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Abstract. We show how the *ad hoc* prescriptions appearing in 2001 for the Lie derivative of Lorentz tensors are a direct consequence of the Kosmann lift defined earlier, in a much more general setting encompassing older results of Y. Kosmann about Lie derivatives of spinors.

1 Introduction

The geometric theory of Lie derivatives of spinor fields is an old and intriguing issue that is relevant in many contexts, among which we quote the applications in Supersymmetry (see [5], [22]) and the problem of separation of variables of Dirac equation (see [10]). It is as well essential for the understanding of the general foundations of the theory of spinor fields and, eventually, of General Relativity as a whole. We stress that despite spinor fields can be endowed with a correct physical interpretation only in a quantum framework, this quantum field theory is obtained by quantization procedures from a classical variational problem. Hence even if a classical field theory describing spinors is not endowed with a direct physical interpretation its variational issues (field equations and conserved quantities) are mathematically interesting on their own as well as they have important consequences on the corresponding quantum field theory.

The situation in Minkowski spacetime (as well as on other maximally symmetric spaces) is pretty well established and it is based on the existence of sufficiently many Killing vectors ξ. The problem of Lie derivatives arises when one wants to generalize these arguments to more general spacetimes, i.e. when Killing vectors are less than enough, or when coupling with gravity, i.e. when the metric background cannot be regarded as being fixed *a priori* but it has to be determined dynamically by field equations. A definition for Lie derivatives of spinors along generic spacetime vector fields, not necessarily Killing ones, on a general curved spacetime was already proposed in 1971 by Y. Kosmann (see [16], [17], [18], [19]) by an *ad hoc* prescription. In 1996 we and coauthors (see also [12]) provided a geometric framework which

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justifies the *ad hoc* prescription within the general framework of Lie derivatives on fiber bundles (see also [24], [23] and [2]) in the explicit context of gauge natural bundles [15] which turn out to be the most appropriate arena for (gauge-covariant) field theories [6].

The key point is the construction of the (generalized) *Kosmann lift* (so-called by us in honour of the original *ad hoc* prescription) which is induced by any spacetime frame. This lift is defined on any principal bundle Σ having the special orthogonal group as structure group in any dimension and signature. According to this prescription a spacetime vector field ξ is uniquely lifted to a bundle vector field $\hat{\xi}_\Sigma$.

This lift $\hat{\xi}_\Sigma$ on the principal bundle Σ defines in turn the Lie derivative operator on sections of *any* fiber bundle associated to Σ, where objects like spinors or spin-connections are defined as sections. Unfortunately, this Lie derivative is not *natural*, in the sense that it does not preserve the commutator unless it is restricted to Killing vectors only. However, we stress that an advantage of this framework consists in showing and definitely explaining why there cannot be and in fact there is no possible natural prescription for the Lie derivative of spinors. As a consequence, one has to choose whether to restrict artificially to Killing vectors (which is certainly physically impossible unless under extremely special conditions) or to learn how to cope with the fact that spinors are non-natural objects. The gauge natural formalism is a possible escape (see [3]). In any case unless restricting to very special situation, one has to define Lie derivatives with respect to arbitrary spacetime vector fields. Furthermore, even in special situations one can *a posteriori* restrict the vector field to be Killing one (if any exists) in order to obtain a unifying view on the matter, in which all Lie derivatives are obtained as a specialization of a general notion.

The very same framework introduced for spinors provides a suitable arena to deal with *Lorentz tensors* in GR. Similar approaches can be found in the literature (see [27]) as well as more recently (see [21]). In GR there are many objects which are endowed with specific transformation rules with respect to Lorentz transformations, even though, of course, in GR these transformations cannot be implemented in general by a subgroup of the whole group of all diffeomorphisms. Let us mention e.g. tetrads and spin connections in a Cartan framework, where pointwise Lorentz transformations act as a gauge group. This framework is also the kinematical arena to define the self-dual formulation of GR that is the starting point of LQG approach.

We shall here review the general theory of Lorentz tensors and their Lie derivative and compare with the direct and *ad hoc* method based on Killing vectors appeared in [22]. The key issue consists in recognizing that Lorentz tensors are, *by definition*, sections of some bundle associated to a suitable principal bundle Σ by means of the appropriate tensorial representation of the appropriate special orthogonal structure group.

2 The Kosmann lift

Let M be a m-dimensional manifold (which will be required to allow global metrics of signature $\eta = (r, s)$, with $m = r + s$). Let us denote by x^μ local coordinates on M, which induce a basis ∂_μ of tangent spaces; let $L(M)$ denote the *general* *frame bundle* of M and set (x^μ, V^μ_a) for fibered coordinates on $L(M)$. We can define a right-invariant basis for vertical vectors on $L(M)$

$$\rho^\mu_\nu = V^\mu_a \frac{\partial}{\partial V^\nu_a}$$

The general frame bundle is natural (see [15]), hence any spacetime vector field $\xi = \xi^\mu \partial_\mu$ defines a natural lift on $L(M)$

$$\hat{\xi} = \xi^\mu \, \partial_\mu + \partial_\mu \xi^\nu \, \rho^\mu_\nu$$

We stress that the lift vector field $\hat{\xi}$ is global whenever ξ is global.

A connection on $L(M)$ is denoted by $\Gamma^\alpha_{\beta\mu}$ and it defines a lift

$$\Gamma : TM \to TL(M) : \xi^\mu \partial_\mu \mapsto \xi^\mu \left(\partial_\mu - \Gamma^\alpha_{\beta\mu} \rho^\beta_\alpha\right)$$

This lift does not in general preserve commutators, unless the connection is flat.

Ordinary tensors are sections of bundles associated to $L(M)$. The connection $\Gamma^\alpha_{\beta\mu}$ induces connections on associated bundles and defines in turn the covariant derivatives of ordinary tensors.

Example 1. For example, tensors of rank $(1,1)$ are sections of the bundle $T^1_1(M)$ associated to $L(M)$ using the appropriate tensor representations, namely

$$\lambda : \mathrm{GL}(m) \times V \to V : (J^\mu_\nu, t^\mu_\nu) \mapsto t'^\mu_\nu = J^\mu_\alpha t^\alpha_\beta \bar{J}^\beta_\nu$$

where the bar denotes the inverse in $\mathrm{GL}(n, \mathbb{R})$.

The connection Γ on $L(M)$ induces on this associated bundle the connection

$$T^1_1(\Gamma) = dx^\mu \otimes \left(\partial_\mu - \left(\Gamma^\alpha_{\gamma\mu} t^\gamma_\beta - \Gamma^\gamma_{\beta\mu} t^\alpha_\gamma\right) \frac{\partial}{\partial t^\alpha_\beta}\right)$$

which in turn defines the standard covariant derivative of such tensors:

$$\nabla_\xi t = Tt(\xi) - T^1_1(\Gamma)(\xi) = \xi^\mu \left(d_\mu t^\alpha_\beta + \Gamma^\alpha_{\gamma\mu} t^\gamma_\beta - \Gamma^\gamma_{\beta\mu} t^\alpha_\gamma\right) \frac{\partial}{\partial t^\alpha_\beta}$$

If a metric $g = g_{\mu\nu} \, dx^\mu \otimes dx^\nu$ is given on M then its Christoffel symbols define the Levi-Civita connection of the metric. Such a connection is torsionless (i.e. symmetric in lower indices) and *compatible with the metric*, i.e. such that $\nabla_\mu g_{\alpha\beta} = 0$.

Let now $(\Sigma, M, \pi, \mathrm{SO}(\eta))$ be a principal bundle over the manifold M and let (x^μ, S^a_b) be (overdetermined) fibered “coordinates” on the principal bundle Σ. We can define a right-invariant pointwise basis σ_{ab} for vertical vectors on Σ by setting

$$\sigma_{ab} = \eta_{d[a} \rho^d_{b]} \qquad\qquad \rho^d_b = S^d_c \frac{\partial}{\partial S^b_c}$$

where η_{ab} is the canonical diagonal matrix of signature $\eta = (r, s)$ and square brackets denote skew-symmetrization over indices.

A connection on Σ is in the form

$$\omega = dx^\mu \otimes \left(\partial_\mu - \omega^{ab}_\mu \sigma_{ab}\right)$$

Also in this case the connection on Σ induces connections on any associated bundle and there defines covariant derivatives of sections.

A *frame* is a bundle map $e : \Sigma \to L(M)$ which preserves the right action, i.e. such

$$\begin{array}{ccc} \Sigma & \xrightarrow{e} & L(M) \\ \downarrow & & \downarrow \\ M & = & M \end{array} \qquad\qquad \begin{array}{ccc} \Sigma & \xrightarrow{e} & L(M) \\ \downarrow R_S & & \downarrow R_{i(S)} \\ \Sigma & \xrightarrow{e} & L(M) \end{array}$$

i.e. $e \circ R_S = R_{i(S)} \circ e$, where R denotes the relevant canonical right actions defined on the principal bundles Σ and $L(M)$ and where $i : \mathrm{SO}(\eta) \to \mathrm{GL}(m)$ is the canonical group inclusion. We stress that on any M which allows global metrics of signature η the bundle Σ can always be chosen so that there exist global frames; see [7]. Locally the frame is represented by invertible matrices e^μ_a and it defines a spacetime metric $g_{\mu\nu} = e^\mu_a \, \eta_{ab} \, e^\nu_b$ which is called the *induced metric*.

As for the Levi-Civita connection, a frame defines a connection on Σ (called the spin-connection of the frame) given by

$$\omega^{ab}_\mu = e^a_\alpha \left(\Gamma^\alpha_{\beta\mu} e^{b\beta} + d_\mu e^{b\alpha}\right) \tag{1}$$

where $\Gamma^\alpha_{\beta\mu}$ denote Christoffel symbols of the induced metric. The spin-connection is compatible with the frame in the sense that

$$\nabla_\mu e^\nu_a = d_\mu e^\nu_a + \Gamma^\nu_{\lambda\mu} e^\lambda_a - \omega^c{}_{a\mu} e^\nu_c \equiv 0$$

In general the (natural) lift $\hat{\xi}$ of a spacetime vector field ξ to $L(M)$ is not adapted to the image $e(\Sigma) \subset L(M)$ and thence it does not define any vector field on Σ. With this notation the Kosmann lift of $\xi = \xi^\mu \partial_\mu$ is defined by $\hat{\xi}_K = \xi^\mu \partial_\mu + \hat{\xi}^{ab} \sigma_{ab}$ (see [4]) where we set:

$$\hat{\xi}^{ab} = e^{[a}_\nu \nabla_\mu \xi^\nu e^{b]\mu} - \omega^{ab}_\mu \xi^\mu \tag{2}$$

and where $e^{a\mu} = \eta^{ac} e^\mu_c$ and e^b_ν denote the inverse frame matrix.

Let us stress that despite appearing so, the Kosmann lift (2) does not in fact depend on the connection, but just on the frame and its first derivatives. The same lift can be written as $\hat{\xi}^{ab} = \nabla^{[b}\xi^{a]} - \omega^{ab}_\mu \xi^\mu$ where we set $\xi^a = \xi^\mu e^a_\mu$ since one can prove that

$$\nabla_b \xi^a = e^a_\nu \nabla_\mu \xi^\nu e^\mu_b$$

Another useful equivalent expression for the Kosmann lift is giving the vertical part of the lift with respect to the spin connection (see [6], pages 288–290), namely

$$\hat{\xi}^{ab}_{(V)} := \hat{\xi}^{ab} + \omega^{ab}_\mu \xi^\mu = e^{[a}_\nu \nabla_\mu \xi^\nu e^{b]\mu} = \nabla^{[b}\xi^{a]} \tag{3}$$

This last expression is useful since it expresses a manifestly covariant quantity.

We have to stress that the Kosmann lift does not preserve commutators. In fact if one considers two spacetime vectors ξ and ζ and computes the Kosmann lift of the commutator $[\xi,\zeta]$ one can easily prove that

$$[\xi,\zeta]\hat{}_K = [\hat{\xi}_K, \hat{\zeta}_K] + \frac{1}{2} e^a_\alpha \pounds_\zeta g^{\alpha\lambda} \pounds_\xi g_{\lambda\beta} e^{b\beta} \sigma_{ab}$$

Thence only if one restricts to Killing vectors (i.e. $\pounds_\xi g = 0$) one recovers that the lift preserves commutators.

3 The Lie Derivative of Lorentz Tensors

Let λ be a representation (of rank (p,q)) of $\mathrm{SO}(\eta)$ over a suitable vector space V. Let E_A be a basis of V so that a point $t \in V$ is given by $t = t^A E_A$ and $\lambda(J,t) = \lambda^A_B(J) t^B$.

Example 2. For example, if $V = T^1_1(\mathbb{R}^m) \sim \mathbb{R}^m \otimes \mathbb{R}^m$ with coordinates t^a_b we may have

$$\lambda : \mathrm{SO}(\eta) \times V \to V : (J,t) \mapsto J^a_c t^c_d \bar{J}^d_b$$

the bar denoting now the inverse in $\mathrm{SO}(\eta)$. This is the tensor representation of rank $(1,1)$.

Then, by definition, a *Lorentz tensor* is a section of the bundle $\Sigma_\lambda = \Sigma \times_\lambda V$ associated to λ through the representation λ. Fibered coordinates on Σ_λ are in the form (x^μ, t^A) and transition functions of Σ act on Σ_λ through the representation λ.

If we consider a global infinitesimal generator of automorphisms over Σ (also called a *Lorentz transformation*) locally expressed as

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^{ab}(x)\sigma_{ab}$$

(which projects over the spacetime vector field $\xi = \xi^\mu\partial_\mu$) this induces a global vector field over Σ_λ locally given by

$$\Xi_\lambda = \xi^\mu(x)\partial_\mu + \xi^A \frac{\partial}{\partial t_A} \qquad \xi^A = \xi^{ab}\partial_{ab}\lambda^A_B(\mathbb{I})t^B$$

Let us remark that this vector field is linear in ξ.

Example 3. For example, if λ is the tensor representation of rank $(1,1)$ given above, then the induced vector field is

$$\Xi_\lambda = \xi^\mu\partial_\mu + \left(\xi^{a\cdot}{}_{c}\, t^c_b - t^a_d\, \xi^{d\cdot}{}_{b}\right) \frac{\partial}{\partial t^a_b}$$

where indices are lowered and raised by η_{ab}.

According to the general framework for Lie derivatives (see [24]) for a section $t : M \to \Sigma_\lambda : x^\mu \mapsto (x, t^A(x))$ of the bundle Σ_λ with respect to the (infinitesimal) Lorentz tranformation Ξ, we find

$$\pounds_\Xi t = Tt(\xi) - \Xi_\lambda \circ t = \left(\xi^\mu d_\mu t^A - \xi^{ab}\partial_{ab}\lambda^A_B(\mathbb{I})t^B\right) \frac{\partial}{\partial t^A} \tag{4}$$

Example 4. For example, if λ is the tensor representation of rank $(1,1)$ given above the Lie derivative of a section reads as

$$\pounds_\Xi t = \left(\xi^\mu d_\mu t^a_b - \xi^a{}_c^{\cdot}\, t^c_b + t^a_d\, \xi^d{}_b^{\cdot}\right)\frac{\partial}{\partial t^a_b} = \left(\xi^\mu \nabla_\mu t^a_b - (\xi_{(V)})^a{}_c^{\cdot}\, t^c_b + t^a_d\, (\xi_{(V)})^d{}_b^{\cdot}\right)\frac{\partial}{\partial t^a_b}$$

where $(\xi_{(V)})^a{}_c^{\cdot} = \xi^a{}_c^{\cdot} + \omega^a{}_{c\mu}\xi^\mu$ denotes the vertical part of Ξ with respect to the same connection used for the covariant derivative $\nabla_\mu t^a_b = d_\mu t^a_b + \omega^a{}_{c\mu} t^c_b - \omega^c{}_{b\mu} t^a_c$. Let us stress that in spite of its convenient connection-dependent expressions the Lie derivative does not eventually depend on any connection (as it may seem from our second expression).

Notice that this definition of Lie derivatives is natural, i.e. it preserves commutators, namely

$$[\pounds_{\Xi_1}, \pounds_{\Xi_2}]\sigma = \pounds_{[\Xi_1,\Xi_2]}\sigma \tag{5}$$

Unfortunately, Lorentz tranformations as introduced above have nothing to do with coordinate transformations (or spacetime diffeomorphisms). They have been introduced as gauge transformations acting pointwise and completely unrelated to spacetime diffeomorphisms. Indeed the Lie derivative (4) can be performed with respect to bundle vector fields Ξ instead of spacetime vector fields and this is completely counterintuitive if compared with what expected for spacetime objects like, for example, spinors. These objects are in fact expected to react to spacetime transformations; on the other hand, on a general spacetime there is nothing like Lorentz transformations.

We shall hence define Lie derivatives of Lorentz tensors with respect to any spacetime vector field and then show that in Minkowski spacetime, where Lorentz trasformations are defined, these reproduce and extend the standard notion. The price to be paid is loosing naturality like (5) (which will be retained only for Killing vectors if Killing vectors exist on M).

Let us restrict to vector fields $\hat{\xi}_K$ of Σ which are the Kosmann lift of a spacetime vector field ξ and define the Lie derivative of the Lorentz tensor t with respect to the spacetime vector field ξ to be

$$\pounds_\xi t \equiv \pounds_{\hat{\xi}_K} t = (\xi^\mu d_\mu t^A - \hat{\xi}^{ab}\partial_{ab}\lambda^A_B(\mathbb{I}) t^B)\frac{\partial}{\partial t^A}$$

where $\hat{\xi}^{ab}$ is expressed in terms of the derivatives of ξ^μ (and the frame) as in (2).

Example 5. For example, for Lorentz tensors of rank $(1,1)$ we have

$$\begin{aligned}
\pounds_\xi t \equiv \pounds_{\hat{\xi}} t &= \left(\xi^\mu d_\mu t^a_b - \hat{\xi}^a{}_c^{\cdot}\, t^c_b + t^a_d\, \hat{\xi}^d{}_b^{\cdot}\right)\frac{\partial}{\partial t^a_b} = \\
&= \left(\xi^\mu \nabla_\mu t^a_b - (\hat{\xi}_{(V)})^a{}_c^{\cdot}\, t^c_b + t^a_d\, (\hat{\xi}_{(V)})^d{}_b^{\cdot}\right)\frac{\partial}{\partial t^a_b} = \\
&= \left(\xi^\mu \nabla_\mu t^a_b - \nabla_c \xi^a\, t^c_b + t^a_d\, \nabla_b \xi^d\right)\frac{\partial}{\partial t^a_b} = \\
&= \left(\nabla_d\left(\xi^d t^a_b\right) - \nabla_c \xi^a\, t^c_b\right)\frac{\partial}{\partial t^a_b}
\end{aligned}$$

For a generic Lorentz tensor of any rank, similar terms arise one for each Lorentz index.

Now since the Kosmann lift on Σ does not preserve commutators these Lie derivatives are not natural unless one artificially restricts ξ to be a Killing vector (of course provided M allows Killing vectors!). In fact, one has generically

$$\pounds_{[\xi,\zeta]}t \equiv \pounds_{[\xi,\zeta]\hat{}_K}t \neq \pounds_{[\hat{\xi}_K,\hat{\zeta}_K]}t = [\pounds_{\hat{\xi}_K}, \pounds_{\hat{\zeta}_K}]t \equiv [\pounds_\xi, \pounds_\zeta]t$$

Example 6. One can try to specialize this to simple cases in order to make non-naturality manifest. For example, if one considers a Lorentz vector v^a and two spacetime vector fields ξ and ζ one can easily check that

$$\pounds_{[\xi,\zeta]}v^a = [\pounds_\xi, \pounds_\zeta]v^a + \frac{1}{4}\left(v^\alpha g^{\beta\rho}e^{a\sigma} - v^\rho g^{\beta\sigma}e^{a\alpha}\right)\pounds_\xi g_{\rho\sigma}\pounds_\zeta g_{\alpha\beta}$$

Let us remark that according to this expression when ξ or ζ are Killing vectors of the metric g commutators are preserved. Moreover, the extra term does not vanish in general.

Of course, there are degenerate cases (e.g. setting $\xi = \zeta$) in which the extra terms vanishes due to coefficients without requiring Killing vectors. However, in this case also the other terms vanish.

4 Properties of Lie Derivatives of Lorentz Tensors

We shall prove here two important properties of Lie derivatives as defined above (see, for example, [11], [14], [25], [26] and references quoted therein)

For the Lie derivative of a frame one has

$$\pounds_\xi e^a_\mu = \xi^\lambda \nabla_\lambda e^a_\mu - \nabla_\mu \xi^\lambda e^a_\lambda + (\hat{\xi}_{(V)})^a_b e^b_\mu$$

If we are using, as we can always choose to do, the spin and the Levi-Civita connections for the relevant covariant derivatives, then $\nabla_\lambda e^a_\mu = 0$. By using the Kosmann lift (3) one easily obtains

$$\begin{aligned}\pounds_\xi e^a_\mu &= -\nabla_\mu \xi^\lambda e^a_\lambda + \nabla^{[b}\xi^{a]} e_{b\mu} = -\nabla_\mu \xi^\lambda e^a_\lambda + \nabla_{[\mu}\xi_{\lambda]} e^{a\lambda} = -\nabla_{(\mu}\xi_{\lambda)} e^{a\lambda} = \\ &= \frac{1}{2}\pounds_\xi g_{\mu\lambda} e^{a\lambda}\end{aligned}$$

This expression holds true for any spacetime vector ξ and of course it proves that the Lie derivative vanishes along Killing vectors.

Let us stress that this last expression, obtained here from the general prescription for the Lie derivative of Lorentz tensors, is trivial in view of the expression on the induced metric as a function of the frame; in fact,

$$\frac{1}{2}\pounds_\xi g_{\mu\lambda} e^{a\lambda} = \pounds_\xi e^c_\mu e_{c\lambda} e^{a\lambda} = \pounds_\xi e^a_\mu$$

For the second property we wish to prove let us first notice that the frame induces an isomorphism between TM (on which one considers (x^μ, v^μ) as fibered coordinates) and the bundle of Lorentz vectors $\Sigma \times_\lambda \mathbb{R}^m$ (on which (x^μ, v^a) are considered as fibered coordinates) by

$$\Phi : TM \to \Sigma \times_\lambda \mathbb{R}^m : v^\mu \mapsto v^a = e^a_\mu v^\mu$$

We can thence express the Lie derivative of a section v of $\Sigma \times_\lambda \mathbb{R}^m$ (i.e. a Lorentz vector) in terms of the Lie derivative of the corresponding section of TM. In fact one has:

$$\begin{aligned}\pounds_\xi v^a &= \xi^\mu \nabla_\mu v^a - (\hat{\xi}_{(V)})^a_b v^b = \xi^b \nabla_b v^a - \nabla^{[b}\xi^{a]} v_b = \pounds_\xi v^\mu e^a_\mu + \nabla^{(b}\xi^{a)} v_b = \\ &= \pounds_\xi v^\mu e^a_\mu - \frac{1}{2} e^a_\mu \pounds_\xi g^{\mu\nu} e^b_\nu v_b = \pounds_\xi v^\mu e^a_\mu + \pounds_\xi e^a_\mu v^\mu \end{aligned} \tag{6}$$

Let us stress that these two properties hold true for any spacetime vector field ξ and they specialize to the ones discussed in [22] for Killing vectors.

The origin and meaning of the Lie derivative (6) can be easily understood: one has to take into account that if one drags ξ^a along a vector field the overall change of the object receives a contribution from how the vector changes but also a contribution from how the frame changes.

Similar properties can be easily found for Lorentz tensors of any rank since the frame transforms ordinary tensors into Lorentz tensors; e.g. one has

$$\Phi : t^\mu_\nu \mapsto t^a_b = e^a_\alpha t^\alpha_\beta e^\beta_b$$

5 Transformation of Lorentz Vectors in Minkowski Spacetime

Let us consider Minkowski spacetime $M = \mathbb{R}^4$ with the metric η; being it contractible any bundle over it is trivial. As a consequence we are forced to choose $\Sigma = \mathbb{R}^4 \times \mathrm{SO}(3,1)$. Since $M \equiv \mathbb{R}^4$ is parallelizable, its frame bundle is trivial, i.e. $L(\mathbb{R}^4) = \mathbb{R}^4 \times \mathrm{GL}(4)$. Let us fix Cartesian coordinates x^μ on $M \equiv \mathbb{R}^4$ and let us fix a frame $e_a = \delta^\mu_a \partial_\mu$; such a frame induces the Minkowski metric $\eta_{\mu\nu}$.

In such notation the Levi-Civita connection vanishes, $\Gamma^\alpha_{\beta\mu} = 0$ and the spin connection too, $\omega^{ab}_\mu = 0$; the Kosman lift hence specializes to

$$\hat{\xi}^{ab}_{(V)} = e^{[b\beta} \nabla_\beta \xi^\alpha e^{a]}_\alpha$$

Let us now consider a vector field ξ the flow which is made of Lorentz coordinate tranformations $x'^\mu = \Lambda^\mu_\nu x^\nu$; since ξ is of course a Killing vector, then the Lie derivative of a Lorentz vector is

$$\pounds_\xi v^a = \pounds_\xi v^\mu e^a_\mu = \left(\xi^\alpha \partial_\alpha v^\mu - v^\alpha \dot{\Lambda}^\mu_\alpha\right) \delta^a_\mu \tag{7}$$

Such a Lie derivative corresponds to the trasformation rules

$$v'^a = \Lambda^a_b v^b \tag{8}$$

which is exactly as a vector is expected to trasform under a Lorentz coordinate transformation.

A similar result can be easily extended to covectors, tensors and, with slight though obvious changes, to spinors. When ξ is not Killing, however, the Lie derivative may not be the infinitesimal counterpart of a finite transformation rule as in (7) and (8); in this case the traditional interpretation of Lie derivatives as a measure of changing of objects dragged along spacetime vector fields fails to hold true. One should however wonder whether such an interpretation is really fundamental to many common uses of Lie derivatives. Our answer is in the negative as one can argue by a detailed analysis of physical quantities containing Lie derivatives.

Lie derivatives appear, e.g., in Noether theorem; in this case they appear naturally as a by-product of variational techniques. Here Noether currents turn out to be expressed in terms of Lie derivatives expressed as in equation (4). The interpretation of such Lie derivatives as measuring infinitesimal changes along symmetry transformations is important since, based on that, one can relate Noether currents to symmetries.

Now the essential point is that there is no reason to expect spacetime vector fields to be the most general (infinitesimal) symmetries in Physics. Fundamentally speaking, symmetries encode the observers' freedom to set their conventions to describe Physical world. While coordinates are certainly necessary conventions for any observer (and hence general covariance principle is a fundamental symmetry that should be expected in any physical system), special systems might need further conventions which might result in independent class of symmetries (as it happens in gauge theories, e.g. electromagnetism).

Of course, since these further conventions are independent of spacetime coordinate fixing, gauge transformations cannot be expressed as spacetime diffeomorphisms, but they are expressed as field transformations. As such they are vector fields on the configuration bundle, not on spacetime. It is hence reasonable and important to have a notion of Lie derivative of fields along bundle vectors, as in (4). It is only in GR where symmetries come from spacetime vector fields that one should expect Lie derivatives along spacetime vector fields and their interpretation as quantities related to the spacetime geometry.

This more general situation, i.e. when the quantities entering Noether theorem are interpreted as Lie derivatives of fields along bundle vectors, can be simply discussed by considering a very well-known physical situation, i.e. covariant electromagnetism.

The electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature of a field A_μ which is usually known as a *quadripotential* and, as it is well known, is a connection on a principal bundle P for the group $U(1)$. This is the standard gauge approach to electromagnetism. The Maxwell Lagrangian is

$$L_M = -\frac{1}{4}\sqrt{g} F_{\mu\nu} F^{\mu\nu}_{\cdot\cdot} \tag{9}$$

By variation one obtains

$$\delta L_M = -\frac{1}{2}\sqrt{g} H_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\mu\left(\sqrt{g}F^{\mu\nu}\right)\delta A_\nu - \nabla_\mu\left(\sqrt{g}F^{\mu\nu}\delta A_\nu\right) \tag{10}$$

where we set $H_{\alpha\beta} = F_{\mu\alpha}F^{\mu}_{\cdot\ \beta} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}_{\cdot\cdot} g_{\mu\nu}$ for the standard energy-momentum tensor of the electromagnetic field. The second term in (10) produces Maxwell equations, namely $\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\right)=0$. The third term relates to conservation laws (see [6]).

The Lagrangian (9) is covariant with respect to the infinitesimal transformations

$$\Xi=\xi^\mu\frac{\partial}{\partial x^\mu}+2\partial_\alpha\xi^\mu g^{\alpha\nu}\frac{\partial}{\partial g^{\mu\nu}}+\left(\partial_\mu\xi-\partial_\mu\xi^\nu A_\nu\right)\frac{\partial}{\partial A_\mu}$$

which correspond to 1-parameter families of gauge transformations

$$\begin{cases} x'^\mu = x'^\mu_{(\epsilon)}(x) \\ g'^{\mu\nu} = \dfrac{\partial x'^\mu_{(\epsilon)}}{\partial x'^\alpha} g^{\alpha\beta}\dfrac{\partial x'^\nu_{(\epsilon)}}{\partial x'^\beta} \\ A'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu_{(\epsilon)}}\left(A_\nu+\partial_\nu\alpha_{(\epsilon)}\right) \end{cases}$$

Here the generator ξ^μ is related to the coordinate change $x'^\mu=x'^\mu_{(\epsilon)}(x)$ while the generator ξ is related to the gauge transformation $\alpha_{(\epsilon)}$.

Let us remark that Ξ is a vector field on the configuration bundle (that is a manifold with coordinates $(x^\mu, g^{\mu\nu}, A_\mu)$), not on spacetime. In a general situation (namely unless the principal bundle P is assumed to be trivial) there is no way of either lifting a spacetime vector field to the configuration bundle or globally setting $\xi=0$ so to split the vector Ξ into a spacetime vector and a *"gauge generator"*. In a physical language one usually says that the condition $\xi=0$ is not gauge covariant and hence local, unless there exist global gauges. (By the way, also when global gauges exist, the condition is not gauge covariant and hence unphysical, from a fundamental viewpoint.)

The Lie derivative of the field A_μ along the symmetry generator Ξ is in this case (see (4))

$$\pounds_\Xi A_\mu=\xi^\lambda F_{\lambda\mu}-\nabla_\mu\left(\xi-\xi^\lambda A_\lambda\right)$$

Noether theorem in this case shows (see again [6]) on-shell conservation of the following Noether current

$$\mathcal{E}^\mu=-\sqrt{g}\left(F^{\mu\nu}\pounds_\Xi A_\nu+\xi^\mu L_M\right)$$

In the special case when $\xi^\mu=0$ one has

$$\mathcal{E}^\mu=\sqrt{g}\left(F^{\mu\nu}\nabla_\mu\xi\right)=\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\xi\right)-\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\right)\xi$$

The second term vanishes on-shell, thus one obtains

$$\mathcal{E}^\mu=\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\xi\right)$$

The corresponding conserved quantity is

$$Q(\xi)=\frac{1}{2}\int_{\partial\Omega}\sqrt{g}F^{\mu\nu}\xi\, ds_{\mu\nu}$$

where $ds_{\mu\nu}$ is the area element on the boundary of the 3-region Ω of spacetime. This is the electric charge defined *à la* Gauss.

This example shows clearly what happens in general when gauge transformations are allowed and symmetry generators live at bundle level: also in this case Noether theorem involves Lie derivatives, though in the generalized sense introduced above. In this case we are not dealing with Lorentz objects so one cannot introduce Kosmann lift (or similar lifts) and reduce everything to spacetime vector fields.

6 Applications

In order to provide an example of concrete aplication of our formalism here introduced in action we shall here consider the application to the so called *Holst's action principle* (see [13]) which is used as an equivalent formulation of GR suitable for developing LQG through the use of the Barbero-Immirzi connection (see [1], [20], [8], [9] as well as references quoted therein).

Let us first consider tetrad-affine formulation of GR: the fundamental fields are a Lorentz connection Γ^{ab}_μ and a vielbein $e^a = e^a_\mu\, dx^\mu$. The connection defines the curvature form $R^{ab} = \frac{1}{2} R^{ab}{}_{\mu\nu} dx^\mu \wedge dx^\nu$. Let us also set $e = \det|e^a_\mu|$, $R^a{}_\mu = R^{ab}{}_{\mu\nu} e^\nu_b$ and $R = R^{ab}{}_{\mu\nu} e^\mu_a e^\nu_b$; here e^ν_b denotes the inverse frame matrix of e^b_ν. The frame also defines a metric $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$ which in turn defines its Levi-Civita spacetime connection $\Gamma^\alpha_{\beta\mu}$.

On a spacetime of dimension 4, let us consider the Lagrangian

$$L_{tA} = R^{ab} \wedge e^c \wedge e^d\, \epsilon_{abcd}$$

By variation we obtain

$$\begin{aligned}\delta L_{tA} = &-2e e^\sigma_a \left(R^a{}_\mu - \frac{1}{2} R e^a_\mu \right) e^\mu_d\, \delta e^d_\mu - \epsilon_{abcd} \nabla_\mu \left(e^c_\rho e^d_\sigma \right) \epsilon^{\mu\nu\rho\sigma} \delta\Gamma^{ab}_\mu + \\ &+ \epsilon_{abcd} \nabla_\mu \left(e^c_\rho e^d_\sigma \delta\Gamma^{ab}_\mu \right) \epsilon^{\mu\nu\rho\sigma}\end{aligned}$$

Thus one obtains field equations

$$\begin{cases} R^a{}_\mu - \dfrac{1}{2} R e^a_\mu = 0 \\ \nabla_{[\mu} \left(e^{[c}_\rho e^{d]}_{\sigma]} \right) = 0 \end{cases}$$

The second field equation forces the connection to be the connection induced by the frame $\Gamma^{ab}_\mu = \omega^{ab}_\mu$ (see eq. (1)); then the first equation forces the induced metric to obey Einstein equations.

This field theory is dynamically equivalent to standard GR, in the sense that it obeys equivalent field equations. However, the theory is in fact richer in its physical interpretation, since the use of different variables and action principles generate larger symmetry and extra conservation laws. In fact, this theory has a bigger symmetry group being generally covariant and Lorentz covariant.

Noether theorem implies then conservation of the current

$$\mathcal{E}^\mu = 4e e^\mu_a e^\nu_b \pounds_\Xi \Gamma^{ab}_\nu - \xi^\mu L_{tA}$$

along any Lorentz gauge generator $\Xi = \xi^\mu \partial_\mu + \xi^{ab}\sigma_{ab}$. The Lie derivative of a connection is given by

$$\pounds_\Xi \Gamma^{ab}_\nu = \xi^\lambda R^{ab}{}_{\lambda\nu} + \nabla_\nu \hat{\xi}^{ab}$$

where we set $\hat{\xi}^{ab} = \xi^{ab} + \xi^\lambda \Gamma^{ab}_\lambda$.

Hence one obtains

$$\mathcal{E}^\mu = 4ee^\mu_a \left(R^a{}_\mu - \frac{1}{2} R e^a_\mu \right) \xi^\lambda - 4\nabla_\nu \left(ee^\mu_a e^\nu_b \right) \hat{\xi}^{ab} + 4\nabla_\nu \left(ee^\mu_a e^\nu_b \hat{\xi}^{ab} \right)$$

The first and second terms vanish on-shell; hence one obtains

$$\mathcal{E}^\mu = 4\nabla_\nu \left(ee^\mu_a e^\nu_b \hat{\xi}^{ab} \right) \tag{11}$$

Let us stress that this current depends only on the Lorentz generator $\hat{\xi}^{ab}$.

Here is the issue with physical interpretation: we have two equivalent formulations of Einstein GR where Noether currents in one case depend on spacetime vector fields while in tetrad-affine formulation Noether currents depend on Lorentz generator which *a priori* has nothing to do with spacetime transformations. Let us stress of course that unless the spacetime is Minkowski, there is no class of spacetime diffeomorphisms representing *Lorentz transformations*.

Considering the dynamical equivalence at level of field equations and solution space, one would like this equivalence to be extended at level of conservation laws. Moreover, some of the conserved quantities in standard GR are known to be related to physical quantities such as energy, momentum and angular momentum, while one would wish to be able to identify the corresponding quantities in the second formulation. Kosmann lift is in fact essential to relate Lorentz generators to spacetime diffeomorphisms and the corresponding conservation laws.

The Noether current (11) can be restricted setting $\Xi = \hat{\xi}_K$ so that one obtains

$$\mathcal{E}^\mu_{tA} = 4\nabla_\nu \left(e \nabla^\mu \xi^\nu \right)$$

which corresponds to the standard conserved quantity associated to spacetime diffeomorphisms in GR written in terms of Komar superpotential. This (and only this) restores the equivalence between standard GR and tetrad-affine formulation at level of conservation laws.

As a further example let us consider the covariant Lagrangian:

$$L_H = L_{tA} + \beta R^{ab} \wedge e_a \wedge e_b$$

which is known as Holst's Lagrangian.

By variations one obtains equations

$$\begin{cases} e^\mu_d \left(R^a_\mu - \frac{1}{2} R e^a_\mu \right) e^\sigma_a - \beta R_{d\rho\mu\nu} \epsilon^{\mu\nu\rho\sigma} = 0 \\ \nabla_{[\mu} \left(e^{[c}_\rho e^{d]}_{\sigma]} \right) = 0 \end{cases}$$

The second equation still imposes $\Gamma^{ab}_{\mu} = \omega^{ab}_{\mu}$; this in turns implies $R^a{}_{[\rho\mu\nu]} = 0$ (first Bianchi identity) and hence Einstein equations. This shows how also Holst's Lagrangian provides an equivalent formulation of standard GR.

It is interesting to check if also in this case the equivalence is preserved also at level of conservation laws. The Noether current is

$$\mathcal{E}^{\mu}_{H} = 4ee^{\mu}_{a}e^{\nu}_{b}\pounds_{\Xi}\Gamma^{ab}_{\nu} + ee^{\mu}_{c}e^{\nu}_{d}\epsilon^{cd\cdot\cdot}{}_{ab}\pounds_{\Xi}\Gamma^{ab}_{\nu} - \xi^{\mu}L_{H}$$

As in the previous case this can be recasted modulo terms vanishing on-shell as follows

$$\mathcal{E}^{\mu}_{H} - \mathcal{E}^{\mu}_{tA} = \nabla_{\nu}\left(ee^{\mu}_{c}e^{\nu}_{d}\epsilon^{cd\cdot\cdot}{}_{ab}\hat{\xi}^{ab}\right)$$

Again this has nothing to do with spacetimes symmetries and in general would affect conserved quantities. When Kosmann lift is again inserted into these conservation laws one obtains

$$\mathcal{E}^{\mu}_{H} - \mathcal{E}^{\mu}_{tA} = \nabla_{\nu}\left(\nabla^{\rho}\xi^{\sigma}\epsilon^{\mu\nu}{}_{\rho\sigma}\right)$$

which vanishes being the divergence of a divergence. Hence once again the correspondence at level of conservation laws is preserved when the Kosmann lift is used.

7 Conclusion

We presented a framework to deal with Lorentz objects and showed how it applies to tetrad-affine formulation and Holst's formulation of GR. In particular we showed that equivalence can be extended at the level of conservation laws if one introduces the Kosmann lift which establishes a correspondence among symmetry generators in different formulations.

One could argue whether the Lie derivatives defined above could be physically interpreted in a correct way. Of course, one could always restrict to situations in which enough Killing vectors exist (or even to Minkowski spacetime $(\mathbb{R}^4, \eta)$); in these cases the standard results are obtained in particular.

However, in a generic spacetime (M, g) one has no Killing vectors and at the end one has to decide whether a physical interpretation of these objects along generic spacetime vector field makes any sense.

The framerwork we introduced for Lorentz tensors provides a rigorous way of investigating formal properties which in our opinion are the only necessary basis for a physical intepretation of Lie derivatives of Lorentz tensors themselves.

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