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Two Free-boundary Problems With Singularities

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The paper discusses methods for the treatment of singularities in boundary-value problems, including problems with some free boundaries, and also introduces a generalization of the Regula Falsi method for the computation of free boundaries.

I. Introduction

In [2] I gave an account of some recent work at the Oxford University Computing Laboratory on elliptic problems with singularities, and how the methods there introduced could be applied to some free-boundary problems. The purpose of this paper is to bring this work up-to-date.

The investigation started with an attempt to solve an elliptic equation of the form

$$\nabla^2 u = g(r, \theta)\, u\,, \tag{1}$$

where the origin 0 of polar coordinates is the intersection of two straight lines forming part of the boundary. With conditions like

$$\alpha u + \frac{\beta}{r} \frac{\partial u}{\partial \theta} = f(r) \tag{2}$$

on these lines, the solution will lack smoothness in the neighbourhood of the origin when the internal angle exceeds the value π at that point. The aim of the method is to find a series solution of (1) and (2) in the neighbourhood of 0, this solution containing a number of arbitrary constants. By suitable truncation of this series, and with the use of finite-difference formulae at mesh points sufficiently remote from 0, it is then possible to find approximate values for the more important arbitrary constants and at the same time to determine approximate values of u at the finite-difference mesh points. Full details are given in Fox and SANKAR [3] and Fox [2].

In the latter paper I also mentioned work in progress on free-boundary problems with singularities, and in the rest of this paper I give details of our work on two such problems. In the first problem we take account of the singularity by methods similar in principle to those already mentioned. Our work on the second problem is somewhat different, because we are here less concerned with the treatment of the singularity, but more with the finding of a successful and rapidly convergent iterative method for the determination of an approximate free boundary.

2. First Free Boundary Problem

In Figure 1 a constant head of water seeps from the left through a permeable wall, with a "free surface" AB, then drips down to C and flows uniformly to the right. Our problem is the determination of the free boundary AB.

The differential equations and boundary conditions, in Cartesian coordinates, are

$$\nabla^2 \Phi = 0 \text{ in } OABCD\,, \tag{3}$$

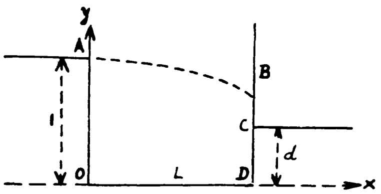

Fig. 1.

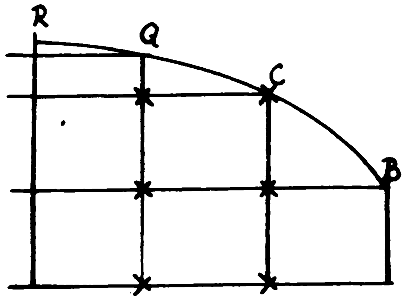

Fig. 2.

$$\Phi = 1 \text{ on } OA\,, \quad \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } OD\,, \quad \Phi = d \text{ on } DC\,, \quad \Phi = y \text{ on } CB\,, \tag{4}$$

and

$$\Phi = y\,, \quad \partial \Phi / \partial \nu = 0 \text{ on } AB\,. \tag{5}$$

Given L and d, the double condition (5) enables us to determine AB.

A successful method achieves this iteratively by first guessing some boundary AB, solving (3) and (4) and the second of (5), and then adjusting the guess so that the first of (5) is temporarily satisfied. In the solution of each boundary-value problem there is a weak singularity at C, which can be treated by the methods of Fox and Sankar, and a stronger singularity at B.

In my 1971 account I already reported that a student Joyce Taylor (now Mrs. Aitchison) had shown that in the vicinity of B the solution is given by

$$z = \frac{W}{\pi} \ln(-iW) + i \sum_{k=1}^{\infty} A_k \exp(\tfrac{1}{2} i \pi k)\, W^k\,, \tag{6}$$

and the free boundary has the shape

$$y = \frac{x}{\pi} \ln(-x) + \sum_{k=1}^{\infty} B_k x^k\,, \quad B_k = (-1)^k A_k\,, \tag{7}$$

the x, y origin now being at the point B, where

$$W = \Phi + i\psi + iz\,, \quad z = x + iy\,, \tag{8}$$

and ψ is the harmonic conjugate of Φ. In a more recent paper (Aitchison [1]) details are given of how (6) and (7) are used to improve the finite-difference solution in the neighbourhood of the singular point B.

Again we use special formulae, in the region of B, for values of Φ at the points marked with a cross in Figure 2, and elsewhere we use simple finite-difference equations here arranged, however, so that mesh points are on the boundary. From a first guess for the boundary we have known values of x and y at boundary points like Q. Taking a truncated form of (7) this enables us to compute the relevant constants B_k and hence the values of Φ at points like Q and also the position of B. Values of Φ at other points near B are then given implicitly by (6), with the same number of constants B_k. This implicit solution of (6), we find, can be obtained quite easily with Newton's method. The obvious resulting iterative method works well, and gives good solutions without the need for a specially small finite-difference interval in the region of B.

3. Second Free-boundary Problem

Our second problem is the determination of the free boundary DC in Figure 3, representing the axi-symmetric cavitational flow of an incompressible fluid past a circular disc placed coaxially inside a cylinder.

The differential equation and boundary conditions are

$$\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = 0\,, \qquad (9)$$

$$\Phi = 1 \text{ on } AB\,, \quad \Phi = 0 \text{ on } FED\,, \quad \frac{\partial \Phi}{\partial z} = 0 \text{ on } AF \text{ and } BC\,, \qquad (10)$$

and

$$\Phi = 0\,, \quad \frac{1}{r}\frac{\partial \Phi}{\partial \nu} = q \text{ on } DC\,. \qquad (11)$$

If all the dimensions are specified in Figure 3 our problem is to determine the value of the constant q and the position of the free boundary DC.

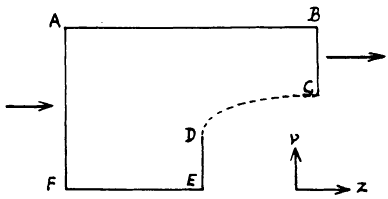

Fig. 3.

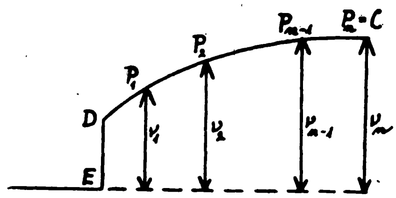

Fig. 4.

There is rather a strong singularity at D, which we have not yet been able to treat properly, but another interesting fact is that it is here difficult to find a satisfactory iterative method for the determination of DC which satisfies the approximating (finite-difference) algebraic problem. In the percolation problem *local* adjustment of the boundary position led to smooth and satisfactory convergence of the iterative method. Here, however, when we take a first guess at DC, solve (9), (10) and the first of (11), and adjust locally at all the boundary mesh points to satisfy the second of (11), we find that successive estimates of DC manifest oscillation of rapidly increasing amplitude.

In Figure 4 our adjusting process tries to find positions for the P_j at which the equations

$$f_j = \frac{1}{r_j} \frac{\partial \Phi_j}{\partial v} - q_e = 0, \quad j = 1, 2, \ldots, n \,, \tag{12}$$

are satisfied for some estimate q_e of q. Each f_j, of course, depends on all the r_j, and instead of trying to adjust each P_j by changing only the local r_j we try to adjust them simultaneously, treating (12) as a system of non-linear simultaneous equations. Newton's method is here unattractive, the relevant derivatives being difficult to compute, but we can apply a generalization of the Regula Falsi method.

Suppose that we start with $n + 1$ different estimates of DC, that is of the vector $r^{(i)}$ of heights $r_1^{(i)}, r_2^{(i)}, \ldots, r_n^{(i)}$, $i = 1, 2, \ldots n + 1$ in Figure 4. After solving $n + 1$ corresponding boundary-value problems we can compute the f_j of (12) for each i. A little manipulation shows that the components of a better verctor r of heights, obtained from the generalized Regula Falsi method, are given by

$$r_j = \frac{| \, r_j^{(i)} \;\; f_1(r^{(i)}) \;\; f_2(r^{(i)}) \ldots f_n(r^{(i)}) \, |}{| \, 1 \;\; f_1(r^{(i)}) \;\; f_2(r^{(i)}) \ldots f_n(r^{(i)}) \, |} \,, \; j = 1, 2, \ldots, n \,, \tag{13}$$

where the numerator and denominator are determinants of order $n + 1$, where the ith row is shown in each case and i goes from 1 to $n + 1$.

With modern computers this computation is by no means prohibitive. In our example, for illustration, we took $n = 9$, giving 10 boundary-value problems to solve for each q_e, and to get a good estimate of q we used 7 different values of q_e, with a large number of determinants of order 10 to be evaluated in (13). All this took less than 20 minutes on the medium-sized *KDF*9 computer.

For each q_e we find the corresponding Regula Falsi curve, and from this we compute $\frac{1}{r} \frac{\partial \Phi}{\partial v}$ at the relevant points and find the average of these quantities and the deviation from the average. When the latter is small we have a satisfactory computation of the free boundary and the constant q. If the deviation is not sufficiently small we take our best q_e and repeat the Regula Falsi process, with our new curve as one of the starting guesses. There is, of course, some danger that when we are near to the true solution both determinants in (13) may be very small, with corresponding uncertainty in the computed r_j.

Full details are given in FOX and SANKAR [4], and the method works extremely well in the sense that, starting from "smooth" guessed positions of the required boundary, our next computed position is also "smooth" and considerably more accurate.

Though we have no guarantee that the method will always converge, and always converge "smoothly", the fact that all the unknowns are involved in determining the adjustment of every point suggests that this generalized Regula Falsi method is likely to succeed quite often when other methods fail. The generalized Regula Falsi, first suggested by Gauss as long ago as 1809, has not been used very much by numerical analysts, and there is clearly a strong case for its resurrection.

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