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ON TOTALIZATION OF THE KURZWEIL-HENSTOCK INTEGRAL IN THE MULTIDIMENSIONAL SPACE

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Abstract. In this paper a full totalization is presented of the *Kurzweil-Henstock* integral in the multidimensional space. A residual function of the total *Kurzweil-Henstock* primitive is defined.

Keywords: totalization, Kurzweil-Henstock integral, primitive

MSC 2010: 26A39, 26B20

1. Introduction

Let F be a differentiable function on an interval E in the m-dimensional space with the derivative f. The problem of recovering F from f is called the problem of primitives. *Cabral* and *Lee* [3] gave an affirmative answer to the question whether we can describe the *Kurzweil-Henstock* primitive F without explicitly involving f. To do this, they first adopted the convention [4, p. 57] that

$$D_cF(x) = \begin{cases} \lim\limits_{|I_c|\to 0^+} \dfrac{F(I_c)}{|I_c|} & \text{if the limit exists,} \\ 0 & \text{if the limit does not exist,} \end{cases}$$

where I_c is a compact cubic subinterval of E, x is a vertex of I_c and $|I_c|$ is the *Lebesgue* measure of I_c. Secondly, they showed that if a real-valued point function f is *Kurzweil-Henstock* integrable on E with a primitive F, then $\lim\limits_{|I_c|\to 0^+} F(I_c)/|I_c| = f(x)$ almost everywhere on E, and after that they answered affirmatively the question of whether it is possible to define a point function $D_cF(x)$ for a *Kurzweil-Henstock* primitive F which is *Kurzweil-Henstock* integrable on E and such that $(\mathcal{KH})\int_I D_cF = F(I)$ for any compact subinterval I of E. One step further in this direction is a question whether we can describe F that is not a *Kurzweil-Henstock* primitive in the sense that the corresponding point function $D_{\mathrm{ex}}F$ defined in what follows is *Kurzweil-Henstock* integrable on E and $(\mathcal{KH})\int_E D_{\mathrm{ex}}F \neq F(E)$. In this paper, we give an affirmative answer to this question.

2. Preliminaries

The set E always refers to a fixed, compact interval in the multidimensional space $\mathbb{R}^m$. The collection $\mathcal{I}(E)$ is the family of compact subintervals I of E. A *Kurzweil-Henstock* partial division $\Delta = \{(x, I)\}$ of E is any finite set (collection) of point-interval pairs (x, I), such that x is a vertex of I, $I \in \mathcal{I}(E)$ and the subintervals I are nonoverlapping. The points x are the tags of Δ [1]. Let Γ be a subset of E. If $x \in \Gamma$ for each $(x, I) \in \Delta$, then Δ is said to be tagged in Γ. A partial division $\Delta = \{(x, I)\}$ of E is called a division of E if the union of the intervals I in Δ is equal to E. Given $\delta\colon E \to (0, 1)$, named a gauge, a partial division Δ of E is said to be δ-*fine* if for each $(x, I) \in \Delta$ the interval I is contained in the open ball $B(x, \delta(x))$ centred at x and of radius $\delta(x)$.

Any real valued function F defined on $\mathcal{I}(E)$ is an interval function. For any collection of nonoverlapping subintervals I_1, $I_2 \in \mathcal{I}(E)$, let $F(I_1 \cup I_2) = F(I_1) + F(I_2)$. This property is called additivity. A function $F\colon \mathcal{I}(E) \to \mathbb{R}$ is said to be differentiable at $x \in E$ with a derivative $f(x)$ if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon\colon E \to (0, 1)$ such that $|F(I) - f(x)|I|| < \varepsilon|I|$ whenever (x, I) is a δ_ε-*fine* point-interval pair and x is a vertex of $I \in \mathcal{I}(E)$. The function F is called a primitive. A function $f\colon E \to \mathbb{R}$ is said to be *Kurzweil-Henstock* integrable to a real number A on E if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon\colon E \to (0, 1)$ such that

$$\Bigl|(\Delta)\sum f(x)|I| - A\Bigr| < \varepsilon \tag{2.1}$$

whenever Δ is a δ_ε-*fine* division of E. In symbols, $A = (\mathcal{KH})\int_E f$. If a function f is *Kurzweil-Henstock* integrable on E and $F(I) = (\mathcal{KH})\int_I f$ for all compact subintervals I of E, then the additive interval function F is called the *Kurzweil-Henstock* primitive of f.

3. A residual function of a primitive

For a given pair of functions F and f let $X \subset E$ be the set of points at which the primitive F is not differentiable and [2]

$$\Gamma_\varepsilon^{\mathcal{KH}} = \{(x, I)\colon\ x \in E \setminus X \text{ is a vertex of } I \text{ and } |F(I) - f(x)|I|| < \varepsilon|I|\}.$$

Then we can define a point function $D_{\text{ex}}F\colon E \to \mathbb{R}$ by extending f from $E \setminus X$ to E by $D_{\text{ex}}F(x) = 0$ for $x \in X$, so that

$$D_{\text{ex}}F = \begin{cases} f(x) & \text{if } x \in E \setminus X, \\ 0 & \text{if } x \in X. \end{cases} \tag{3.1}$$

From the collection of all δ_ε-*fine* point-interval pairs $(x, I) \in \Gamma_\varepsilon^{\mathcal{KH}}$, the subset $E \setminus X$ of E may be obtained, as follows.

Definition 3.1. The set $\{x \in E\colon$ for every $\varepsilon > 0$ there exists a δ_ε-fine $(x, I) \in \Gamma_\varepsilon^{\mathcal{KH}}\}$ denoted by $(vp)E$ is said to be the set of regular points of F on E.

Given $\varepsilon > 0$, in the set

$$\Omega_\varepsilon^{\mathcal{KH}} = \{(x, I)\colon\ x \in X \text{ is a vertex of } I \text{ and } |F(I)| \geqslant \varepsilon|I|\} \tag{3.2}$$

we isolate two subsets:

$$\Omega_{<\varepsilon}^{\mathcal{KH}} = \{(x, I)\colon\ x \in X \text{ is a vertex of } I \text{ and } \varepsilon|I| \leqslant |F(I)| < \varepsilon\} \quad \text{and}$$
$$\Omega_{\geqslant\varepsilon}^{\mathcal{KH}} = \{(x, I)\colon\ x \in X \text{ is a vertex of } I \text{ and } |F(I)| \geqslant \varepsilon\}.$$

Now, from the collection of all δ_ε-*fine* point-interval pairs $(x, I) \in \Omega_\varepsilon^{\mathcal{KH}}$, two subsets of X may be obtained, as follows.

Definition 3.2. The set $\{x \in E\colon$ for every $\varepsilon > 0$ there exists a δ_ε-fine $(x, I) \in \Omega_{<\varepsilon}^{\mathcal{KH}}\}$ denoted by $(vss)E$ is said to be the set of seeming singular points of F on E.

Definition 3.3. The set $\{x \in E\colon$ for every $\varepsilon > 0$ there exists a δ_ε-fine $(x, I) \in \Omega_{\geqslant\varepsilon}^{\mathcal{KH}}\}$ denoted by $(vs)E$ is said to be the set of singular points of F on E.

Accordingly, we are now in a position to define the notion of a residue of the primitive F at $x \in E$.

Definition 3.4. A function $F\colon \mathcal{I}(E) \to \mathbb{R}$ is said to have a residue at $x \in E$ with the residual value $\mathcal{R}(x)$ if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon\colon E \to (0, 1)$ such that

$$|F(I) - \mathcal{R}(x)| < \varepsilon \tag{3.3}$$

whenever (x, I) is a δ_ε-fine point-interval pair and x is a vertex of $I \in \mathcal{I}(E)$.

A real-valued point function $\mathcal{R}$ defined on E is called a residual function of the primitive F on E.

Definition 3.5. Let $F\colon \mathcal{I}(E) \to \mathbb{R}$ and $\Gamma \subseteq E$. The residual function $\mathcal{R}$ of F is said to be basically summable (BS_{δ_ε}) on Γ with the sum $S \in \mathbb{R}$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon\colon E \to (0,1)$ such that $|(\Delta)\sum F(I) - S| < \varepsilon$ whenever $\Delta = \{(x, I)\}$ is a δ_ε-fine partial division tagged in Γ. The residual function $\mathcal{R}$ of F is BSG_{δ_ε} on Γ if Γ can be written as a countable union of sets on each of which F is BS_{δ_ε}. In symbols, $S = \sum_{x\in\Gamma} \mathcal{R}(x)$.

Remark 3.1. By Definition 5.11 in [1], if $S = 0$ in the above definition, then F has negligible variation on Γ. On the contrary, if there is a subset Γ of E of variation zero (this means, given $\varepsilon > 0$ there is a gauge δ_ε such that $(\Delta)\sum |I| < \varepsilon$ whenever $\Delta = \{(x, I)\}$ is δ_ε-fine partial division tagged in Γ, [2]) on which $\mathcal{R}$ of F is BS_{δ_ε} with $S \neq 0$, then F does not satisfy the variational Strong *Lusin* condition ($SL_v(E)$) on E. On the other hand, since for every $\varepsilon > 0$ there exists a gauge δ_ε such that $|F(I)| < \varepsilon$ whenever (x, I) is a δ_ε-fine point-interval pair tagged in $(vp)E \cup (vss)E$ and x is a vertex of $I \in \mathcal{I}(E)$, it follows immediately that $\mathcal{R}(x) \equiv 0$ on $(vp)E \cup (vss)E$. In addition, for a given pair of functions F and $\mathcal{R}$, if F is an additive function and $\mathcal{R}$ is BS_{δ_ε} on E, then, by Definition 3.5, $S = F(E)$, that is $\sum_{x\in E} \mathcal{R}(x) = F(E)$. So, if F is the *Kurzweil-Henstock* primitive, then, in spite of the fact that $\mathcal{R}(x)$ vanishes identically on E, for any compact interval $I \in \mathcal{I}(E)$ we have

$$\sum_{x\in I} \mathcal{R}(x) = (\mathcal{KH}) \int_I D_{\mathrm{ex}} F.$$

4. The total Kurzweil-Henstock primitive

If there are compact subintervals I of E such that $F(I) \neq (\mathcal{KH}) \int_I D_{\mathrm{ex}} F$, then a question that arises is whether we can describe the primitive F, in this emphasized case, too. The affirmative answer comes from the following definition.

Definition 4.1. If $F\colon \mathcal{I}(E) \to \mathbb{R}$ is an additive function, then $D_{\mathrm{ex}}F$ is totally Kurzweil-Henstock integrable to $F(E)$ on E. In symbols,

$$F(E) = (\mathcal{KH})vt \int_E D_{\mathrm{ex}} F. \tag{4.1}$$

By the total *Kurzweil-Henstock* integral the *Kurzweil-Henstock* primitive is totalized, in the sense that any additive interval function F defined on $\mathcal{I}(E)$ is the total *Kurzweil-Henstock* primitive of $D_{\mathrm{ex}}F$ on E. This means that $F(I) = (\mathcal{KH})vt \int_I D_{\mathrm{ex}} F$ for any compact subinterval I of E. Therefore, if $D_{\mathrm{ex}}F$ is *Kurzweil-Henstock* integrable on E and $F(E) \neq (\mathcal{KH}) \int_E D_{\mathrm{ex}} F$, then there is a real number

S such that $F(E) = (\mathcal{KH})vt\int_E D_{\rm ex}F = (\mathcal{KH})\int_E D_{\rm ex}F + S$. Clearly, in this case, the integral equality $F(I) = (\mathcal{KH})\int_I D_{\rm ex}F$, which is not valid for all compact subintervals I of E, must be replaced by $F(I) = (\mathcal{KH})vt\int_I D_{\rm ex}F$. If $D_{\rm ex}F$ is not *Kurzweil-Henstock* integrable on E, then the sum $(\mathcal{KH})\int_E D_{\rm ex}F + S$ reduces to the so-called indeterminate expression $\infty - \infty$ that, in this particular case, takes the value $F(E)$. However, in this case, too, $F(I) = (\mathcal{KH})vt\int_I D_{\rm ex}F$ for any compact subinterval I of E.

Our main result reads as follows.

Theorem 4.1. *Let $F\colon \mathcal{I}(E) \to \mathbb{R}$ be an additive function such that $D_{\rm ex}F$ is Kurzweil-Henstock integrable on E. Then F is a total Kurzweil-Henstock primitive if and only if its residual function $\mathcal{R}$ is BS_{δ_ε} on $\Omega_\varepsilon^{\mathcal{KH}}$.*

Proof. By the definition of $D_{\rm ex}F$ at a point $x \in (vp)E$, given $\varepsilon > 0$ there is a gauge $\delta_\varepsilon^\star\colon E \to (0,1)$ such that $|F(I) - D_{\rm ex}F(x)|I|| < \varepsilon|I|$ whenever $(x,I) \in \Gamma_\varepsilon^{\mathcal{KH}}$ is a $\delta_\varepsilon^\star$-*fine* point-interval pair.

($\Longrightarrow$) Suppose that F is a total *Kurzweil-Henstock* primitive. Since $D_{\rm ex}F(x)$ is both totally *Kurzweil-Henstock* integrable with a primitive F and *Kurzweil-Henstock* integrable on E, it is true that there exists a real number S with the following property: for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon^*\colon E \to (0,1)$ such that $|(\Delta)\sum[F(I) - D_{\rm ex}F(x)|I|] - S| < \varepsilon$ whenever Δ is a δ_ε^*-*fine* division of E. A gauge δ_ε may be chosen such that $\delta_\varepsilon(x) = \min(\delta_\varepsilon^\star(x), \delta_\varepsilon^*(x))$ if $x \in (vp)E$ and $\delta_\varepsilon(x) = \delta_\varepsilon^*(x)$ otherwise on E. Therefore, for any δ_ε-*fine* division Δ of E (remember: if $x \in E \setminus (vp)E$, then $D_{\rm ex}F(x) = 0$)

$$\Big|(\Delta\cap\Omega_\varepsilon^{\mathcal{KH}})\sum F(I) - S\Big| \leqslant \Big|(\Delta)\sum[F(I) - D_{\rm ex}F(x)|I|] - S\Big| + \Big|(\Delta\setminus\Omega_\varepsilon^{\mathcal{KH}})\sum[F(I) - D_{\rm ex}F(x)|I|]\Big| < \varepsilon(1+|E|).$$

($\Longleftarrow$) Let $\mathcal{R}$ of F be $BS_{\delta_\varepsilon^*}$ on $\Omega_\varepsilon^{\mathcal{KH}}$. If $\Delta = \{(x,I)\}$ is a δ_ε-*fine* division of E such that $\delta_\varepsilon(x) = \delta_\varepsilon^\star(x)$ if $x \in (vp)E$ and $\delta_\varepsilon(x) = \delta_\varepsilon^*(x)$ otherwise, then

$$\Big|(\Delta)\sum[F(I) - D_{\rm ex}F(x)|I|] - S\Big| \leqslant (\Delta\setminus\Omega_\varepsilon^{\mathcal{KH}})\sum|F(I) - D_{\rm ex}F(x)|I|| + \Big|(\Delta\cap\Omega_\varepsilon^{\mathcal{KH}})\sum F(I) - S\Big| < \varepsilon(|E|+1).$$

□

Remark 4.1. By the preceding theorem

$$F(E) = (\mathcal{KH})\int_E D_{\rm ex}F + \sum_{x\in\Omega_\varepsilon^{\mathcal{KH}}}\mathcal{R}(x),$$

that is

$$(\mathcal{KH})vt\int_E D_{\rm ex}F = (\mathcal{KH})\int_E D_{\rm ex}F + \sum_{x\in\Omega_\varepsilon^{\mathcal{KH}}} \mathcal{R}(x).$$

Since, by *Hake's* theorem [1], $(\mathcal{KH})\int_E D_{\rm ex}F = (\mathcal{KH})vp\int_E D_{\rm ex}F$, where the so called *principal value* of $(\mathcal{KH})\int_E D_{\rm ex}F$ is denoted by vp, the sum $\sum_{x\in\Omega_\varepsilon^{\mathcal{KH}}} \mathcal{R}(x)$ may be conditionally called the *singular value* of $(\mathcal{KH})\int_E D_{\rm ex}F$ $(\mathcal{KH})$. In symbols, $\sum_{x\in\Omega_\varepsilon^{\mathcal{KH}}} \mathcal{R}(x) = (\mathcal{KH})vs\int_E D_{\rm ex}F$. Accordingly,

$$(\mathcal{KH})vt\int_E D_{\rm ex}F = (\mathcal{KH})vp\int_E D_{\rm ex}F + (\mathcal{KH})vs\int_E D_{\rm ex}F.$$

If $F(I) := (\mathcal{KH})\int_{\partial I} F$, where ∂I is the boundary of $I \in \mathcal{I}(E)$, and if $D_{\rm ex}F$ vanishes identically on $(vp)E$, then

$$(\mathcal{KH})\int_{\partial E} F = \sum_{x\in\Omega_\varepsilon^{\mathcal{KH}}} \mathcal{R}(x),$$

which is an extension of *Cauchy's* residue theorem result in $\mathbb{R}^m$.

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