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POSITIVE PERIODIC SOLUTION FOR RATIO-DEPENDENT n-SPECIES DISCRETE TIME SYSTEM*

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Abstract. In this paper, sharp a priori estimate of the periodic solutions is obtained for the discrete analogue of the continuous time ratio-dependent predator-prey system, which is governed by nonautonomous difference equations, modelling the dynamics of the $n-1$ competing preys and one predator having nonoverlapping generations. Based on more precise a priori estimate and the continuation theorem of the coincidence degree, an easily verifiable sufficient criterion of the existence of positive periodic solutions is established. The result obtained in this paper greatly improves the existing results.

Keywords: ratio-dependent, predator-prey system, periodic solution, a priori estimate

MSC 2010: 34C25

1. Introduction

The traditional Lotka-Volterra type predator-prey model has received great attention from both theoretical and mathematical biologists, and has been well studied [9]. Recently, models with the prey-dependent-only response function have been facing challenges from biology and physiology communities [1]. Basing on growing biological and physiological evidence some biologists have argued that, in many situations, especially when predators have to search for food (and therefore, have to share or compete for food), the functional response in a prey-predator model should be ratio-dependent, which can be roughly stated as that the per capita predator growth rate

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should be a function of the ratio of prey to predator abundance. This has been strongly supported by numerous field and laboratory experiments and observations. Starting from this argument and the traditional prey-dependent-only model, Arditi and Ginzburg [1] first studied the ratio-dependent predator-prey model.

Ratio-dependent models have not been well studied yet in the sense that most results are for models with constant environment [5]. This means that the models have been assumed to be autonomous, that is, all biological or environmental parameters have been assumed to be constant in time. However, this is rarely the case in real life, because many biological and environmental parameters do vary in time (e.g., naturally subject to seasonal fluctuations). When this is taken into account, a model must be nonautonomous, which is, of course, more difficult to analyze in general. But, in doing so, one can and should also take advantage of the properties of those varying parameters. For example, one may assume the parameters are periodic or almost periodic for seasonal reasons [2].

Though much progress has been seen in the ratio-dependent predator-prey theories, such systems are not well studied in the sense that most results concern the continuous time cases (see, for example, [7], [10]–[14]). Many authors have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. It is well known that, compared to the continuous time systems, the discrete time ratio-dependent predator-prey systems are more difficult to deal with. It is highly nontrivial to attack the existence of positive periodic solutions of this type systems. However, few works have been done for discrete time ratio-dependent predator-prey systems. With the help of differential equations with piece constant arguments, Fan et al. ([4], [6]) proposed first a discrete analogue of the continuous time ratio-dependent predator-prey system and gave some new sufficient conditions for the existence of a positive periodic solution. In order to improve and extend the results in [4], Ding and Lu [3] proposed the ratio-dependent n-species predator-prey system

$$\tag{1} \begin{cases} x_i'(t) = x_i(t)[a_i(t) - a_{ii}(t)x_i(t)] - \displaystyle\sum_{j=1, j\neq i}^{n-1} a_{ij}(t)x_j(t)x_i(t) \\ \qquad\qquad - \dfrac{a_{in}(t)x_i(t)x_n(t)}{m_{in}(t)x_n(t) + x_i(t)}, \\ x_n'(t) = x_n(t)\left\{-a_n(t) + \displaystyle\sum_{l=1}^{n-1} \dfrac{a_{nl}(t)x_l(t)}{m_{in}(t)x_n(t) + x_l(t)}\right\}, \end{cases}$$

where $x_n(t)$, living on $x_1(t), x_2(t), \ldots, x_{n-1}(t)$, represents the predator density at t;

$x_i(t)$, $a_i(t)$ stand for the densities and intrinsic growth rate of the ith prey, respectively; $a_{ii}(t)$ and $a_{ij}(t)$ denote the intra-specific competition rates of the ith prey and the inter-specific competition rates of ith prey to the jth prey, respectively; $a_{in}(t)$ is the maximal predator per capita consumption rate, i.e., the maximum number of the ith prey that can be eaten by a predator in each time unit and $m_{in}(t)$ is the number of the ith prey necessary to achieve one-half of the maximum rate $a_{in}(t)$; $a_{ni}(t)/a_{in}(t)$ is the measure of the food quality that the ith prey provides for the conversion into the predator birth; $a_n(t)$ is the death rate of the predator; The predator consumes the ith prey according to the functional response $x_i(t)/[m_{in}(t)x_n(t) + x_i(t)]$, $i = 1, 2, \ldots, n-1$. Furthermore, Ding and Lu in [3] established some sufficient criteria for the existence of a positive periodic solution of the following discrete time analogue of system (1):

$$
(2)\qquad \begin{cases} x_i(k+1) = x_i(k)\exp\left\{a_i(k) - \displaystyle\sum_{j=1}^{n-1} a_{ij}(k)x_j(k) - \frac{a_{in}(k)x_n(k)}{m_{in}(k)x_n(k) + x_i(k)}\right\}, \\ \hfill i = 1, 2, \ldots, n-1, \\ x_n(k+1) = x_n(k)\exp\left\{-a_n(k) + \displaystyle\sum_{l=1}^{n-1} \frac{a_{nl}(k)x_l(k)}{m_{ln}(k)x_n(k) + x_l(k)}\right\}, \end{cases}
$$

where $k \in \mathbb{Z}_+$; $a_{ii}\colon \mathbb{Z} \to \mathbb{R}_+$ $(i = 1, 2, 3, \ldots, n-1)$; $a_{nj}\colon \mathbb{Z} \to \mathbb{R}_+$ $(j = 1, 2, 3, \ldots, n-1)$; $m_{ln}\colon \mathbb{Z} \to \mathbb{R}_+$ $(l = 1, 2, 3, \ldots, n-1)$; $a_{ij}(k) \geqslant 0$ ($i \neq j$ and $i, j = 1, 2, 3, \ldots, n-1$) and $\bar{a}_i > 0$ $(i = 1, 2, 3, \ldots, n)$; $a_{ij}(k)$ and $a_i(k)$ are ω-periodic functions.

Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{R}_+$ and $\mathbb{R}^n$ denote the sets of all integers, positive integers, positive real numbers and n-dimensional Euclidean vector space, respectively.

For convenience, throughout the paper we will use the notation

$$
\omega \in \mathbb{Z}_+, \quad \bar{g} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} g(k), \quad \overline{G} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} |g(k)|, \quad I_\omega = \{0, 1, \ldots, \omega-1\},
$$

where $g(k)$ is an ω-periodic sequence of real numbers defined for $k \in \mathbb{Z}$. For a given ω-periodic function y_i, denote

$$
y_i(\xi_i) = \min_{k\in I_\omega} y_i(k), \quad y_i(\eta_i) = \max_{k\in I_\omega} y_i(k).
$$

In this paper, by using some analysis skill, an important inequality is first proved and applied to obtain the improved a priori estimate of the periodic solution. Based on sharp a priori estimate and the related continuation theorem of the coincidence degree, a verifiable sufficient condition is established for the existence of positive periodic solutions of a discrete time nonautonomous ratio-dependent n-species predator-prey system. This sufficient condition improves the main result obtained in [3]. The paper also corrects some mistakes in [3].

2. Existence of positive periodic solution

Let X, Y be normed vector spaces, $L\colon \operatorname{Dom} L \subset X \to Y$ a linear mapping, and $N\colon X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q)$, then $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}\colon (I-P)X \to \operatorname{Im} L$ is invertible, so we denote the inverse of this map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on Ω if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N\colon \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 1 ([8])**.** *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$,*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

Ding and Lu in [3] proved the following important lemmas.

Lemma 2 ([3])**.** *$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n)^\top \colon x_i > 0,\ i = 1,2,3,\ldots,n\}$ is positive invariant with respect to equation* (2).

Lemma 3 ([3])**.** *If $\bar{a}_{n1} > \bar{a}_n$, then the system of algebraic equations*

$$\tag{3} \begin{cases} \bar{a}_i - \bar{a}_{ii} v_i = 0, \quad i = 1,2,\ldots,n-1, \\ \bar{a}_n - \dfrac{1}{\omega} \displaystyle\sum_{k=0}^{\omega-1} \sum_{j=1}^{n-1} \frac{a_{nj}(k) v_j}{m_{jn}(k) v_n + v_j} = 0 \end{cases}$$

has a unique solution $(v_1^*, v_2^*, \ldots, v_n^*)$ with $v_l^* > 0$, $l = 1,2,\ldots,n$.

Considering the biological significance of system (2), we specify $(x_1(0), x_2(0), \ldots, x_n(0))^\top \in \mathbb{R}^n_+$.

Lemma 4. *Suppose $g\colon \mathbb{Z} \to \mathbb{R}$, $g(k+\omega) = g(k)$, $\omega \in \mathbb{Z}_+$. Then for any fixed $k_1 \in I_\omega = \{0,1,2,\ldots,\omega-1\}$ and any $k \in \mathbb{Z}$, we have*

$$\tag{4} g(k_1) - \frac{1}{2} \sum_{s=0}^{\omega-1} |g(s+1) - g(s)| \leqslant g(k) \leqslant g(k_1) + \frac{1}{2} \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

Proof. Since g is ω-periodic, we only need to show that inequality (4) is valid for any $k \in I_\omega$.

If $k = k_1$, then inequality (4) is clearly true.

If $k > k_1$, then

$$g(k) - g(k_1) = \sum_{s=k_1}^{k-1} [g(s+1) - g(s)]. \tag{5}$$

So

$$|g(k) - g(k_1)| = \left| \sum_{s=k_1}^{k-1} [g(s+1) - g(s)] \right| \leqslant \sum_{s=k_1}^{k-1} |g(s+1) - g(s)|. \tag{6}$$

Furthermore,

$$g(k) - g(k_1) = g(k) - g(k_1+\omega) = -[g(k_1+\omega) - g(k)] = -\sum_{s=k}^{k_1+\omega-1} [g(s+1) - g(s)]. \tag{7}$$

Thus,

$$|g(k) - g(k_1)| = \left| -\sum_{s=k}^{k_1+\omega-1} [g(s+1) - g(s)] \right| \leqslant \sum_{s=k}^{k_1+\omega-1} |g(s+1) - g(s)|. \tag{8}$$

Combining inequality (6) with inequality (8) gives that

$$\begin{aligned} 2|g(k) - g(k_1)| &\leqslant \sum_{s=k_1}^{k-1} |g(s+1) - g(s)| + \sum_{s=k}^{k_1+\omega-1} |g(s+1) - g(s)| \\ &= \sum_{s=k_1}^{k_1+\omega-1} |g(s+1) - g(s)| = \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|. \end{aligned} \tag{9}$$

Therefore,

$$|g(k) - g(k_1)| \leqslant \frac{1}{2} \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|. \tag{10}$$

Similarly, we can prove that inequality (10) holds if $k < k_1$.

Obviously, inequality (10) implies inequality (4). This completes the proof. $\square$

Fan et al. [4] proved the following lemma.

Lemma. *Let $g\colon \mathbb{Z} \to \mathbb{R}$ be ω-periodic, i.e., $g(k+\omega) = g(k)$. Then for any fixed $k_1, k_2 \in I_\omega$ and any $k \in \mathbb{Z}$, one has*

$$g(k) \leqslant g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|, \quad g(k) \geqslant g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

Basing on the above lemma, Fan et al. [4] and Ding et al. [3] studied the existence of a positive periodic solution for the predator-prey system. Obviously, Lemma 4 in our paper greatly improves Lemma 2 in [3] and Lemma 3.2 [4]. We can obtain more precise a priori estimate of the periodic solution by using Lemma 4.

Next, we will investigate the existence of a positive periodic solution for the predator-prey system (2).

Define

$$l_n = \{y = \{y(k)\}\colon\ y(k) \in \mathbb{R}^n,\ k \in \mathbb{Z}\}.$$

Take the usual norm

$$\|y\| = (|y_1|_0^2 + |y_2|_0^2 + \ldots + |y_n|_0^2)^{1/2},$$

where $|y_i|_0 = \max\limits_{k\in I_\omega} |y_i(k)|$, $i = 1, 2, \ldots, n$. Let $l^\omega \subset l_n$ denote the subspace of all ω-periodic sequences equipped with the usual norm, then it is easy to prove that l^ω is a finite-dimensional Banach space.

Let

$$l_0^\omega = \left\{y = \{y(k)\} \in l^\omega\colon\ \frac{1}{\omega}\sum_{k=0}^{\omega-1} y(k) = 0\right\},$$
$$l_c^\omega = \bigl\{y = \{y(k)\} \in l^\omega\colon\ y(k) = h,\ h \in \mathbb{R}^n,\ k \in \mathbb{Z}\bigr\}.$$

Then it follows that both l_0^ω and l_c^ω are closed linear subspaces of l^ω and

$$l^\omega = l_0^\omega \oplus l_c^\omega, \quad \dim l_c^\omega = n.$$

Theorem 1. *System* (2) *has at least one ω-periodic solution with strictly positive components if*

$$\bar{a}_i - \sum_{j=1, j\neq i}^{n-1} \bar{a}_{ij} \mathrm{e}^{L_j} - \overline{\left(\frac{a_{in}}{m_{in}}\right)} > 0, \quad i = 1, 2, \ldots, n-1, \tag{11}$$

and

$$\bar{a}_{n1}\mathrm{e}^{d_1} - \bar{a}_n \mathrm{e}^{L_1} > 0, \tag{12}$$

where $L_j = \ln(\bar a_j/\bar a_{jj}) + \frac{1}{2}(\bar A_j + \bar a_j)\omega$, $j = 1, 2, \ldots, n-1$,

$$d_1 = \ln \frac{\bar a_1 - \sum_{j=2}^{n-1} \bar a_{1j} \mathrm{e}^{L_j} - \overline{\left(\frac{a_{1n}}{m_{1n}}\right)}}{\bar a_{11}} - \frac{1}{2}(\bar A_1 + \bar a_1)\omega,$$

$\bar A_i = \omega^{-1} \sum_{k=0}^{\omega-1} |a_i(k)|$.

Proof. Let $x_i(k) = \exp\{y_i(k)\}$, $i = 1, 2, \ldots, n$. Then equation (2) becomes

$$\begin{cases} y_i(k+1) - y_i(k) = a_i(k) - \sum_{j=1}^{n-1} a_{ij}(k) \exp\{y_j(k)\} \\ \qquad - \dfrac{a_{in}(k)\exp\{y_n(k)\}}{m_{in}(k)\exp\{y_n(k)\} + \exp\{y_i(k)\}}, \quad i = 1, 2, \ldots, n-1, \\ y_n(k+1) - y_n(k) = -a_n(k) + \sum_{l=1}^{n-1} \dfrac{a_{nl}(k)\exp\{y_l(k)\}}{m_{ln}(k)\exp\{y_n(k)\} + \exp\{y_l(k)\}}, \end{cases} \tag{13}$$

where $k \in \mathbb{Z}_+$.

Let $X = Y = l^\omega$. For any $y \in X$ and $k \in \mathbb{Z}$, define $(Ly)(k) = y(k+1) - y(k)$ and

$$Ny(k) = \begin{cases} a_i(k) - \sum_{j=1}^{n-1} a_{ij}(k) \exp\{y_j(k)\} - \dfrac{a_{in}(k)\exp\{y_n(k)\}}{m_{in}(k)\exp\{y_n(k)\} + \exp\{y_i(k)\}}, \\ \qquad\qquad i = 1, 2, \ldots, n-1, \\ -a_n(k) + \sum_{l=1}^{n-1} \dfrac{a_{nl}(k)\exp\{y_l(k)\}}{m_{ln}(k)\exp\{y_n(k)\} + \exp\{y_l(k)\}}. \end{cases} \tag{14}$$

Then L is a bounded linear operator and

$$\operatorname{Ker} L = l^\omega_c, \quad \operatorname{Im} L = l^\omega_0, \quad \dim \operatorname{Ker} L = n = \operatorname{codim} \operatorname{Im} L.$$

So, L is a Fredholm mapping of index zero.

Define

$$Py = \frac{1}{\omega} \sum_{k=0}^{\omega-1} y(k), \ \ y \in X, \qquad Qz = \frac{1}{\omega} \sum_{k=0}^{\omega-1} z(k), \ \ z \in Y.$$

It is not difficult to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Furthermore, the generalized inverse (to L) $K_P\colon \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$K_P(z) = \sum_{s=0}^{\omega-1} z(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)z(s). \tag{15}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, it is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus N is L-compact on $\overline{\Omega}$. Since $\operatorname{Im} Q = \operatorname{Ker} L$, the isomorphic mapping J from $\operatorname{Im} Q$ to $\operatorname{Ker} L$ is I. Corresponding to the operator equation $Ly = \lambda N y$, $\lambda \in (0,1)$, we have

$$\begin{cases} y_i(k+1) - y_i(k) = \lambda\biggl(a_i(k) - \displaystyle\sum_{j=1}^{n-1} a_{ij}(k)\exp\{y_j(k)\} \\ \qquad\qquad - \dfrac{a_{in}(k)\exp\{y_n(k)\}}{m_{in}(k)\exp\{y_n(k)\} + \exp\{y_i(k)\}}\biggr), \quad i = 1,2,\dots,n-1, \\ y_n(k+1) - y_n(k) = \lambda\biggl(-a_n(k) + \displaystyle\sum_{l=1}^{n-1} \dfrac{a_{nl}(k)\exp\{y_l(k)\}}{m_{ln}(k)\exp\{y_n(k)\} + \exp\{y_l(k)\}}\biggr), \end{cases} \tag{16}$$

for $\lambda \in (0,1)$. Assume that $y = \{y(k)\} = \{(y_1(k), y_2(k), \dots, y_n(k))^\top\} \in X$ is an arbitrary solution of (16) for a certain $\lambda \in (0,1)$. Summing both sides of (16) from 0 to $\omega - 1$ with respect to k gives

$$\begin{cases} \bar{a}_i\omega = \displaystyle\sum_{k=0}^{\omega-1}\biggl(\sum_{j=1}^{n-1} a_{ij}(k)\exp\{y_j(k)\} + \dfrac{a_{in}(k)\exp\{y_n(k)\}}{m_{in}(k)\exp\{y_n(k)\} + \exp\{y_i(k)\}}\biggr), \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1,2,\dots,n-1, \\ \bar{a}_n\omega = \displaystyle\sum_{k=0}^{\omega-1}\sum_{l=1}^{n-1} \dfrac{a_{nl}(k)\exp\{y_l(k)\}}{m_{ln}(k)\exp\{y_n(k)\} + \exp\{y_l(k)\}}. \end{cases} \tag{17}$$

Equation (16) and (17) imply

$$\sum_{k=0}^{\omega-1} |y_i(k+1) - y_i(k)| \leqslant \sum_{k=0}^{\omega-1} |a_i(k)| + \bar{a}_i\omega = (\bar{A}_i + \bar{a}_i)\omega, \quad i = 1,2,\dots,n-1. \tag{18}$$

From (17) we have

$$\bar{a}_i\omega \geqslant \sum_{k=0}^{\omega-1} a_{ii}(k)\exp\{y_i(k)\} \geqslant \sum_{k=0}^{\omega-1} a_{ii}(k)\mathrm{e}^{y_i(\xi_i)}. \tag{19}$$

So

$$y_i(\xi_i) \leqslant \ln \frac{\bar{a}_i}{\bar{a}_{ii}}. \tag{20}$$

From Lemma 4 and (18) it follows that

$$\begin{aligned} y_i(k) &\leqslant y_i(\xi_i) + \frac{1}{2}\sum_{s=0}^{\omega-1} |y_i(s+1) - y_i(s)| \\ &\leqslant \ln \frac{\bar{a}_i}{\bar{a}_{ii}} + \frac{1}{2}(\bar{A}_i + \bar{a}_i)\omega =: L_i, \quad i = 1, 2, \ldots, n-1. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} \bar{a}_i\omega &\leqslant \sum_{k=0}^{\omega-1}\biggl(\sum_{j=1}^{n-1} a_{ij}(k)\exp\{y_j(k)\} + \frac{a_{in}(k)}{m_{in}(k)}\biggr) \\ &\leqslant \omega\biggl(\bar{a}_{ii}\mathrm{e}^{y_i(\eta_i)} + \sum_{j=1, j\neq i}^{n-1} \bar{a}_{ij}\mathrm{e}^{L_j} + \overline{\Bigl(\frac{a_{in}}{m_{in}}\Bigr)}\biggr), \quad i = 1, 2, \ldots, n-1. \end{aligned} \tag{22}$$

Then

$$y_i(\eta_i) \geqslant \ln \frac{\bar{a}_i - \displaystyle\sum_{j=1, j\neq i}^{n-1} \bar{a}_{ij}\mathrm{e}^{L_j} - \overline{\Bigl(\frac{a_{in}}{m_{in}}\Bigr)}}{\bar{a}_{ii}}. \tag{23}$$

Therefore, Lemma 4 and (18) imply

$$\begin{aligned} y_i(k) \geqslant \ln \frac{\bar{a}_i - \displaystyle\sum_{j=1, j\neq i}^{n-1} \bar{a}_{ij}\mathrm{e}^{L_j} - \overline{\Bigl(\frac{a_{in}}{m_{in}}\Bigr)}}{\bar{a}_{ii}} - \frac{1}{2}(\bar{A}_i + \bar{a}_i)\omega =: d_i, \\ i = 1, 2, \ldots, n-1. \end{aligned} \tag{24}$$

From the nth equation of (17) we infer

$$\bar{a}_n\omega \leqslant \sum_{k=0}^{\omega-1}\sum_{l=1}^{n-1} \frac{a_{nl}(k)\exp\{y_l(k)\}}{m_{ln}(k)\exp\{y_n(k)\}} \leqslant \sum_{k=0}^{\omega-1}\sum_{l=1}^{n-1} \frac{a_{nl}(k)\exp\{L_l\}}{m_{ln}(k)\exp\{y_n(\xi_n)\}}. \tag{25}$$

So

$$y_n(\xi_n) \leqslant \ln \frac{\displaystyle\sum_{l=1}^{n-1} \overline{\Bigl(\frac{a_{nl}}{m_{ln}}\Bigr)}\,\mathrm{e}^{L_l}}{\bar{a}_n}. \tag{26}$$

Using Lemma 4 again yields

$$
\begin{aligned}
(27)\qquad y_n(k) &\leqslant y_n(\xi_n) + \frac{1}{2}\sum_{s=0}^{\omega-1} |y_n(s+1) - y_n(s)| \\
&\leqslant \ln \frac{\sum_{l=1}^{n-1} \overline{\left(\frac{a_{nl}}{m_{ln}}\right)} \mathrm{e}^{L_l}}{\bar{a}_n} + \frac{1}{2}(\bar{A}_n + \bar{a}_n)\omega =: L_n.
\end{aligned}
$$

Denote $m_{1n}^u = \max\{m_{1n}(k),\, k \in I_\omega\}$.

From (17) we have

$$
\begin{aligned}
(28)\qquad \bar{a}_n\omega &\geqslant \sum_{k=0}^{\omega-1} \frac{a_{n1}(k)\exp\{y_1(k)\}}{m_{1n}(k)\exp\{y_n(k)\} + \exp\{y_1(k)\}} \\
&\geqslant \sum_{k=0}^{\omega-1} \frac{a_{n1}(k)\mathrm{e}^{d_1}}{m_{1n}^u \mathrm{e}^{y_n(\eta_n)} + \mathrm{e}^{L_1}} \geqslant \frac{\omega\bar{a}_{n1}\mathrm{e}^{d_1}}{m_{1n}^u \mathrm{e}^{y_n(\eta_n)} + \mathrm{e}^{L_1}},
\end{aligned}
$$

that is,

$$
(29)\qquad y_n(\eta_n) \geqslant \ln \frac{\bar{a}_{n1}\mathrm{e}^{d_1} - \bar{a}_n\mathrm{e}^{L_1}}{\bar{a}_n m_{1n}^u}.
$$

Hence,

$$
(30)\qquad y_n(k) \geqslant \ln \frac{\bar{a}_{n1}\mathrm{e}^{d_1} - \bar{a}_n\mathrm{e}^{L_1}}{\bar{a}_n m_{1n}^u} - \frac{1}{2}(\bar{A}_n + \bar{a}_n)\omega =: d_n.
$$

Let $H_i = \max\{|L_i|, |d_i|,\, i = 1, 2, \ldots, n\}$. From inequalities (21), (24), (27), and (30), we have

$$
(31)\qquad |y_i(k)| \leqslant H_i, \quad \|y\| \leqslant \left(\sum_{i=1}^{n} H_i^2\right)^{1/2} =: M_0.
$$

Obviously, M_0 is independent of λ.

Now consider the algebraic equations

$$
(32)\qquad \begin{cases}
\displaystyle\sum_{k=0}^{\omega-1}\Bigg(a_i(k) - a_{ii}(k)\exp\{y_i\} - \mu\sum_{j=1, j\neq i}^{n-1} a_{ij}(k)\exp\{y_j\} \\
\displaystyle\qquad -\mu\frac{a_{in}(k)\exp\{y_n\}}{m_{in}(k)\exp\{y_n\} + \exp\{y_i\}}\Bigg) = 0, \quad i = 1, 2, \ldots, n-1, \\
\displaystyle\sum_{k=0}^{\omega-1}\left(-a_n(k) + \sum_{l=1}^{n-1}\frac{a_{nl}(k)\exp\{y_l\}}{m_{ln}(k)\exp\{y_n\} + \exp\{y_l\}}\right) = 0,
\end{cases}
$$

for $(y_1, y_2, \ldots, y_n)^\top \in \mathbb{R}^n$, in which $\mu \in [0,1]$ is a parameter. Using technique similar to the previous one for estimating values, we get $\bar{d}_i \leqslant y_i \leqslant \bar{L}_i$, $i = 1, 2, \ldots, n$, where

$$\begin{aligned}
\bar{L}_i &= \ln \frac{\bar{a}_i}{\bar{a}_{ii}}, \quad i = 1, 2, \ldots, n-1,\\
\bar{d}_i &= \ln \frac{\bar{a}_i - \sum\limits_{j=1, j\neq i}^{n-1} \bar{a}_{ij} \mathrm{e}^{\bar{L}_j} - \overline{\left(\dfrac{a_{in}}{m_{in}}\right)}}{\bar{a}_{ii}}, \quad i = 1, 2, \ldots, n-1,\\
\bar{L}_n &= \ln \frac{\sum\limits_{l=1}^{n-1} \overline{\left(\dfrac{a_{nl}}{m_{ln}}\right)} \mathrm{e}^{\bar{L}_l}}{\bar{a}_n},\\
\bar{d}_n &= \ln \frac{\bar{a}_{n1} \mathrm{e}^{\bar{d}_1} - \bar{a}_n \mathrm{e}^{\bar{L}_1}}{\bar{a}_n m_{1n}^u}.
\end{aligned}$$

Taking $\overline{H}_i = \max\{|\bar{L}_i|, |\bar{d}_i|, i = 1, 2, \ldots, n\}$, $\overline{H}_i$ is independent of μ. Let $M_1 = \Big(\sum\limits_{i=1}^{n} \overline{H}_i^2\Big)^{1/2}$. Then M_1 is independent of μ and $\|y\| \leqslant M_1$ for every solution $(y_1, y_2, \ldots, y_n)^\top$ of (32).

Since $\bar{a}_{n1} \mathrm{e}^{d_1} - \bar{a}_n \mathrm{e}^{L_1} > 0$ and $d_1 < L_1$, we have $\bar{a}_{n1} > \bar{a}_n$. By Lemma 3, equation (3) has a unique positive solution $(v_1^*, v_2^*, \ldots, v_n^*)^\top$ with $v_i^* > 0$. Now, let $M = M_0 + M_1 + M_2$ where M_2 is taken sufficiently large such that $\Big[\sum\limits_{i=1}^{n} (\ln v_i^*)^2\Big]^{1/2} \leqslant M_2$.

We define

$$\Omega = \{y = \{y(k)\} \in X, \ \|y\| < M\}. \tag{33}$$

Then it is clear that Ω is an open, bounded set in X and verifies the requirement (a) of Lemma 1. When $y \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^n$, y is a constant vector in $\mathbb{R}^n$ with $\|y\| = M$. Note that

$$QNy = \begin{cases} \dfrac{1}{\omega} \sum\limits_{k=0}^{\omega-1} \Big(a_i(k) - \sum\limits_{j=1}^{n-1} a_{ij}(k) \exp\{y_j\} - \dfrac{a_{in}(k) \exp\{y_n\}}{m_{in}(k) \exp\{y_n\} + \exp\{y_i\}} \Big), \\ \hfill i = 1, 2, \ldots, n-1, \\ \dfrac{1}{\omega} \sum\limits_{k=0}^{\omega-1} \Big(-a_n(k) + \sum\limits_{l=1}^{n-1} \dfrac{a_{nl}(k) \exp\{y_l\}}{m_{ln}(k) \exp\{y_n\} + \exp\{y_l\}} \Big). \end{cases} \tag{34}$$

So

$$QNy \neq 0.$$

Now let us consider homotopic $h_\mu(y) = \mu QNy + (1-\mu)Gy$, $\mu \in [0,1]$, where

$$(35)\qquad Gy = \frac{1}{\omega}\sum_{k=0}^{\omega-1}\begin{pmatrix} a_1(k) - a_{11}(k)\exp\{y_1\} \\ a_2(k) - a_{22}(k)\exp\{y_2\} \\ \vdots \\ a_{n-1}(k) - a_{n-1,n-1}(k)\exp\{y_{n-1}\} \\ -a_n(k) + \displaystyle\sum_{l=1}^{n-1}\frac{a_{nl}(k)\exp\{y_l\}}{m_{ln}(k)\exp\{y_n\}+\exp\{y_l\}} \end{pmatrix}.$$

From (32)–(35) we have $0 \notin h_\mu(\partial\Omega\cap\operatorname{Ker} L)$ and $\deg\{QN, \Omega\cap\operatorname{Ker} L, 0\} = \deg\{G, \Omega\cap\operatorname{Ker} L, 0\}$. Direct calculation shows that

$$(36)\quad \deg\Big\{G, \Omega\cap\operatorname{Ker} L, 0\Big\} = \operatorname{sgn}\Big\{(-1)^n\prod_{i=1}^{n-1}\frac{\bar a_{ii}v_i^*}{\omega}\sum_{k=0}^{\omega-1}\sum_{l=1}^{n-1}\frac{a_{nl}(k)m_{ln}(k)v_l^*v_n^*}{m_{ln}(k)v_n^*+v_l^*}\Big\} \neq 0.$$

Note that $J = I$, $\deg\{JQN, \Omega\cap\operatorname{Ker} L, 0\} \neq 0$. By Lemma 1, equation (16) has at least one ω-periodic solution. Therefore, equation (2) has at least one positive ω-periodic solution. □

3. Conclusion

In this paper, based on improved a priori estimate of the periodic solution, a new sufficient condition is established for the existence of positive periodic solutions of a class of nonautonomous discrete time food web model of $n-1$ competing preys and one predator. The result obtained in this paper greatly improves the existing results. This paper also corrects some mistakes in [3].

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