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Complex Oscillation Theory of Differential Polynomials

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Abstract

In this paper, we investigate the relationship between small functions and differential polynomials $g_f(z) = d_2f'' + d_1f' + d_0f$, where $d_0(z)$, $d_1(z)$, $d_2(z)$ are entire functions that are not all equal to zero with $\rho(d_j) < 1$ $(j = 0, 1, 2)$ generated by solutions of the differential equation $f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F$, where a, b are complex numbers that satisfy $ab(a - b) \neq 0$ and $A_j(z) \not\equiv 0$ $(j = 0, 1)$, $F(z) \not\equiv 0$ are entire functions such that $\max\{\rho(A_j),\ j = 0, 1,\ \rho(F)\} < 1$.

Key words: linear differential equations, differential polynomials, entire solutions, order of growth, exponent of convergence of zeros, exponent of convergence of distinct zeros

2000 Mathematics Subject Classification: 34M10, 30D35

1 Introduction and statement of results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [12], [17]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r, \varphi) = o(T(r, f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r, f)$ is the Nevanlinna characteristic function of f.

Definition 1.1 ([12], [15]) Let f be a meromorphic function with order $0 < \rho(f) < \infty$. Then the type of f is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{T(r, f)}{r^{\rho(f)}}. \tag{1.1}$$

If f is an entire function, then the type of f is defined by

$$\sigma_M(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\rho(f)}},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Remark 1.1 There exist entire functions f which satisfy $\sigma_M(f) \neq \sigma(f)$. For example, if $f(z) = e^z$, then we have $\sigma_M(f) = 1$ and $\sigma(f) = \frac{1}{\pi}$.

To give the precise estimate of fixed points, we define:

Definition 1.2 ([16], [19]) Let f be a meromorphic function and let $z_1, z_2, \ldots$ ($|z_j| = r_j$, $0 < r_1 \leq r_2 \leq \ldots$) be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of f is defined by

$$\overline{\tau}(f) = \inf\left\{\tau > 0\colon \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\right\}.$$

Clearly,

$$\overline{\tau}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}, \tag{1.2}$$

where $\overline{N}\left(r, \frac{1}{f-z}\right)$ is the counting function of distinct fixed points of $f(z)$ in $\{z\colon |z| < r\}$.

For the second order linear differential equation

$$f'' + e^{-z} f' + B(z) f = 0, \tag{1.3}$$

where $B(z)$ is an entire function, it is well-known that each solution f of equation (1.3) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1.3), then by [9], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1.3) will have infinite order. But equation (1.3) with $B(z) = -(1 + e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of equation (1.3) has infinite order? Many authors, Frei [10], Ozawa [18], Amemiya–Ozawa [1] and Gundersen [11], Langley [14] have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$, then every solution $f \not\equiv 0$ of (1.3) has infinite order.

In 2002, Z. X. Chen [7] considered the question: What conditions on $B(z)$ when $\rho(B) = 1$ will guarantee that every nontrivial solution of (1.3) has infinite order? He proved the following results, which improved results of Frei, Amemiya–Ozawa, Ozawa, Langley and Gundersen.

Theorem A ([7]) *Let $A_j(z)$ $(\not\equiv 0)$, $j = 0,1$, and $D_j(z)$, $j = 0,1$, be entire functions with $\max\{\rho(A_j)\,(j=0,1),\ \rho(D_j)\,(j=0,1)\} < 1$, and let a, b be complex numbers that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$, $0 < c < 1$. Then every solution $f \not\equiv 0$ of the equation*

$$f'' + (D_1(z) + A_1(z)e^{az})\,f' + \big(D_0(z) + A_0(z)e^{bz}\big)\,f = 0 \tag{1.4}$$

is of infinite order.

Setting $D_j \equiv 0$, $j = 0,1$, in Theorem A, we obtain the following result.

Theorem B *Let $A_j(z)$ $(\not\equiv 0)$, $j = 0,1$, be entire functions with $\max\big\{\rho(A_j),\ j = 0,1\big\} < 1$, and let a, b be complex numbers that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$, $0 < c < 1$. Then every solution $f \not\equiv 0$ of the equation*

$$f'' + A_1(z)e^{az} f' + A_0(z)e^{bz} f = 0 \tag{1.5}$$

is of infinite order.

Theorem C ([7]) *Let $A_j(z)$ $(\not\equiv 0)$, $j = 0,1$, be entire functions with $\rho(A_j) < 1$, $j = 0,1$, and let a, b be complex numbers that satisfy $ab \neq 0$ and $a = cb$ $(c > 1)$. Then every solution $f \not\equiv 0$ of equation (1.5) is of infinite order.*

Combining Theorems B and C, we obtain the following result.

Theorem D *Let $A_j(z)$ $(\not\equiv 0)$, $j = 0,1$, be entire functions with $\rho(A_j) < 1$, $j = 0,1$, and let a, b be complex numbers that satisfy $ab \neq 0$ and $a \neq b$. Then every solution $f \not\equiv 0$ of equation (1.5) is of infinite order.*

Consider the second order non-homogeneous linear differential equation

$$f'' + A_1(z)e^{az} f' + A_0(z)e^{bz} f = F, \tag{1.6}$$

where a, b are complex numbers and $A_j(z)$ $(\not\equiv 0)$, $j = 0,1$, $F(z)$ are entire functions with $\max\{\rho(A_j),\ j = 0,1,\ \rho(F)\} < 1$. In [20], J. Wang and I. Laine have investigated the growth of solutions of (1.6) and have obtained the following result.

Theorem E ([20]) *Let $A_j(z)$ $(\not\equiv 0)$, $j = 0,1$, and $F(z)$ be entire functions with $\max\{\rho(A_j),\ j = 0,1,\ \rho(F)\} < 1$, and let a, b be complex numbers that satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution f of equation (1.6) is of infinite order.*

Remark 1.2 Independently in [4], the authors have studied equation (1.6) and have obtained the same result as in Theorem E but the proof is quite different (see [4]).

Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [21]). However, there are few studies on the fixed points of solutions of differential equations. It was in the year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of second order linear differential equations with entire coefficients (see [6]). In [19], Wang and Yi investigated fixed points and hyper order of differential polynomials generated by solutions of second order linear differential equations with meromorphic coefficients. In [13], Laine and Rieppo gave an improvement of the results of [19] by considering fixed points and iterated order. In [16], Liu and Zhang have investigated the fixed points and hyper order of solutions of some higher order linear differential equations with meromorphic coefficients and their derivatives. Recently, in [2], [3], Belaïdi gave an extension of the results of [16].

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations.

The main purpose of this paper is to study the relation between small functions and some differential polynomials generated by solutions of the second order linear differential equation (1.6). We obtain some estimates of their distinct fixed points.

Theorem 1.1 *Let $A_j(z)$ $(\not\equiv 0)$, $j = 0, 1$, and $F \not\equiv 0$ be entire functions with $\max\{\rho(A_j), j = 0, 1, \rho(F)\} < 1$, and let a, b be complex numbers that satisfy $ab(a - b) \neq 0$. Let $d_0(z), d_1(z), d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j) < 1$, $j = 0, 1, 2$, and let $\varphi(z)$ be an entire function with finite order. If f is a solution of (1.6) then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty$.*

Corollary 1.1 *Let $A_j(z)$, $j = 0, 1$, $F(z)$, $d_j(z)$, $j = 0, 1, 2$, a, b satisfy the additional hypotheses of Theorem 1.1. If f is a solution of (1.6), then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ has infinitely many fixed points and satisfies $\overline{\tau}(g_f) = \tau(g_f) = \infty$.*

Theorem 1.2 *Let $A_j(z)$, $j = 0, 1$, $F(z)$, a, b, $\varphi(z)$ satisfy the additional hypotheses of Theorem 1.1. If f is a solution of (1.6), then*

$$\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \rho(f) = +\infty. \tag{1.7}$$

Remark 1.3 If $\rho(F) \geq 1$, then equation (1.6) can possesses solution of finite order. For instance the equation

$$f'' + e^{-z} f' + e^{z} f = 1 + e^{2z}$$

satisfies $\rho(F) = \rho\left(1 + e^{2z}\right) = 1$ and has a finite order solution $f(z) = e^{z} - 1$.

Now, let us denote

$$\alpha_1 = d_1 - d_2 A_1 e^{az}, \qquad \alpha_0 = d_0 - d_2 A_0 e^{bz}, \tag{1.8}$$

$$\beta_1 = d_2 A_1^2 e^{2az} - \left((d_2A_1)' + ad_2A_1 + d_1A_1\right) e^{az} - d_2A_0e^{bz} + d_0 + d_1', \quad (1.9)$$

$$\beta_0 = d_2A_0A_1e^{(a+b)z} - ((d_2A_0) + bd_2A_0 + d_1A_0)e^{bz} + d_0', \quad (1.10)$$

$$h = \alpha_1\beta_0 - \alpha_0\beta_1, \quad (1.11)$$

$$\psi = \frac{\alpha_1\left(\varphi' - (d_2F)' - \alpha_1 F\right) - \beta_1\left(\varphi - d_2F\right)}{h}. \quad (1.12)$$

Theorem 1.3 *Let $A_j(z)$, $j = 0, 1$, $d_j(z)$, $j = 0, 1, 2$, a, b satisfy the additional hypotheses of Theorem 1.1, and $F(z)$ be an entire function such that $\rho(F) \geq 1$. Let $\varphi(z)$ be an entire function with finite order such that $\psi(z)$ is not a solution of equation (1.6). If $f(z)$ is a solution of (1.6), then the differential polynomial $g_f = d_2f'' + d_1f' + d_0f$ satisfies $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ with at most one finite order solution f_0.*

Next, we investigate the relation between small functions and differential polynomials of a pair of non-homogeneous linear differential equations and we obtain the following result.

Theorem 1.4 *Let $A_j(z)$, $j = 0, 1$, $d_j(z)$, $j = 0, 1, 2$, a, b satisfy the additional hypotheses of Theorem 1.1. Let $F_1 \not\equiv 0$ and $F_2 \not\equiv 0$ be entire functions such that $\max\{\rho(F_j), j = 1, 2\} < 1$ and $F_1 - CF_2 \not\equiv 0$ for any constant C, and let $\varphi(z)$ be an entire function with finite order. If f_1 is a solution of the equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F_1 \quad (1.13)$$

and f_2 is a solution of the equation

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F_2, \quad (1.14)$$

then the differential polynomial

$$g_{f_1 - Cf_2}(z) = d_2\left(f_1'' - Cf_2''\right) + d_1\left(f_1' - Cf_2'\right) + d_0\left(f_1 - Cf_2\right)$$

satisfies $\overline{\lambda}(g_{f_1-Cf_2} - \varphi) = \lambda(g_{f_1-Cf_2} - \varphi) = \infty$ for any constant C.

2 Preliminary lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 2.1 ([5]) *Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = F, \quad (2.1)$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 2.2 *Let f, g be meromorphic functions with orders $0 < \rho(f), \rho(g) < \infty$ and types $0 < \sigma(f), \sigma(g) < \infty$. Then the following statements hold:*
(i) If $\rho(g) < \rho(f)$, then

$$\sigma(f+g) = \sigma(fg) = \sigma(f). \tag{2.2}$$

(ii) If $\rho(f) = \rho(g)$ and $\sigma(g) \neq \sigma(f)$, then

$$\rho(f+g) = \rho(fg) = \rho(f). \tag{2.3}$$

Proof (i) By the definition of the type, we have

$$\sigma(f+g) = \limsup_{r\to+\infty} \frac{T(r, f+g)}{r^{\rho(f+g)}} \leq \limsup_{r\to+\infty} \frac{T(r,f) + T(r,g) + O(1)}{r^{\rho(f+g)}}. \tag{2.4}$$

Since $\rho(g) < \rho(f)$, then $\rho(f+g) = \rho(f)$. Thus, from (2.4), we obtain

$$\sigma(f+g) \leq \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\rho(f)}} + \limsup_{r\to+\infty} \frac{T(r,g) + O(1)}{r^{\rho(f)}} = \sigma(f). \tag{2.5}$$

Since

$$\rho(f+g) = \rho(f) > \rho(g), \tag{2.6}$$

then by (2.5)

$$\sigma(f) = \sigma(f+g-g) \leq \sigma(f+g). \tag{2.7}$$

From (2.5) and (2.7), we obtain $\sigma(f+g) = \sigma(f)$.

Now we prove $\sigma(fg) = \sigma(f)$. Since $\rho(g) < \rho(f)$, then $\rho(fg) = \rho(f)$. By the definition of the type, we have

$$\begin{aligned} \sigma(fg) &= \limsup_{r\to+\infty} \frac{T(r, fg)}{r^{\rho(fg)}} \leq \limsup_{r\to+\infty} \frac{T(r,f) + T(r,g)}{r^{\rho(f)}} \\ &\leq \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\rho(f)}} + \limsup_{r\to+\infty} \frac{T(r,g)}{r^{\rho(f)}} = \sigma(f). \end{aligned} \tag{2.8}$$

Since

$$\rho(fg) = \rho(f) > \rho(g) = \rho\left(\frac{1}{g}\right), \tag{2.9}$$

then by (2.8)

$$\sigma(f) = \sigma\left(fg\frac{1}{g}\right) \leq \sigma(fg). \tag{2.10}$$

From (2.8) and (2.10), we obtain $\sigma(fg) = \sigma(f)$.

(ii) Without lost of generality, we suppose that $\rho(f) = \rho(g)$ and $\sigma(g) < \sigma(f)$. Then, we have

$$\rho(f+g) \leq \max\{\rho(f), \rho(g)\} = \rho(f) = \rho(g). \tag{2.11}$$

If we suppose that $\rho(f+g)<\rho(f)=\rho(g)$, then by (2.2), we get

$$\sigma(g)=\sigma(f+g-f)=\sigma(f)$$

and this is a contradiction. Hence $\rho(f+g)=\rho(f)=\rho(g)$.

Now, we prove that $\rho(fg)=\rho(f)=\rho(g)$. Also we have

$$\rho(fg)\leq\max\{\rho(f),\rho(g)\}=\rho(f)=\rho(g). \tag{2.12}$$

If we suppose that $\rho(fg)<\rho(f)=\rho(g)=\rho\left(\frac{1}{f}\right)$, then by (2.2), we can write

$$\sigma(g)=\sigma\left(fg\frac{1}{f}\right)=\sigma\left(\frac{1}{f}\right)=\sigma(f)$$

and this is a contradiction. Hence $\rho(fg)=\rho(f)=\rho(g)$. □

Lemma 2.3 ([8]) *Let a,b be complex numbers such that $ab\neq 0$ and $\arg a\neq\arg b$ or $a=cb$, $0<c<1$. We denote index set by $\Lambda_1=\{0,a,b,2a,a+b\}$. If H_j, $j\in\Lambda_1$, and $H_{2b}\not\equiv 0$ are all meromorphic functions of orders that are less than 1, setting $\Psi_1(z)=\sum_{j\in\Lambda_1}H_j(z)e^{jz}$, then $\Psi_1(z)+H_{2b}e^{2bz}\not\equiv 0$.*

Lemma 2.4 *Let a,b be complex numbers that satisfy $ab\neq 0$ and $a=cb$, $c>1$. We denote index set by $\Lambda_2=\{0,a,b,a+b,2a,2b\}$. Let H_j $(j\in\Lambda_2)$ be meromorphic functions of orders that are less than 1, setting $\Psi_2(z)=\sum_{j\in\Lambda_2}H_j(z)e^{jz}$. If there exists $j\in\Lambda_2-\{0\}$ such that $H_j\not\equiv 0$ then $\Psi_2(z)\not\equiv 0$.*

Proof By Lemma 2.2, we have $\rho(\Psi_2)=1$. Hence, $\Psi_2(z)\not\equiv 0$. □

Lemma 2.5 *Let $A_j(z)$ $(\not\equiv 0)$, $j=0,1$, and $F\not\equiv 0$ be entire functions with $\max\{\rho(A_j), j=0,1, \rho(F)\}<1$, and let a,b be complex numbers that satisfy $ab(a-b)\neq 0$. Let $d_0(z),d_1(z),d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j)<1$, $j=0,1,2$. If f is a solution of equation (1.6), then the differential polynomial $g_f=d_2f''+d_1f'+d_0f$ satisfies $\rho(g_f)=\rho(f)=\infty$.*

Proof Suppose that $f(z)$ is a solution of the equation (1.6). Then by Theorem E, we have $\rho(f)=\infty$. Now, we prove $\rho(g_f)=\rho(d_2f''+d_1f'+d_0f)=\infty$.

Suppose that $\arg a\neq\arg b$ or $a=cb$, $0<c<1$. First we suppose that $d_2\not\equiv 0$. Substituting $f''=F-A_1e^{az}f'-A_0e^{bz}f$ into g_f, we get

$$g_f-d_2F=(d_1-d_2A_1e^{az})f'+(d_0-d_2A_0e^{bz})f. \tag{2.13}$$

Differentiating both sides of the equation (2.13) and replacing f'' with

$$f''=F-A_1e^{az}f'-A_0e^{bz}f,$$

we obtain

$$\begin{aligned}&g_f'-(d_2F)'-(d_1-d_2A_1e^{az})F\\&=\left[d_2A_1^2e^{2az}-\left((d_2A_1)'+ad_2A_1+d_1A_1\right)e^{az}-d_2A_0e^{bz}+d_0+d_1'\right]f'\\&\quad+\left[d_2A_0A_1e^{(a+b)z}-\left((d_2A_0)'+bd_2A_0+d_1A_0\right)e^{bz}+d_0'\right]f.\end{aligned} \tag{2.14}$$

Then by (2.13), (2.14), (1.8), (1.9) and (1.10), we have

$$\alpha_1 f' + \alpha_0 f = g_f - d_2 F, \tag{2.15}$$

$$\beta_1 f' + \beta_0 f = g_f' - (d_2 F)' - (d_1 - d_2 A_1 e^{az}) F. \tag{2.16}$$

Set

$$\begin{aligned} h &= \alpha_1 \beta_0 - \alpha_0 \beta_1 \\ &= (d_1 - d_2 A_1 e^{az}) \left[d_2 A_0 A_1 e^{(a+b)z} - ((d_2 A_0)' + b d_2 A_0 + d_1 A_0) e^{bz} + d_0' \right] \\ &- \left(d_0 - d_2 A_0 e^{bz}\right) \left[d_2 A_1^2 e^{2az} - ((d_2 A_1)' + a d_2 A_1 + d_1 A_1) e^{az} - d_2 A_0 e^{bz} + d_0 + d_1' \right]. \end{aligned} \tag{2.17}$$

Now check all the terms of h. Since the term $d_2^2 A_1^2 A_0 e^{(2a+b)z}$ is eliminated, by (2.17) we can write $h = \Psi_1(z) - d_2^2 A_0^2 e^{2bz}$, where $\Psi_1(z)$ is defined as in Lemma 2.3. By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemma 2.3, we see that $h \not\equiv 0$.

Suppose now $a = cb$, $c > 1$. By (2.17), we can write

$$h = \Psi_2(z) = H_0 + H_a e^{az} + H_b e^{bz} + H_{a+b} e^{(a+b)z} + H_{2a} e^{2az} + H_{2b} e^{2bz},$$

where $H_0, H_a, H_b, H_{a+b}, H_{2a}, H_{2b}$ are entire functions of orders less than 1. By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ we have

$$H_{2b} = -d_2^2 A_0^2 \not\equiv 0.$$

Then by Lemma 2.4, we have $h \not\equiv 0$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$. Using a similar reasoning as above we get $h \not\equiv 0$.

Finally, if $d_2 \equiv 0$, $d_1 \equiv 0$, $d_0 \not\equiv 0$, then we have $h = -d_0^2 \not\equiv 0$. Hence, $h \not\equiv 0$. By (2.15), (2.16) and (2.17), we obtain

$$f = \frac{\alpha_1 \left(g_f' - (d_2 F)' - \alpha_1 F\right) - \beta_1 (g_f - d_2 F)}{h}. \tag{2.18}$$

If $\rho(g_f) < \infty$, then by (2.18) we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho(g_f) = \infty$. □

3 Proof of Theorem 1.1

Suppose that f is a solution of equation (1.6). Then by Theorem E, we have $\rho(f) = \infty$. Set $w = d_2 f'' + d_1 f' + d_0 f - \varphi$. Since $\rho(\varphi) < \infty$, by Lemma 2.5 we have $\rho(w) = \rho(g_f) = \rho(f) = \infty$. In order to prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$, we need to prove $\overline{\lambda}(w) = \lambda(w) = \infty$. By $g_f = w + \varphi$, we get from (2.18)

$$f = \frac{\alpha_1 \left(w' + \varphi' - (d_2 F)' - \alpha_1 F\right) - \beta_1 (w + \varphi - d_2 F)}{h}. \tag{3.1}$$

So,

$$f = \frac{\alpha_1 w^{'} - \beta_1 w}{h} + \psi, \tag{3.2}$$

where

$$\psi(z) = \frac{\alpha_1 \left(\varphi' - (d_2 F)' - \alpha_1 F\right) - \beta_1 (\varphi - d_2 F)}{h}.$$

Substituting (3.2) into equation (1.6), we obtain

$$\frac{\alpha_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = F - \left(\psi'' + A_1(z) e^{az} \psi' + A_0(z) e^{bz} \psi\right) = A,$$

where ϕ_j, $j = 0, 1, 2$, are meromorphic functions with $\rho(\phi_j) < \infty$, $j = 0, 1, 2$. Since $\rho(\psi) < \infty$, by Theorem E it follows that $A \not\equiv 0$. By $\alpha_1 \not\equiv 0$, $h \not\equiv 0$ and Lemma 2.1, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, i.e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$.

4 Proof of Theorem 1.2

Suppose that f is a solution of equation (1.6). Then by Theorem E we have $\rho(f) = \rho(f') = \rho(f'') = \infty$. Since $\rho(\varphi) < \infty$, then $\rho(f - \varphi) = \rho(f' - \varphi) = \rho(f'' - \varphi) = \infty$. By using a proof similar to that of Theorem 1.1, we obtain Theorem 1.2.

5 Proof of Theorem 1.3

Assume that f_0 is a solution of (1.6) with $\rho(f_0) = \rho < \infty$. If f_1 is a second finite order solution of (1.6), then $\rho(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous equation (1.5) of (1.6), but $\rho(f_1 - f_0) = \infty$ from Theorem D. This is a contradiction. Hence (1.6) has at most one finite order solution f_0 and all other solutions f_1 of (1.6) satisfy $\rho(f_1) = \infty$. By hypothesis of Theorem 1.3, $\psi(z)$ is not a solution of equation (1.6). Then

$$F - \left(\psi'' + A_1(z) e^{az} \psi' + A_0(z) e^{bz} \psi\right) \not\equiv 0.$$

By reasoning similar to that in the proof of Theorem 1.1, we can prove Theorem 1.3.

Remark 5.1 The condition "$\psi(z)$ is not a solution of equation (1.6)" in Theorem 1.3, is necessary because if $\psi(z)$ is a solution of equation (1.6), then we have $F - \left(\psi'' + A_1(z) e^{az} \psi' + A_0(z) e^{bz} \psi\right) \equiv 0$.

6 Proof of Theorem 1.4

Suppose that f_1 is a solution of equation (1.13) and f_2 is a solution of equation (1.14). Set $w = f_1 - C f_2$. Then w is a solution of the equation

$$w'' + A_1(z) e^{az} w' + A_0(z) e^{bz} w = F_1 - C F_2.$$

By $\rho(F_1 - CF_2) < 1$, $F_1 - CF_2 \not\equiv 0$ and Theorem E, we have $\rho(w) = \infty$. Thus, by Theorem 1.1, we obtain that $\overline{\lambda}(g_{f_1 - Cf_2} - \varphi) = \lambda(g_{f_1 - Cf_2} - \varphi) = \infty$.