_2^{(i)}, y_2^{(j)})_2 = \\ &= \Big(\sum_{i=1}^{m} \alpha_i \varphi(x_1^{(i)}, x_2^{(i)}), \sum_{j=1}^{n} \beta_j \varphi(T_1^+ y_1^{(j)}, T_2^+ y_2^{(j)})\Big) = \\ &= (x, (T_1^+ \otimes T_2^+) y)\,. \end{aligned}$$

Hence $\mathsf{D}(T_1^+ \otimes T_2^+) \subset \mathsf{D}((T_1 \otimes T_2)^+)$ and $(T_1^+ \otimes T_2^+)\, y = (T_1 \otimes T_2)^+ y$ for all $y \in \mathsf{D}(T_1^+ \otimes T_2^+)$, which is (3.3).

(c) Because of (a) we only have to verify $T_1 \otimes T_2 \subset (T_1 \otimes T_2)^+$. This relation is easily obtained from (3.3) using the obvious implication

$$(3.5)\qquad T_r \subset S_r\,(r = 1, 2) \;\Rightarrow\; T_1 \otimes T_2 \subset S_1 \otimes S_2 \quad \text{for } S_r = T_r^+\,.$$

*) Elements of the theory of unbounded operators, e.g. definitions and properties of the adjoint operator, of closed, closable, symmetric operators etc. can be found in refs. [2], [3], [5].

**) This is analogous to be relation $(TS)^+ \supset S^+T^+$ that is valid for any densely defined operators T, S on $\mathscr{H}$ for which $(TS)^+$ exists. This relation becomes equality if T is bounded. We shall see in the next section that equality holds in (3.3) if both T_1, T_2 are bounded.

(d) This statement is a direct consequence of the following general property of densely defined operators: T is closable if and only if T^+ is densely defined. Now, if T_1 and T_2 are closable then (a) together with (3.3) implies that $(T_1 \otimes T_2)^+$ is densely defined, i.e. $T_1 \otimes T_2$ is closable. ■

Remark 1: Suppose T_1, T_2 are self-adjoint. Then, according to (c), $T_1 \otimes T_2$ is symmetric. However, $T_1 \otimes T_2$ is in general not self-adjoint. In fact $T_1 \otimes T_2$ is essentially self-adjoint (e.s.a.), i.e. its closure is self-adjoint (see the second part of this paper or ref. [3]).

Remark 2: It is well known from linear algebra that for linear operators on a finite-dimensional $\mathscr{H}$ the following "arithmetical" rules hold:

(3.6) $$(T_1 + S_1) \otimes T_2 = T_1 \otimes T_2 + S_1 \otimes T_2\,,$$

(3.7) $$(T_1 S_1) \otimes (T_2 S_2) = (T_1 \otimes T_2)\,(S_1 \otimes S_2)\,.$$

We shall show in the next section that these rules remain valid in infinite-dimensional Hilbert spaces as far as only bounded operators T_r, S_r $(r = 1, 2)$ are considered. For unbounded operators domains must be taken into account, so that only the following weaker "rules" can be easily obtained in the general case:

(3.6a) $$(T_1 + S_1) \otimes T_2 \subset T_1 \otimes T_2 + S_1 \otimes T_2\,,$$

(3.7a) $$(T_1 S_1) \otimes (T_2 S_2) \subset (T_1 \otimes T_2)\,(S_1 \otimes S_2)\,.$$

Remark 3: According to statement (d) $\overline{T_1 \otimes T_2}$ exists if T_1 and T_2 are closable. In this case some authors (cf.ref. [3]) do not make difference between $T_1 \otimes T_2$ and its closure and call both these operators "tensor product of T_1 and T_2". We shall even in this case understand by tensor product the operator determined in Definition 2 and use explicit notation $\overline{T_1 \otimes T_2}$ for its closure except the case when T_1 and T_2 are bounded: we shall see in the next section that $T_1 \otimes T_2$ is then bounded and, according to the usual convention, we understand by $T_1 \otimes T_2$ its continuous extension to $\overline{\mathsf{D}_{1\otimes 2}} = \mathscr{H}$, i.e.

(3.8) $$T_1 \otimes T_2 = \overline{T_1 \otimes T_2} \quad \text{if} \quad T_1, T_2 \quad \text{are bounded.}$$

We shall conclude this section by showing that there is a close connection between operators $T_1 \otimes T_2$ obtained for different realizations of $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Theorem 5: Let $T_1 \otimes T_2$ and $(T_1 \otimes T_2)'$ be tensor products of T_1, T_2 obtained for realizations $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ and $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H}',\varphi'}$ respectively. Then $T_1 \otimes T_2$ and $(T_1 \otimes T_2)'$ are unitarily equivalent.

Proof: According to Theorem 3 there exists a unitary (isometric) operator V from $\mathscr{H}$ onto $\mathscr{H}'$ such that $\varphi'(x, y) = V\varphi(x, y)$ for all $x \in \mathscr{H}_1$, $y \in \mathscr{H}_2$. For any $x \in \mathsf{D}_{1\otimes 2}$, $x = \sum_{i=1}^{n} \alpha_i \varphi(x_1^{(i)}, x_2^{(i)})$, one has $Vx = \sum_{i=1}^{n} \alpha_i \varphi'(x_1^{(i)}, x_2^{(i)}) \in \mathsf{D}'_{1\otimes 2}$ and similarly one proves that each vector $x' \in D'_{1\otimes 2}$ can be written as Vx where $x \in D_{1\otimes 2}$, i.e. $x = V^{-1}x'$. Thus $D'_{1\otimes 2} = VD_{1\otimes 2}$. It holds further for any $x' \in D'_{1\otimes 2}$, $x' = \sum_{i=1}^{n} \alpha_i \varphi'(x_1^{(i)}, x_2^{(i)})$:

$$(T_1 \otimes T_2)'x' = \sum_{i=1}^{n} \alpha_i \varphi'(T_1 x_1^{(i)}, T_2 x_2^{(i)}) = \sum_{i=1}^{n} \alpha_i V\varphi(T_1 x_1^{(i)}, T_2 x_2^{(i)}) = $$
$$= V(T_1 \otimes T_2)\, V^{-1} x' . \quad \blacksquare$$

Remark: If T_1, T_2 are closable then $(\overline{T_1 \otimes T_2})$ and $(\overline{T_1 \otimes T_2})'$ are also unitarily equivalent, since V, V^{-1} are continuous and any vector $x \in D(\overline{T_1 \otimes T_2})$ is the limit of a sequence $\{x_n\} \subset D_{1\otimes 2}$ such that $(T_1 \otimes T_2)\, x_n \to (\overline{T_1 \otimes T_2})\, x$.

If T and T' are unitarily equivalent, $T' = VTV^{-1}$, then all properties of T' can be easily derived from those of T (if V is known). Moreover, some important characteristics such as norm (if T, T' are bounded) or spectrum (if T, T' are self-adjoint) are identical. Thus, although one gets for different realizations of $\mathscr{H}_1 \otimes \mathscr{H}_2$ different tensor products of given two self-adjoint operators A_1, A_2, they (or strictly speaking their closures) all have the same spectrum, eigenvalues etc. Hence the notions like "spectrum of $\overline{A_1 \otimes A_2}$" etc. are meaningful and depend only on A_1, A_2. These topics will be discussed in the second part of this paper.

4. Tensor products of bounded operators

We shall now discuss the tensor products $T_1 \otimes T_2$ of bounded T_r's. There are two reasons for considering this case separately:

(i) the general properties of tensor products derived in the preceding section become much simpler and easier applicable,

(ii) several classes of bounded operators are of basic importance for quantum theory.

Throughout this section we assume that each bounded operator is defined everywhere in its corresponding Hilbert space and use the following notation:

B, C	......	general bounded operator
A	....	Hermitian
U	....	unitary
E	....	projection
W	....	trace class, in particular statistical operator.

Operators on the starting Hilbert spaces $\mathscr{H}_r$ $(r = 1, 2, \ldots)$ are always marked by a lower index (e.g. B_r, C_r), especially unit and zero operator in $\mathscr{H}_r$ are denoted by I_r and O_r, respectively. Operators on $\mathscr{H}_1 \otimes \mathscr{H}_2$ are written either in the "explicit form" $B_1 \otimes B_2$ or they are printed in bold face; in particular symbols $\mathbf{B}_r$ have the following meaning

(4.1) $$\mathbf{B}_1 = B_1 \otimes I_2 , \quad \mathbf{B}_2 = I_1 \otimes B_2$$

and similarly in the case of n-fold tensor product

(4.1a) $$\mathbf{B}_r = I_1 \otimes I_2 \otimes \dots \otimes I_{r-1} \otimes B_r \otimes I_{r+1} \otimes \dots \otimes I_n .$$

Theorem 6: The tensor product $B_1 \otimes B_2$ of bounded operators B_r on $\mathscr{H}_r (r = 1, 2)$ is bounded and

(4.2) $$\|B_1 \otimes B_2\| = \|B_1\|_1 \cdot \|B_2\|_2 \quad \star).$$

Proof: Let $\mathscr{E}_r = \{e_r^{(i)}\}$ be a complete orthonormal set in $\mathscr{H}_r$ and denote $\mathscr{E}^{(\lambda)} = (\varphi(\mathscr{E}_1 \times \mathscr{E}_2))_\lambda$. Since $\mathsf{D}(B_r) = \mathscr{H}_r$, we have $\mathsf{D}_{1\otimes 2} \supset \mathscr{E}^{(\lambda)}$ and also $\mathsf{D}(\mathbf{B}_1) \subset \mathscr{E}^{(\lambda)}$, $\mathsf{D}(\mathbf{B}_2) \supset \mathscr{E}^{(\lambda)}$. Let us further denote by $(B_1 \otimes B_2)_0$ the operator $(B_1 \otimes B_2) \restriction \mathscr{E}^{(\lambda)}$, the symbols $(\mathbf{B}_1)_0$, $(\mathbf{B}_2)_0$ having the analogous meaning for operators $\mathbf{B}_1$, $\mathbf{B}_2$, respectively. For any $x \in \mathscr{E}^{(\lambda)}$,

$$x = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \varphi(e_1^{(i)}, e_1^{(j)}), \quad \text{we get} \quad \|x\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_{ij}|^2$$

and thus

$$\|\mathbf{B}_1 x\|^2 = \| \sum_{j=1}^{n} (\sum_{i=1}^{m} \alpha_{ij} \varphi(B_1 e_1^{(i)}, e_2^{(j)}))\|^2 =$$
$$= \sum_{j=1}^{n} \| \sum_{i=1}^{m} \alpha_{ij} \varphi(B_1 e_1^{(i)}, e_2^{(j)})\|^2 =$$
$$= \sum_{j=1}^{n} \|\varphi(B_1 \sum_{i=1}^{m} \alpha_{ij} e_1^{(i)}, e_2^{(j)})\|^2 \leq$$
$$\leq \|B_1\|_1^2 \sum_{j=1}^{n} \| \sum_{i=1}^{m} \alpha_{ij} e_1^{(i)}\|_1^2 = \|B_1\|_1 \, \|x\|^2 ,$$

i.e. $(\mathbf{B}_1)_0$ is bounded and $\|(\mathbf{B}_1)_0\| \leq \|B_1\|_1$. In the same way one gets $\|(\mathbf{B}_2)_0\| \leq \leq \|B_2\|_2$. Further, since for any $x \in \mathscr{E}^{(\lambda)}$

$$(B_1 \otimes B_2)\, x = \mathbf{B}_1 \mathbf{B}_2 x ,$$

it holds

$$(B_1 \otimes B_2)_0 = (\mathbf{B}_1)_0 \, (\mathbf{B}_2)_0 .$$

Hence $(B_1 \otimes B_2)_0$ is bounded and

$$\|(B_1 \otimes B_2)_0\| \leq \|(\mathbf{B}_1)_0\| \, \|(\mathbf{B}_2)_0\| \leq \|B_1\|_1 \, \|B_2\|_2 .$$

Now $\overline{\mathscr{E}^{(\lambda)}} = \mathscr{H}$ (see Lemma 2.2) and therefore the standard continuous extension of $(B_1 \otimes B_2)_0$ gives a bounded operator defined everywhere in $\mathscr{H}$. In the sense of Remark 3 to Theorem 5 we identify this extension with $B_1 \otimes B_2$; thus

$$\|B_1 \otimes B_2\| \leq \|B_1\|_1 \, \|B_2\|_2 .$$

In order to prove the opposite inequality we take, for a given $\varepsilon > 0$, unit vectors $e_r \in \mathscr{H}_r$ such that

(*) $$\|B_r e_r\|_r > \|B_r\|_r - \varepsilon .$$

$\star$) We denote by $\| \,.\, \|_r$ the norm on the Banach space $\mathfrak{L}(\mathscr{H}_r)$ that is formed by all bounded operators on $\mathscr{H}_r$.

Now $\varphi(e_1, e_2)$ is a unit vector in $\mathscr{H}$ and hence it must hold

$$||B_1 \otimes B_2|| \geq ||(B_1 \otimes B_2)\, \varphi(e_1, e_2)|| = ||B_1 e_1||_1 \, ||B_2 e_2||_2 \,.$$

Using $(\star)$ we further get

$$||B_1 \otimes B_2|| \geq ||B_1||_1 \, ||B_2||_2 - \varepsilon(||B_1||_1 + ||B_2||_2) + \varepsilon^2 \,.$$

Since ε is an arbitrary positive number, this relation implies

$$||B_1 \otimes B_2|| \geq ||B_1||_1 \, ||B_2||_2 \,. \quad \blacksquare$$

Remark: Although $||\,.\,||$ on the left-hand side of (4.2) refers to some realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$ by means of which $B_1 \otimes B_2$ is defined, we see from the right-hand-side expression that $||B_1 \otimes B_2||$ depends only on the $||B_r||_r$'s. In fact, it follows immediately from Theorem 5 that the norm of any bounded $B_1 \otimes B_2$ is realization-independent.

Lemma 4.1: Let B_r, C_r be bounded operators on $\mathscr{H}_r$. Then

(a) the following "arithmetical" rules hold

(4.3) $\quad (\alpha B_1 \otimes B_2) = (B_1 \otimes \alpha B_2) = \alpha(B_1 \otimes B_2)$ for any $\alpha \in \mathbf{C}$,

(4.4) $\quad (B_1 + C_1) \otimes (B_2 + C_2) = B_1 \otimes B_2 + C_1 \otimes B_2 + B_1 \otimes C_2 + C_1 \otimes C_2$,

(4.5) $\quad (B_1 C_1) \otimes (B_2 C_2) = (B_1 \otimes B_2)\,(C_1 \otimes C_2)$;

(b) $B_1 \otimes B_2 = 0$ if and only if at least one of B_1, B_2 is zero operator;

(c) if $B_1 \otimes B_2 \neq 0$ and $B_1 \otimes B_2 = C_1 \otimes C_2$, then

$$C_1 = \alpha B_1, \; C_2 = \frac{1}{\alpha} B_2 \text{ for some } \alpha \in \mathbf{C}, \; \alpha \neq 0.$$

Proof: (a) Let us remind that two bounded operators B, C on $\mathscr{H}$, which satisfy $Bx = Cx$ for all vectors x of a complete set $\mathsf{M} \subset \mathscr{H}$, are equal. It is therefore sufficient to verify that (4.3) – (4.5) hold for any $x \in \varphi(\mathscr{H}_1 \times \mathscr{H}_2)$, which is obvious.

(b) This statement follows immediately from (4.2) and (4.3).

(c) For any $x_r \in \mathscr{H}_r$ it holds

$$(\star) \qquad \varphi(B_1 x_1, B_2 x_2) = \varphi(C_1 x_1, C_2 x_2) \,.$$

Condition $B_1 \otimes B_2 \neq 0$ together with (b) implies that both B_1 and B_2 are non-zero, i.e. that there exist vectors $x_r^{(0)} \in \mathscr{H}_r \; (r = 1, 2)$ such that $B_r x_r^{(0)} \neq 0$. It then follows from $(\star)$ and Lemma 2.1a

$$(B_r x_r^{(0)}, C_r x_r^{(0)})_r \neq 0 \,.$$

Substituting $x_1 = x_1^{(0)}$ into $(\star)$ and taking scalar product with $\varphi\left(\frac{B_1 x_1^{(0)}}{||B_1 x_1^{(0)}||_1^2}, y_2\right)$ we get

$$(B_2 x_2, y_2)_2 = \frac{(C_1 x_1^{(0)}, B_1 x_1^{(0)})_1}{||B_1 x_1^{(0)}||_1^2} (C_2 x_2, y_2)_2$$

and, since x_2, y_2 are arbitrary,

$$B_2 = \alpha C_2$$

where

$$\alpha = \frac{(C_1 x_1^{(0)}, B_1 x_1^{(0)})_1}{\|B_1 x_1^{(0)}\|_1^2} .$$

Similarly

$$B_1 = \alpha' C_1 \quad \text{where} \quad \alpha' = \frac{(C_2 x_2^{(0)}, B_2 x_2^{(0)})_2}{\|B_2 x_2^{(0)}\|_2^2} = \frac{1}{\alpha} . \qquad \blacksquare$$

Corollary: Operators $\mathbf{B}_1, \mathbf{B}_2$ commute for any B_1, B_2:

(4.6) $$\mathbf{B}_1 \mathbf{B}_2 = \mathbf{B}_2 \mathbf{B}_1 = B_1 \otimes B_2 . \quad ^{*)}$$

The list of "arithmetical" rules should also contain a relation between $(B_1 \otimes B_2)^{-1}$ and B_1^{-1}, B_2^{-1} assuming all the inverses exist. Such a relation can be easily derived for regular B_r's**). The reason for this restriction is that for a regular bounded B_r the inverse B_r^{-1} not only exists but is also bounded (see refs. [2], [3]), and thus $B_1^{-1} \otimes B_2^{-1}$ is bounded. Then we get from (4.5):

$$(B_1 \otimes B_2)(B_1^{-1} \otimes B_1^{-1}) = (B_1^{-1} \otimes B_2^{-1})(B_1 \otimes B_2) = \mathbf{I} ,$$

which is equivalent to the following statement:

Lemma 4.2: If B_1, B_2 are regular bounded operators, then $B_1 \otimes B_2$ is regular and

(4.7) $$(B_1 \otimes B_2)^{-1} = B_1^{-1} \otimes B_2^{-1} .$$

The tensor product $B_1 \otimes B_2$, like any bounded operator, has a unique bounded adjoint that is determined as follows (cf. Theorem 4):

Lemma 4.3: Let B_r be bounded operators on $\mathscr{H}_r$. Then

(4.8) $$(B_1 \otimes B_2)^+ = B_1^+ \otimes B_2^+ .$$

Proof: Operators B_r^+ are bounded and so is $B_1^+ \otimes B_2^+$, i.e. $\mathrm{D}(B_1^+ \otimes B_2^+) = \mathscr{H}$. The rest follows from (3.3). $\blacksquare$

Corollary: If A_1, A_2 are Hermitian, so is $A_1 \otimes A_2$.

We shall now use the above three lemmas to derive some properties of $B_1 \otimes B_2$ for several important classes of B_r's.

*) Using this one can formulate precisely statements of the following type that occur currently in textbooks on quantum mechanics: "If a_1, a_2 are observables referring to particles *1* and *2* respectively, then the corresponding (Hermitian) operators A_1, A_2 commute". In fact, operators A_1, A_2 act on different Hilbert spaces and thus their product and commutator is not defined. Applications of the tensor-product formalism for description of composite systems will be discussed in the second part of this paper.

**) A linear operator T on $\mathscr{H}$ is regular if it maps bijectively $\mathscr{H}$ onto $\mathscr{H}$.

Lemma 4.4: (a) If A_1, A_2 are positive*) so is $A_1 \otimes A_2$. Then

$$\sqrt{A_1 \otimes A_2} = \sqrt{A_1} \otimes \sqrt{A_2} \; . \tag{4.9}$$

(b) If U_1, U_2 are unitary so is $U_1 \otimes U_2$.

(c) If E_1, E_2 are projections so is $E_1 \otimes E_2$; the range of $E_1 \otimes E_2$ is then given by

$$(E_1 \otimes E_2)\mathscr{H} = \overline{(\varphi(E_1\mathscr{H}_1 \times E_2\mathscr{H}_2))_\lambda} \; . \tag{4.10}$$

Proof: (a) To any $A \geq 0$ there exists a unique positive $\sqrt{A}$ such that $\sqrt{A}\sqrt{A} = A$. Using (4.5) for $B_r = C_r = \sqrt{A_r}$ we see that $A_1 \otimes A_2$ is the square of Hermitian operator $\sqrt{A_1} \otimes \sqrt{A_2}$ and hence $A_1 \otimes A_2$ is positive and formula (4.9) holds ($\sqrt{A_1 \otimes A_2}$ is unique!).

(b) This follows immediately from Lemmas 4.2 and 4.3.

(c) $E_1 \otimes E_2$ is a projection since it is Hermitian and idempotent, i.e. $(E_1 \otimes E_2)^2 = E_1^2 \otimes E_2^2 = E_1 \otimes E_2$. For proving (4.10) let us introduce the following notation:

$$E_r^{(1)} \equiv E_r \, , \quad E_r^{(2)} \equiv I_r - E_r$$

$$\mathscr{G}_r^{(i)} \equiv E_r^{(i)}\mathscr{H}_r \;\; (i = 1, 2), \;\; \text{i.e.} \;\; \mathscr{G}_r^{(1)} = (\mathscr{G}_r^{(2)})^\perp \, .$$

Further, let

$$\mathbf{E}_{ij} \equiv E_1^{(i)} \otimes E_2^{(j)} \, , \quad \mathscr{G}_{ij} \equiv \overline{(\varphi(\mathscr{G}_1^{(i)} \times \mathscr{G}_2^{(j)}))_\lambda} \quad (i, j = 1, 2) \, .$$

Then (4.10) becomes

$$\mathbf{E}_{11}\mathscr{H} = \mathscr{G}_{11} \, .$$

Now

$$E_r^{(i)} E_r^{(j)} = \delta_{ij}E_r^{(i)} \, , \quad E_r^{(1)} + E_r^{(2)} = I_r, \quad \text{and thus}$$

$$\mathbf{E}_{ik}\mathbf{E}_{jl} = \delta_{ij}\delta_{kl}\mathbf{E}_{ik} \quad \text{and} \quad \sum_{i,j=1}^{2} \mathbf{E}_{ij} = \mathbf{I} \, .$$

These relations imply

$$\sum_{i,j=1}^{2} \dotplus \mathbf{E}_{ij}\mathscr{H} = \mathscr{H}, \;\; \text{i.e.} \;\; (\mathbf{E}_{11}\mathscr{H})^\perp = \mathbf{E}_{12}\mathscr{H} \dotplus \mathbf{E}_{21}\mathscr{H} \dotplus \mathbf{E}_{22}\mathscr{H} \, . \tag{*}$$

On the other hand, the subspaces $\mathscr{G}_{ij}$ are orthogonal; using Lemma 2.3 we find

$$\sum_{i,j=1}^{2} \dotplus \mathscr{G}_{ij} = \mathscr{H}, \;\; \text{i.e.} \;\; (\mathscr{G}_{11})^\perp = \mathscr{G}_{12} \dotplus \mathscr{G}_{21} \dotplus \mathscr{G}_{22} \, . \tag{**}$$

For any $x_1 \in \mathscr{G}_1^{(i)}$, $x_2 \in \mathscr{G}_2^{(j)}$ one has

$$\mathbf{E}_{ij}\varphi(x_1, x_2) = \varphi(E_1^{(i)} x_1, E_2^{(j)} x_2) = \varphi(x_1, x_2) \, ,$$

so that

$$\varphi(\mathscr{G}_1^{(i)} \times \mathscr{G}_2^{(j)}) \subset \mathbf{E}_{ij}\mathscr{H}$$

*) A bounded operator A is positive if $(Ax, x) \geq 0$ for all x. Notice that each positive operator is Hermitian — this follows immediately from the "polarization identity" (see refs. [2], [3].)

and consequently

(***) $$\mathscr{G}_{ij} \subset \mathbf{E}_{ij}\mathscr{H} \, .$$

Relation (4.10) is obtained if one observes that (*), (**) and (***) imply

$$(\mathscr{G}_{11})^{\perp} \subset (\mathbf{E}_{11}\mathscr{H})^{\perp}, \text{ i.e. } \mathscr{G}_{11} \supset \mathbf{E}_{11}\mathscr{H} \, . \quad \blacksquare$$

Corollary 1: If $A_r \geq A'_r \geq 0$, then $A_1 \otimes A_2 \geq A'_1 \otimes A'_2 \geq 0$.

Proof: This follows immediately from statement (a) if one writes $A_1 \otimes A_2 - A'_1 \otimes A'_2 = (A_1 - A'_1) \otimes A_2 + A'_1 \otimes (A_2 - A'_2)$ and takes into account that the sum of positive operators is positive. ■

Corollary 2: Let B_r be bounded operators. Then

(4.11) $$|B_1 \otimes B_2| = |B_1| \otimes |B_2| \, .$$

Proof: Let us remind that for any bounded B one denotes by $|B|$ the positive operator $\sqrt{B^+B}$. Then (4.11) is easily obtained from (4.9). ■

Corollary 3: Let E_r, E'_r be non-zero projections. Then

(a) $E_1 \otimes E_2 + E'_1 \otimes E'_2$ is a projection if and only if $E_1 + E'_1$ *or* $E_2 + E'_2$ is a projection,

(b) $E_1 \otimes E_2 - E'_1 \otimes E'_2$ is a projection if and only if $E_1 - E'_1$ *and* $E_2 - E'_2$ are projections*),

(c) $(E_1 \otimes E_2)(E'_1 \otimes E'_2)$ is a projection if and only if $E_1E'_1$ *and* $E_2E'_2$ are projections**).

Proof: (a) The "if" part is simple. The "only if" part can be proved in the same way as it is done for "single" projections (see e.g. ref. [2]).

(b) If E, E' are projections, then the following three statements are equivalent [2]:

(i) $E - E'$ is a projection
(ii) $E - E' \geq 0$
(iii) $EE' = E'E = E'$.

Now, if $E_1 \otimes E_2 - E'_1 \otimes E'_2$ is a projection, i.e. $(E_1 \otimes E_2)(E'_1 \otimes E'_2) = E'_1 \otimes E'_2$ then, according to Lemma 4.1, $E_1E'_1 = \alpha E'_1$, $E_2E'_2 = \frac{1}{\alpha} E'_2$. Multiplying the first relation by E_1 from the left and reminding that E_1, E'_1 are non-zero, we find $\alpha = 1$, i.e. $E_r - E'_r$ are projections. Conversely, if $E_r - E'_r$ are projections, then (ii) and Corollary 1 imply that $E_1 \otimes E_2 - E'_1 \otimes E'_2$ is so.

*) We do not specify ranges since nothing more can be said about them than what holds for "single" projections: $(E + E')\mathscr{H} = (E\mathscr{H}) \dotplus (E'\mathscr{H})$ and $(E - E')\mathscr{H}$ is the orthogonal complement of $E'\mathscr{H}$ in $E\mathscr{H}$.

**) Concerning the range of $(E_1 \otimes E_2)(E_1' \otimes E_2')$ see the following Remark.

(c) If both E_1E_1' and E_2E_2' are projections, then

$$(\star)\qquad E_1E_1' \otimes E_2E_2' = (E_1 \otimes E_2)(E_1' \otimes E_2')$$

is a projection. Conversely, assuming that the non-zero $E_1 \otimes E_2$ and $E_1' \otimes E_2'$ commute, we obtain from ($\star$) and Lemma 4.1

$$E_1E_1' = \alpha E_1'E_1 \quad \text{and} \quad E_2E_2' = \frac{1}{\alpha} E_2'E_2 .$$

One easily finds (e.g. multiplying the first relation by E_1' from the both sides) that $\alpha = 1$, i.e. E_r and E_r' commute and thus E_rE_r' are projections. ■

Remark: If E_1E_1' and E_2E_2' are projections, then using the rule

$$(EE')\mathscr{H} = (E\mathscr{H}) \cap (E'\mathscr{H})$$

we get from ($\star$) the following interesting formula which supplements Lemma 2.3

$$\begin{aligned}(4.12)\qquad & \overline{(\varphi(\mathscr{G}_1 \times \mathscr{G}_2))_\lambda} \cap \overline{(\varphi(\mathscr{G}_1' \times \mathscr{G}_2'))_\lambda} = \\ & = \overline{(\varphi((\mathscr{G}_1 \cap \mathscr{G}_1') \times (\mathscr{G}_2 \cap \mathscr{G}_2')))_\lambda} = \overline{(\varphi((\mathscr{G}_1 \times \mathscr{G}_2) \cap (\mathscr{G}_1' \times \mathscr{G}_2')))_\lambda}\end{aligned}$$

where we denote

$$\mathscr{G}_r = E_r\mathscr{H}_r , \quad \mathscr{G}_r' = E_r'\mathscr{H}_r .$$

Bounded operators on a separable Hilbert space*) have many properties similar to those of linear operators on a finite-dimensional space, for example one can represent them by infinite-dimensional matrices, generalize the notion of trace for some of them etc.

Suppose now the $\mathscr{H}_r$'s separable (i.e. any space in which $\mathscr{H}_1 \otimes \mathscr{H}_2$ is realized is separable). We shall derive formulae expressing elements of a matrix representation of $B_1 \otimes B_2$, its trace, etc. by means of corresponding quantities for the B_r's. Let $\mathscr{E}_r = \{e_r^{(i)}\}_{i=1}^{\infty}$ be ortonormal bases in $\mathscr{H}_r$; then vectors $e_{(i,j)} = \varphi(e_1^{(i)}, e_2^{(j)})$ $(i, j = 1, 2, \ldots)$ in some fixed order form an orthonormal basis $\mathscr{E}$ in $\mathscr{H}$. Denote by $b^{(r)}$ the matrix representation of B_r with respect to $\mathscr{E}_r$, i.e. $b_{ik}^{(r)} = (B_re_r^{(k)}, e_r^{(i)})_r$. Then the matrix representation $b^{(1)} \otimes b^{(2)}$ of $B_1 \otimes B_2$ with respect to $\mathscr{E}$ is determined as follows:

$$\begin{aligned}(4.13)\qquad (b^{(1)} \otimes b^{(2)})_{(i,j)(k,l)} &= ((B_1 \otimes B_2)\, e_{(k,l)}, e_{(i,j)}) = \\ &= (B_1e_1^{(k)}, e_1^{(i)})_1 (B_2e_2^{(l)}, e_2^{(j)})_2 = b_{ik}^{(1)}b_{jl}^{(2)} .\end{aligned}$$

We thus see that the well-known formula of linear algebra concerning the direct product of matrices remains unchanged in the infinite-dimensional case.

Before proceeding let us remind some points from the theory of bounded operators on separable $\mathscr{H}$:

1. *Absolute norm* (see refs. [2], [5]).

*) We assume for convenience everywhere in the following $\dim \mathscr{H} = \infty$. All the formulae which will be derived are of course valid also in the finite dimensional case.

The quantity

$$\mathrm{N}(B) = \Big(\sum_{k=1}^{\infty} \|Be_k\|^2\Big)^{1/2}$$

does not depend on the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ by means of which it is expressed. If $\mathrm{N}(B) < \infty$ then B is called Hilbert-Schmidt operator and $\mathrm{N}(B)$ is its absolute norm. Each Hilbert-Schmidt operator is compact. Hermitian compact operators are of special importance since they have a pure point spectrum, i.e. their eigenvectors form an orthonormal basis.

2. *Trace classes* (see ref. [3]).

The trace of a bounded operator is obtained by extending the usual definition of the trace to the infinite-dimensional matrix representation of bounded operators. However one must add some further conditions to ensure convergence of the series $\sum_{k=1}^{\infty} (Be_k, e_k)$ and its independence of the basis. It appears that these requirements are satisfied by each bounded operator W for which $\mathrm{N}(\sqrt{|W|}) < \infty$. Such an operator is called trace class. Since $\sqrt{|W|}$ is Hermitian one can rewrite the above condition as follows

$$\sum_{k=1}^{\infty} (|W|\, e_k, e_k) < \infty\,. \quad \star) \tag{4.14}$$

The sum of this series is again basis-independent and (4.14) further implies:

(i) the series

$$\sum_{k=1}^{\infty} (We_k, e_k)$$

is absolutely convergent and its sum is basis-independent. One can therefore define the trace of W by

$$\mathrm{Tr}\, W = \sum_{k=1}^{\infty} (We_k, e_k)\,. \tag{4.14a}$$

(ii) WB and BW are trace classes for any bounded B and

$$\mathrm{Tr}\, WB = \mathrm{Tr}\, BW\,. \tag{4.15}$$

3. *Statistical operators* (see refs. [1], [2]).

Let W be a positive trace class. From the relation

$$(Ax, x)\,(Ay, y) \geq |(Ax, y)|^2\,,$$

which is valid for any positive A and all $x, y \in \mathscr{H}$, one gets using (4.14)

$$\mathrm{N}(W) \leq \mathrm{Tr}\, W < \infty\,.$$

Hence each positive trace class has a finite absolute norm and is therefore compact. A positive trace class W is called statistical operator (density matrix) if it is "normalized", i.e. if

$$\mathrm{Tr}\, W = 1\,.$$

$\star$) Notice that owing to the uniqueness of the square root it holds $|W| = W$ if $W \geq 0$.

Let us now return to tensor products and conclude this section by the following lemma that will be used in the second part of this paper when we shall discuss how a composite quantum system can be described by means of its subsystems.

Lemma 4.5: Let $\mathscr{H}_r\,(r = 1, 2)$ be separable. Then

(a) if B_1, B_2 are Hilbert-Schmidt operators so is $B_1 \otimes B_2$ and

$$\mathrm{N}(B_1 \otimes B_2) = \mathrm{N}(B_1)\,\mathrm{N}(B_2)\,; \tag{4.16}$$

(b) if W_1, W_2 are trace classes so is $W_1 \otimes W_2$ and

$$\mathrm{Tr}((W_1 \otimes W_2)\,(B_1 \otimes B_2)) = \mathrm{Tr}(W_1B_1)\,\mathrm{Tr}(W_2B_2) \tag{4.17}$$

for any bounded B_1, B_2.

Proof: (a) Let us take arbitrary orthonormal bases $\mathscr{E}_r \subset \mathscr{H}_r\,(r = 1, 2)$ and $\mathscr{E} = \varphi(\mathscr{E}_1 \times \mathscr{E}_2) \subset \mathscr{H}$ (see the text preceding (4.13)). Since $\mathrm{N}(B_1 \otimes B_2)$ is basis-independent, we can use $\mathscr{E}$ for calculating it:

$$\mathrm{N}^2(B_1 \otimes B_2) = \sum_{(i,j)} \|(B_1 \otimes B_2)\, e_{(i,j)}\|^2 = \sum_{(i,j)} \|B_1e_1^{(i)}\|_1^2\, \|B_2e_2^{(j)}\|_2^2\,.$$

The order of summation is given by the order of vectors in $\mathscr{E}$. However, since all terms in the series are non-negative, summation can be carried out in any order. Thus

$$\mathrm{N}(B_1 \otimes B_2) = \Big(\sum_{i=1}^{\infty} \|B_1e_1^{(i)}\|_1^2 \cdot \sum_{j=1}^{\infty} \|B_2e_2^{(j)}\|_2^2 \Big)^{1/2} = \mathrm{N}(B_1)\,\mathrm{N}(B_2)\,.$$

(b) Using (4.9), (4.11), (4.14) and (4.16) we find

$$\mathrm{Tr}\,|W_1 \otimes W_2| = \mathrm{N}^2\big(\sqrt{|W_1 \otimes W_2|}\,\big) = \mathrm{N}^2\big(\sqrt{|W_1|} \otimes \sqrt{|W_2|}\big) =$$
$$= \mathrm{N}^2\big(\sqrt{|W_1|}\,\big)\,\mathrm{N}^2\big(\sqrt{|W_2|}\big) = \mathrm{Tr}\,|W_1|\,\mathrm{Tr}\,|W_2|\,,$$

i.e. $W_1 \otimes W_2$ is a trace class. Then operator $(W_1 \otimes W_2)\,(B_1 \otimes B_2) = (W_1B_1) \otimes (W_2B_2)$ is also a trace class and the series

$$\sum_{(i,j)} (W_1B_1e_1^{(i)}, e_1^{(i)})_1\, (W_2B_2e_2^{(j)}, e_2^{(j)})_2$$

is absolutely convergent. Applying the same argument as in (a) we get

$$\mathrm{Tr}((W_1 \otimes W_2)\,(B_1 \otimes B_2)) = \mathrm{Tr}(W_1B_1 \otimes W_2B_2) =$$
$$= \sum_{i=1}^{\infty} (W_1B_1e_1^{(i)}, e_1^{(i)})_1 \sum_{j=1}^{\infty} (W_2B_2e_2^{(j)}, e_2^{(j)})_2 =$$
$$= \mathrm{Tr}(W_1B_1)\,\mathrm{Tr}(W_2B_2)\,. \quad \blacksquare$$

Corollary: If W_1, W_2 are statistical operators so is $W_1 \otimes W_2$.

Acknowledgment

The authors are truly indebted to Dr. M. Havlíček for helpful and inspiring discussions.

References

[1] J. M. JAUCH: Foundations of Quantum Mechanics, Addison-Wesley Publ. Co, Reading, Mass. (1968).

[2] J. BLANK, P. EXNER, M. HAVLÍČEK: Selected topics of mathematical physics: Theory of linear operators on Hilbert space I (in Czech), SPN, Prague (1975).

[3] M. REED, B. SIMON: Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York (1972).

[4] A. N. KOLMOGOROV, S. V. FOMIN: Elements of the theory of functions and functional analysis (in Russian), Nauka, Moscow (1972).

[5] N. I. ACHIEZER, I. M. GLAZMAN: Theory of linear operators on Hilbert space (in Russian), Nauka, Moscow (1966).

[6] A. MESSIAH: Quantum mechanics I, II, North-Holland Publ. Co, Amsterdam (1964).

[7] S. S. SCHWEBER: An Introduction to Relativistic Quantum Field Theory, Row, Peterson & Co, Elmsford, N.Y. (1961).

[8] J. FORMÁNEK: Introduction to quantum theory I, II (in Czech), SPN, Prague (1973, 1974).

[9] N. J. VILENKIN: Special functions and the theory of group representations (in Russian), Nauka, Moscow (1965).

[10] G. E. ŠILOV: Mathematical Analysis: Finite-dimensional vector spaces (in Russian), Nauka, Moscow (1969).

[11] S. MCLANE, G. BIRKHOFF: Algebra, The Macmillan Comp., New York (1968).