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Treatment of Boundary Singularities in Two Dimensional Elliptic Problems by Galerkin Methods

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Methods of subtracting off singular terms and for refining the mesh locally are proposed for adapting the standard Galerkin method so that it may produce accurate approximations to the solutions of mixed boundary problems for Poisson's equation, when these contain boundary singularities.

In this short communication two approaches are considered for adapting the standard Galerkin method when it is used to produce accurate approximations to the solutions of two dimensional Poisson problems containing boundary singularities. Such singularities can occur when the boundary contains a re-entrant corner, and are due to the combination of boundary shape and boundary conditions. The two approaches are first that of substracting off from the solution of the problem a term having the form of the singularity, and second that of refining the mesh locally in the neighbourhood of the point at which the singularity occurs.

We consider the problem where the function $u(x, y)$ satisfies

$$\begin{aligned} -\Delta[u(x, y)] &= g(x, y), \quad && (x, y) \in \Omega\,, \\ u(x, y) &= f_1(x, y), && (x, y) \in \partial\Omega_1\,, \\ \frac{\partial u(x, y)}{\partial \nu} &= 0, && (x, y) \in \partial\Omega_2\,, \end{aligned} \tag{1}$$

where Δ is the Laplacian operator, $\Omega \subset E^2$ is a simply connected open bounded domain with polygonal boundary, $\partial\Omega_1$ is the union of some of the sides of $\partial\Omega$, $\partial\Omega_2 = \partial\Omega - \partial\Omega_1$, $g \in L_2(\Omega), f_1 \in L_2(\partial\Omega_1)$, $\bar{\Omega} = \Omega \cup \partial\Omega$ and $\partial/\partial\nu$ is the derivative in the direction of the outward normal to the boundary.

Let $W_2^1(\Omega)$ be the usual Sobolev space and $W_2^1(\Omega) \cap (\partial\Omega_1)_0$ be the subspace of functions such that for $v \in W_2^1(\Omega) \cap (\partial\Omega_1)_0$, $v \in W_2^1(\Omega)$ and $v = 0$ on $\partial\Omega_1$. A weak problem corresponding to (1) is formed and it is the solution $u(x, y) \in \psi_1 + W_2^1(\Omega)$ of this which is approximated with the Galerkin solution $U(x, y) \in S^h$. The region Ω is discretized into triangular (or rectangular) elements and the notation $u \in \psi_1 + W_2^1(\Omega)$ means that $u = \psi_1 + v$ where $v \in W_2^1(\Omega) \cap (\partial\Omega_1)_0$, $\psi_1 \in W_2^1(\bar{\Omega})$ is such that $\psi_1 = f_1$ on $\partial\Omega_1$ and S^h is a finite dimensional set elements of which are in $W_2^1(\Omega)$ and take the values of f_1 at a finite set of points (nodes) on $\partial\Omega_1$. Details of the weak problem and the Galerkin method can be found in [3].

Error bounds of the type

$$\|u - U\|_{W_2^1(\Omega)} \leqq hK\,|u|_2\,, \tag{2}$$

where

$$|u|_2 = \left\{ \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_2}^2 + \left\| \frac{\partial^2 u}{\partial x \partial y} \right\|_{L_2}^2 + \left\| \frac{\partial^2 u}{\partial y^2} \right\|_{L_2}^2 \right\}^{\frac{1}{2}},$$

can be derived, see [1] and [3].

We consider the situation where the part $\partial\Omega_2$ of the boundary $\partial\Omega$ contains a re-entrant corner of internal angle greater than π and the boundary conditions on both arms of the corner are that $\partial u/\partial v = 0$. It is found [1] that in this case $u \in W_2^1(\Omega) - W_2^2(\Omega)$ so that the error bound (2) does not now apply and also that the calculated approximation $U(x, y)$ is inaccurate in the neighbourhood of the singularity.

The two approaches mentioned above are now followed. In the first S^h is augmented with functions from $W_2^1(\Omega)$ which have the form of the boundary singularity so that the space $Aug\ S^h$ is formed. If $\widehat{U} \in Aug\ S^h$ is the new Galerkin approximation to u, then a bound on $\|u - \widehat{U}\|$ having the form of (2) can be calculated; see [3]. In addition, it is found [1] that the inclusion of the singular terms improves the accuracy of the Galerkin solution in the neighbourhood of the singularity.

When the technique of local mesh refinement in the neighbourhood of the singularity is used, the bound (2) is again not applicable. However, recently Gregory and Whiteman [2] for problems in rectangular regions have used local mesh refinement with rectangular elements. The refinement is achieved by successive halving of the mesh near the singularities, and "hanging-points", familiar in finite-difference methods, are introduced. There are thus some elements which are rectangular with five nodes; the four corners and the mid-point of one side. In [2] a conforming, $C^\circ(\Omega)$, element for this situation has been constructed and used to obtain Galerkin approximations of much improved accuracy.

References

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