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Eisa Mahmoudi; Hojatollah Zakerzadeh

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AN ADMISSIBLE ESTIMATOR OF A LOWER-BOUNDED SCALE PARAMETER UNDER SQUARED-LOG ERROR LOSS FUNCTION

Eisa Mahmoudi and Hojatollah Zakerzadeh

Estimation in truncated parameter space is one of the most important features in statistical inference, because the frequently used criterion of unbiasedness is useless, since no unbiased estimator exists in general. So, other optimally criteria such as admissibility and minimaxity have to be looked for among others. In this paper we consider a subclass of the exponential families of distributions. Bayes estimator of a lower-bounded scale parameter, under the squared-log error loss function with a sequence of boundary supported priors is obtained. An admissible estimator of a lower-bounded scale parameter, which is the limiting Bayes estimator, is given. Also another class of estimators of a lower-bounded scale parameter, which is called the truncated linear estimators, is considered and several interesting properties of the estimators in this class are studied. Some comparisons of the estimators in this class with an admissible estimator of a lower-bounded scale parameter are presented.

Keywords: admissibility, Bayes estimator, truncated parameter spaces, squared-log error loss

Classification: 62C10, 62C15, 62C20

1. INTRODUCTION

The problems of inference in truncated parameter spaces occur e. g. in randomized response models (Moors, [9]), in regression problems when the regression function is assumed to be monotone without further assumptions on its shape (Robertson et al., [15]), in estimators for regression problems where the regression function is linear with a slope of known sign (Moors and van Houwelingen, [10]), and in question of robustness of Bayesian methods and in group-Bayes analysis (van Eeden and Zidek, [21, 22]).

The study of truncated parameter spaces is of interest for the following reasons: (i) They often occur in practice. In many cases certain parameter values can be excluded from the parameter space. Nearly all problems in practice have a truncated parameter space and it is almost impossible to argue in practice that a parameter is not bounded. (ii) In truncated parameter space, the commonly used estimators of θ such as the maximum likelihood estimators are inadmissible. Even more characteristic is the fact that boundary rules are mostly inadmissible, where a boundary estimator is an estimator which takes, with positive probability for some $\theta \in \Theta$, values on or near the boundary of Θ. (iii) In truncated parameter space, the frequently used criterion of unbiasedness is useless, since no unbiased estimator exists in general. So, other optimality criteria have to be looked for, such as admissibility, minimaxity and invariance among others.

Estimation in truncated parameter space is one of the most important features in statistical inference. As mentioned above, most of the frequently used estimators such as the MLE are inadmissible in truncated parameter space. One way to avoid this difficulties is using other criteria of optimality, such as minimaxity. Minimaxity of estimators is not only of own interest, but it serves also as a useful benchmark to measure the performance of estimators.

Minimaxity and admissibility results for lower-bounded parameters can be found in Katz [7], Berry [1] and van Eeden [19, 20]. Using squared error loss, Katz [7] gives an admissible minimax estimator of a normal mean μ where $\mu \in [a, \infty)$, and Berry [1] obtains an admissible minimax estimator for the case of the exponential distribution with support $[\theta, \infty)$, where $\theta \in (-\infty, a]$, $\theta \in [b, \infty)$ or $\theta \in [a, b]$. Using scale-invariant squared error loss, van Eeden [19] gives an admissible minimax estimator of the scale parameter θ of a gamma distribution with known shape parameter where $\theta \in [a, \infty)$, and van Eeden [20] find a minimax estimator of the scale parameter of an F-distribution when the parameter is lower bounded.

Jafari Jozani et al. [6] extended the results of van Eeden [19]. They obtained an admissible minimax estimator of a bounded scale parameter in a subclass of the exponential family under scale-invariant squared error loss. Also, they studied the admissibility and minimaxity in the family of transformed chi-square distributions due to Rahman and Gupta [14].

Assume the asymmetric squared-log error loss of the form

$$L(\theta, \delta) = (\ln \delta - \ln \theta)^2 = \left(\ln \frac{\delta}{\theta}\right)^2, \tag{1}$$

which was introduced by Brown [3].

The main reasons for choosing the squared-log error loss function in this study is that (i) unlike the squared error loss function, the squared-log error loss is more useful to bring statistical model nearer to practical situations. (ii) This is balanced loss function in the sense that $\lim L(\theta, \delta) = \infty$ as $\delta \to 0$ or $\delta \to \infty$, while the squared error loss function is out of balance for scale parameter. (iii) The squared-log error loss is not always convex, it is convex for $\delta/\theta \le e$ and concave otherwise, but its risk function has unique minimum with respect to δ. (iv) The squared-log error loss penalizes over estimation more heavily than under estimation. (v) In the problem of estimating an unknown scale parameter the best invariant estimator for squared error loss is biased, see Pitman [12]. (vi) For each given value of the scale parameter σ the squared error loss function gives equal importance to equal differences between δ (estimator of σ) and σ while squared-log error loss function weigh equal proportional errors equally. Although the squared error loss function is relatively easy to handle analytically it seems somewhat inappropriate for a broad range of applications because of the above disturbing features of this loss function in the scale parameter problems.

In this paper an admissible estimator of a lower-bounded scale parameter, under the squared-log error loss function, is obtained. The rest of the paper is as follow: In Section 2, under the squared-log error loss, we derive the Bayes estimator of the lower-bounded scale parameter with respect to a sequence of boundary supported priors in a subclass of the exponential families of distributions. Section 3 is concerned with the admissible estimator of a lower-bounded scale parameter, which is the limiting Bayes estimator. In the exponential family of distributions a class of reasonable estimators of the scale parameter in the classical (untruncated) problem are the linear estimators which arise as (proper or generalized) Bayes estimators. Hence it is natural to consider a truncated version of such linear estimators. In Section 4, another class of estimators of a lower-bounded scale parameter, which is called the truncated linear estimators, is considered and several interesting properties of the estimators in this class is studied. Some comparisons of the estimators in this class with an admissible estimator of a lower-bounded scale parameter, introduced in Section 3, are presented in the end of this section.

2. BAYES ESTIMATION OF A LOWER-BOUNDED SCALE PARAMETER

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m from distribution with density $(1/\tau)\, g\,(x/\tau)$, where g is known and τ is an unknown scale parameter. The joint density of $X_1, X_2, \ldots, X_m$ is denoted by $f\,(\mathbf{x};\tau) = \frac{1}{\tau^m} f\left(\frac{\mathbf{x}}{\tau}\right)$. In some cases this joint density reduces to

$$f(\mathbf{x};\eta) = c(\mathbf{x}, m)\eta^{-\nu} e^{-T(\mathbf{x})/\eta}, \tag{2}$$

where $c(\mathbf{x}, m)$ is a function of $\mathbf{x} = (x_1, \ldots, x_m)$ and m, $\eta = \tau^r$ for some r, ν is a function of m and $T(\mathbf{x})$ is a complete sufficient statistic for η with gamma(ν, η) distribution. Examples of such models are

(i) Gamma(α, β) distribution with α known,

$$\eta = \beta,\ \ \nu = m\alpha,\ \ T(\mathbf{X}) = \textstyle\sum_{i=1}^m X_i, \quad c(\mathbf{x}, m) = \prod_{i=1}^m \frac{x_i^{\alpha-1}}{\Gamma(\alpha)}.$$

(ii) $\mathrm{N}(0, \sigma^2)$ distribution with

$$\eta = \sigma^2,\ \ \nu = m/2,\ \ T(\mathbf{X}) = \tfrac{1}{2}\textstyle\sum_{i=1}^m X_i^2, \quad c(\mathbf{x}, m) = (2\pi)^{-m/2}.$$

Some distributions belonging to this subclass of the exponential family are listed by Jafari Jozani et al. [6]. This class contains many distributions such as Exponential, Gamma, Weibull, Pareto, Normal, Lognormal, Rayleigh, Maxwell and Inverse Gaussian. Some properties of this family of distributions and an admissible linear estimator of $\eta = \tau^r$ in this family, under the entropy loss function, can be found in Parsian and Nematollahi [11].

Sanjari Farsipour and Zakerzadeh [16] studied estimation of the scale parameter in the family of distributions (2). They derived the explicit form of minimum risk scale-equivariant estimator under the squared-log error loss function. Also, the admissibility and inadmissibility of a class of linear estimators of T has been considered.

In this section, the Bayes estimator of η and it's admissibility, in the family of distributions (2), w.r.t. a sequence of proper prior distributions, is given when η is restricted to $\eta \in [a, +\infty)$.

The random variable T has gamma(ν, η) distribution (ν is a function of m) with density function

$$f_T(t|\eta) = \frac{t^{\nu-1} e^{-\frac{t}{\eta}}}{\Gamma(\nu)\,\eta^{\nu}} \qquad t > 0, \quad \eta \geq a. \tag{3}$$

According to van Eeden [19], a sequence of proper prior distributions for η has the form

$$\pi_n(\eta) = \frac{a^{1/n}}{n\eta^{1+1/n}} \qquad \eta \geq a, \quad a > 0. \tag{4}$$

The estimator of η, δ say, considered in this paper is defined as

$$\delta(\mathbf{x}) = \lim_{n\to\infty} \delta_n(\mathbf{x}),$$

where, for $n = 1, 2, \ldots$, δ_n is the Bayes estimator of η with respect to prior density is given in (4), and δ satisfies

$$P_\eta(\delta(\mathbf{X}) \geq a) = 1, \qquad \text{for all } \eta \geq a.$$

It is easy to verify that the posterior distribution of η is given by

$$\pi_n(\eta|T) = \frac{T^{\nu+1/n} e^{-T/\eta}}{\eta^{\nu+1/n+1}\Gamma_{T/a}(\nu + 1/n)} \qquad \eta \geq a,$$

where $\Gamma_x(\alpha) = \int_0^x y^{\alpha-1} e^{-y} \mathrm{d}y$, is the incomplete gamma function. Consider the incomplete digamma function $\Psi_x(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha} \ln \Gamma_x(\alpha)$. Then we have the following theorem.

Theorem 2.1. For the family of distributions (3), the loss function (1), and sequence of prior distributions (4), the Bayes estimator of η, is given by

$$\delta_n(\mathbf{x}) = T(\mathbf{x}) \exp\left[-\Psi_{T(\mathbf{x})/a}(\nu + 1/n)\right]. \tag{5}$$

Further,

$$r_n(\delta_n) < \infty \qquad \text{for all } n = 1, 2, \ldots.$$

Proof. We know that $L(\delta, \eta) = \left[\ln\left(\frac{\delta}{\eta}\right)\right]^2$ and so

$$R(\delta_n, \eta) = E_{T|\eta}\left[\ln\left(\frac{\delta_n}{\eta}\right)\right]^2 = E_{T|\eta}\left[\ln \delta_n - \ln \eta\right]^2.$$

The Bayes estimator of η, which is the value of δ_n that minimizes

$$\int_a^\infty \left[\ln \delta_n - \ln \eta\right]^2 \pi_n(\eta|T)\, \mathrm{d}\eta,$$

is given by $\delta_n = \exp\left[E_{\eta|T}\left[\ln \eta|T\right]\right]$. Differentiating both sides of

$$\Gamma_{bT}(\alpha) = \int_0^{bT} y^{\alpha-1} e^{-y} \mathrm{d}y = \int_0^{b} T^{\alpha} y^{\alpha-1} e^{-yT} \mathrm{d}y, \tag{6}$$

w.r.t. α, and taking $\eta^{-1} = y$, $\alpha = \nu + 1/n$ gives

$$E\left[\ln \eta|T\right] = \int_0^b \ln y \frac{y^{\alpha-1} T^{\alpha} e^{-yT}}{\Gamma_{bT}(\alpha)} \mathrm{d}y = \ln T - \Psi_{bT}(\nu + 1/n),$$

where ν as a function of m is free from n. Therefore

$$\delta_n = T \exp\left[-\Psi_{T/a}(\nu + 1/n)\right],$$

is the Bayes estimator of η under loss (1) and prior distributions (4).

The risk of δ_n, as an estimator of η, under the loss function (1) is given by

$$R(\delta_n, \eta) = \int_0^{\infty} \left(\ln \frac{t}{\eta e^{\Psi_{t/a}(\nu+1/n)}}\right)^2 \frac{t^{\nu-1} e^{-t/\eta}}{\eta^{\nu} \Gamma(v)} \mathrm{d}t.$$

By transforming $\lambda = 1/\eta$ and $b = 1/a$, the Bayes risk of δ_n is given by

$$\begin{aligned} r_n(\delta_n) = & \int_0^b \int_0^{\infty} \left[\ln^2 \lambda t + \Psi_{bt}^2(\nu + 1/n) - 2\Psi_{bt}(\nu + 1/n) \ln \lambda t\right] \\ & \times \tfrac{t^{\nu-1} e^{-\lambda t} \lambda^{\nu}}{\Gamma(\nu)} \tfrac{\lambda^{1/n-1}}{n b^{1/n}} \mathrm{d}t \mathrm{d}\lambda = A_1 + A_2 + A_3, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A_1 &= \int_0^b \int_0^{\infty} \left[\ln \lambda t\right]^2 \frac{t^{\nu-1} e^{-\lambda t} \lambda^{\nu}}{\Gamma(\nu)} \frac{\lambda^{1/n-1}}{n b^{1/n}} \mathrm{d}t \mathrm{d}\lambda \\ &= \int_0^{\infty} \frac{1}{n b^{1/n} \Gamma(\nu) t^{1/n+1}} \left[\int_0^b \ln^2(\lambda t) t^{\nu+1/n} \lambda^{\nu+1/n-1} e^{-\lambda t} \mathrm{d}\lambda\right] \mathrm{d}t \\ &= \int_0^{\infty} \frac{1}{n b^{1/n} \Gamma(\nu) t^{1/n+1}} \Gamma_{bt}^{''}(\nu + 1/n) \mathrm{d}t, \end{aligned} \tag{8}$$

$$\begin{aligned} A_2 &= \int_0^b \int_0^{\infty} \Psi_{bt}^2(\nu + 1/n) \frac{t^{\nu-1} e^{-\lambda t} \lambda^{\nu}}{\Gamma(\nu)} \frac{\lambda^{1/n-1}}{n b^{1/n}} \mathrm{d}t \mathrm{d}\lambda \\ &= \int_0^{\infty} \frac{\Psi_{bt}^2(\nu + 1/n)}{n b^{1/n} \Gamma(\nu) t^{1/n+1}} \left[\int_0^b t^{\nu+1/n} \lambda^{\nu+1/n-1} e^{-\lambda t} \mathrm{d}\lambda\right] \mathrm{d}t \\ &= \int_0^{\infty} \frac{\Psi_{bt}^2(\nu + 1/n)}{n b^{1/n} \Gamma(\nu) t^{1/n+1}} \Gamma_{bt}(\nu + 1/n) \mathrm{d}t \\ &= \int_0^{\infty} \frac{1}{n \Gamma(\nu) b^{1/n} t^{1/n+1}} \left[\frac{(\Gamma_{bt}'(\nu + 1/n))^2}{\Gamma_{bt}(\nu + 1/n)}\right] \mathrm{d}t, \end{aligned} \tag{9}$$

and

$$\begin{aligned} A_3 &= \int_0^b \int_0^\infty -2\Psi_{bt}(\nu+1/n)\ln\lambda t \frac{t^{\nu-1}e^{-\lambda t}\lambda^\nu}{\Gamma(\nu)}\frac{\lambda^{1/n-1}}{nb^{1/n}}\mathrm{d}t\mathrm{d}\lambda \\ &= -2\int_0^\infty \frac{\Psi_{bt}(\nu+1/n)}{nb^{1/n}\Gamma(\nu)t^{1/n+1}}\int_0^b \ln(\lambda t)t^{\nu+1/n}\lambda^{\nu+1/n-1}e^{-\lambda t}\mathrm{d}\lambda\mathrm{d}t \\ &= -2\int_0^\infty \frac{\Psi_{bt}(\nu+1/n)}{nb^{1/n}\Gamma(\nu)t^{1/n+1}}\Gamma'_{bt}(\nu+1/n)\,\mathrm{d}t \\ &= -2\int_0^\infty \frac{1}{nb^{1/n}\Gamma(\nu)t^{1/n+1}}\left[\frac{\left(\Gamma'_{bt}(\nu+1/n)\right)^2}{\Gamma_{bt}(\nu+1/n)}\right]\mathrm{d}t. \end{aligned} \tag{10}$$

Note that

$$\begin{aligned} \Gamma_{bt}(\alpha) &= \int_0^b t^\alpha y^{\alpha-1}e^{-yt}\mathrm{d}y, \\ \Gamma'_{bt}(\alpha) &= \int_0^b (\ln yt)\,t^\alpha y^{\alpha-1}e^{-yt}\mathrm{d}y, \\ \Gamma''_{bt}(\alpha) &= \int_0^b (\ln yt)^2\,t^\alpha y^{\alpha-1}e^{-yt}\mathrm{d}y. \end{aligned}$$

Substituting Equations (8) – (10) into (7) gives

$$\begin{aligned} r_n(\delta_n) &= \int_0^\infty \frac{1}{nb^{1/n}\Gamma(\nu)t^{1/n+1}}\Big\{\Gamma''_{bt}(\nu+1/n) \\ &\quad + \tfrac{(\Gamma'_{bt}(\nu+1/n))^2}{\Gamma_{bt}(\nu+1/n)} - 2\tfrac{\left(\Gamma'_{bt}(\nu+1/n)\right)^2}{\Gamma_{bt}(\nu+1/n)}\Big\}\mathrm{d}t \\ &= \int_0^\infty \frac{1}{nb^{1/n}\Gamma(\nu)t^{1/n+1}}\left[\Gamma''_{bt}(\nu+1/n) - \frac{(\Gamma'_{bt}(\nu+1/n))^2}{\Gamma_{bt}(\nu+1/n)}\right]\mathrm{d}t \\ &\le \int_0^\infty \frac{1}{nb^{1/n}\Gamma(\nu)t^{1/n+1}}\Gamma''_{bt}(\nu+1/n)\,\mathrm{d}t \\ &= \int_0^\infty \frac{1}{nb^{1/n}\Gamma(\nu)t^{1/n+1}}\int_0^b \ln^2(yt)t^{\nu+1/n}y^{\nu+1/n-1}e^{-yt}\mathrm{d}y\mathrm{d}t \\ &= \int_0^b \frac{y^{1/n-1}}{nb^{1/n}\Gamma(\nu)}\left[\int_0^\infty \ln^2(yt)t^{\nu-1}y^\nu e^{-yt}\mathrm{d}t\right]\mathrm{d}y \\ &= \int_0^b \frac{y^{1/n-1}}{nb^{1/n}\Gamma(\nu)}\Gamma''(\nu)\,\mathrm{d}y = \frac{\Gamma''(\nu)}{\Gamma(\nu)}, \end{aligned}$$

which is finite for all positive ν and for all $n = 1, 2, \ldots$, since ν as a function of m is free from n. Thus the proof is completed. □

Remark 2.2. The Bayes estimator (5) is admissible.

P r o o f. According to Theorem 2.1, $0 \le r_n(\delta_n) \le \frac{\Gamma''(\nu)}{\Gamma(\nu)}$. Thus the Bayes risk is finite and independent of n and η, since ν as a function of m, is completely free from n. Then the risk function $R(\delta_n, \eta)$ is finite. For proof the admissibility of δ_n, suppose that δ_n is inadmissible then there exists an estimator d^* such that

$$\begin{aligned} R(\eta, d^*) &\le R(\eta, \delta_n) \qquad \text{for all } \eta \ge a, \\ R(\eta, d^*) &< R(\eta, \delta_n) \qquad \text{for some } \eta \ge a. \end{aligned}$$

Therefore there exists an $\varepsilon > 0$, and an interval $(\eta', \eta'') \subset [a, \infty)$ such that for $a \le \eta' < \eta < \eta''$

$$R(\eta, \delta_n) - R(\eta, d^*) > \varepsilon,$$

which implies

$$\begin{aligned} [r_n(\delta_n) - r_n(d^*)] \quad &> \int_{\eta'}^{\eta''} [R(\eta, \delta_n) - R(\eta, d^*)]\, \pi_n(\eta) \mathrm{d}\eta \\ &> \varepsilon \int_{\eta'}^{\eta''} \pi_n(\eta) \mathrm{d}\eta > 0, \end{aligned}$$

which contradicts the fact that δ_n is Bayes estimator with respect to $\pi_n(\eta)$. Hence δ_n is an admissible and the proof is completed. □

The estimator $\delta(\mathbf{x})$, of $\eta(\ge a)$ considered in the next section is the limit of the Bayes estimators $\delta_n(\mathbf{x})$ of $\eta(\ge a)$ in Equation (5), i. e.,

$$\delta(\mathbf{x}) = \lim_{n \to \infty} \delta_n(\mathbf{x}) = T(\mathbf{x}) \exp\left[-\Psi_{T(\mathbf{x})/a}(\nu)\right], \tag{11}$$

where $\mathbf{x} = (x_1, \ldots, x_m)$ and ν is only a function of m. Also in the next section we show the admissibility of the limiting Bayes estimator $\delta(\mathbf{x})$ and prove that it coincides with the Pitman estimator of η.

3. ADMISSIBILITY OF THE LIMITING BAYES ESTIMATOR

This section is devoted to the admissibility of the limiting Bayes estimator $\delta = \lim_{n \to \infty} \delta_n$, under squared-log error loss function. Also we show that the limiting Bayes estimator δ coincides with the Pitman estimator δ_P for η, where the Pitman estimator can be found by transforming the distribution of T into a location parameter. For more details about the Pitman estimator, one can refer to Stein [18], Hoaglin [5], Pitman [13] and Lehmann and Casella [8]. The foregoing problem is identical to estimating a location parameter $\eta' = \ln \eta$, for the distribution $T' = \ln T$ when the loss function is

$$L'(\eta', \delta') = L(e^{\eta'}, e^{\delta'}) = L(e^{\delta' - \eta'}) = (\delta' - \eta')^2.$$

If we define $L'(x) = L(e^x)$, then $L'(\eta', \delta') = (\delta' - \eta')^2$. Note that the risk of δ, as an estimator of η, under squared-log error loss function L in (1) is $R(\delta, \eta) =$

$E(\ln\delta - \ln\eta)^2$, while the risk of δ', as an estimator of η', under squared error loss function has the form $R(\delta', \eta') = E(\delta' - \eta')^2$. Estimating $\eta' = \ln\eta$ by $\delta' = \ln\delta$ under squared error loss is equivalent to estimate η by δ under squared-log error loss function. Therefore if δ_* is better than δ_{**} under squared-log error loss L, then $\delta'_* = \ln\,\delta_*$ is better than $\delta'_{**} = \ln\,\delta_{**}$ under squared error loss L'.

Remark 3.1. $\delta' = \ln\delta$ is an admissible (inadmissible) estimator of $\eta' = \ln\eta$ under squared error loss function if and only if δ is an admissible (inadmissible) estimator of η under squared-log error loss function.

Lemma 3.2. The Pitman estimator δ_P of $\eta(\geq a)$ is the same as the limiting Bayes estimator δ in (11).

Proof. Suppose that $T' = \ln T$. Since T has gamma(ν, η) distribution, the distribution of T' is given by

$$g_{T'}(t') = \frac{e^{\nu(t'-\eta')}\exp(-e^{t'-\eta'})}{\Gamma(\nu)} = g(t' - \eta') \qquad t' \in R, \quad \eta' \geq \ln a = c,$$

and so, η' is a location parameter. The Pitman estimator of η' under the squared error loss is

$$\delta'_P = \frac{\displaystyle\int_{-\infty}^{\infty} \eta' g(t' - \eta') d\eta'}{\displaystyle\int_{-\infty}^{\infty} g(t' - \eta') \mathrm{d}\eta'} = \frac{\displaystyle\int_{c}^{\infty} \eta' e^{\nu(t'-\eta')}\exp(-e^{t'-\eta'})\mathrm{d}\eta'}{\displaystyle\int_{c}^{\infty} e^{\nu(t'-\eta')}\exp(-e^{t'-\eta'})\mathrm{d}\eta'}.$$

By transforming $y = e^{t'-\eta'}$, we have

$$\delta'_P = \frac{\displaystyle\int_0^{e^{t'-c}} (x - \ln y)\, y^{\nu-1}\exp(-y)\mathrm{d}y}{\displaystyle\int_0^{e^{t'-c}} y^{\nu-1}\exp(-y)\mathrm{d}y} = t' - \frac{\Gamma'_{e^{t'-c}}(\nu)}{\Gamma_{e^{t'-c}}(\nu)} = t' - \Psi_{e^{t'-c}}(\nu),$$

therefore substituting $t' = \ln t$, $c = \ln a$ and $\delta'_P = \ln\delta_P$, gives

$$\delta_P(T) = \frac{T}{\exp(\Psi_{\frac{T}{a}}(\nu))}.$$

Note that the Pitman estimator $\delta_P(T)$ is the same as the limiting Bayes estimator δ in (11). The proof is completed. □

For admissibility of the limiting Bayes estimator $\delta(T) = T(\mathbf{x})\exp\left[-\Psi_{T(\mathbf{x})/a}(\nu)\right]$, using limiting Bayes argument, due to Blyth (1951), [see Lehmann and Casella, [8], p. 265], we have the following theorem.

Theorem 3.3. The limiting Bayes estimator $\delta(T)$ is admissible under squared-log error loss function, for every $\eta \geq a$ and $\nu > 0$.

P r o o f. First note that $R(\eta, \delta)$ is continuous function in η, for each δ and each $\eta \geq a$ for which $R(\eta, \delta) < \infty$, [see, Ferguson, [4], Theorem 2, p. 139]. Now suppose that the limiting Bayes estimator δ is inadmissible, then there exists an estimator δ'' such that

$$\begin{aligned} R(\eta, \delta'') &\leq R(\eta, \delta) \qquad \text{for all } \eta \geq a, \\ R(\eta, \delta'') &< R(\eta, \delta) \qquad \text{for some } \eta \geq a. \end{aligned}$$

One can find an $\varepsilon > 0$, and an interval $(\eta_1, \eta_2) \subset [a, \infty)$ such that for $a \leq \eta_1 < \eta < \eta_2$

$$R(\eta, \delta) - R(\eta, \delta'') > \varepsilon, \tag{12}$$

which implies

$$\begin{aligned} n\left[r_n(\delta) - r_n(\delta'')\right] &> n \int_{\eta_1}^{\eta_2} \left[R(\eta, \delta) - R(\eta, \delta'')\right] \pi_n(\eta) \mathrm{d}\eta \\ &> \varepsilon n \int_{\eta_1}^{\eta_2} \frac{a^{1/n}}{n\eta^{1+1/n}} \mathrm{d}\eta \\ &= \varepsilon n a^{1/n} (\eta_1^{-1/n} - \eta_2^{-1/n}). \end{aligned} \tag{13}$$

Therefore $\lim_{n\to\infty} n\left[r_n(\delta) - r_n(\delta'')\right] > 0$, due to l'Hôpital's rule. Now, if it can be shown that

$$\lim_{n\to\infty} n\left[r_n(\delta) - r_n(\delta_n)\right] = 0, \tag{14}$$

where δ_n is given in (5), then from (13)

$$\frac{r_n(\delta) - r_n(\delta'')}{r_n(\delta) - r_n(\delta_n)} \to \infty, \qquad \text{as} \quad \text{n} \to \infty,$$

which implies that for sufficiently large n

$$r_n(\delta) - r_n(\delta'') > r_n(\delta) - r_n(\delta_n). \tag{15}$$

But (15) contradicts the fact that δ_n is the Bayes estimator of η with respect to the prior π_n in (4), for each $n = 1, 2, \ldots$. Thus the limiting Bayes estimator $\delta(T)$ is admissible.

For the proof of (14) we have

$$\begin{aligned}
&r_n(\delta) - r_n(\delta_n)\\
&= E_\eta(R(\delta,\eta)) - E_\eta(R(\delta_n,\eta))\\
&= E_\eta[E_{T|\eta}(L(\delta,\eta))] - E_\eta[E_{T|\eta}(L(\delta_n,\eta))]\\
&= E_\eta[E_{T|\eta}(\ln \tfrac{T}{\eta e^{\Psi_{T/a}(\nu)}})^2] - E_\eta\left[E_{T|\eta}(\ln \tfrac{T}{\eta e^{\Psi_{T/a}(\nu+1/n)}})^2\right]\\
&= E_\eta\{E_{T|\eta}[\left(\ln \tfrac{T}{\eta}\right)^2 - 2\left(\ln \tfrac{T}{\eta}\right)\Psi_{T/a}(\nu) + \Psi^2_{T/a}(\nu)\\
&- \left(\ln \tfrac{T}{\eta}\right)^2 + 2\left(\ln \tfrac{T}{\eta}\right)\Psi_{T/a}(\nu+1/n) + \Psi^2_{T/a}(\nu+1/n)]\}\\
&= E_T\left[E_{\eta|T}\Big\{\left(\Psi_{T/a}(\nu+1/n) - \Psi_{T/a}(\nu)\right)(2\left(\ln \tfrac{T}{\eta}\right) - \Psi_{T/a}(\nu+1/n) - \Psi_{T/a}(\nu))\Big\}\right]\\
&= E_T\Big\{\left(\Psi_{T/a}(\nu+1/n) - \Psi_{T/a}(\nu)\right)(2E_{\eta|T}\left(\ln \tfrac{T}{\eta}\right) - \Psi_{T/a}(\nu+1/n) - \Psi_{T/a}(\nu))\Big\},
\end{aligned}$$

where

$$\begin{aligned}
E_{\eta|T}\left(\ln \tfrac{T}{\eta}\right) &= \int_0^\infty \left(\ln \frac{T}{\eta}\right) \frac{T^{\nu+1/n} e^{-T/n}}{\eta^{\nu+1/n+1}\Gamma_{T/a}(\nu+1/n)} \mathrm{d}\eta\\
&= \int_0^{T/a} (\ln y) \frac{y^{\nu+1/n-1} e^{-y}}{\Gamma_{T/a}(\nu+1/n)} \mathrm{d}y\\
&= \tfrac{\Gamma'_{T/a}(\nu+1/n)}{\Gamma_{T/a}(\nu+1/n)}\\
&= \Psi_{T/a}(\nu+1/n).
\end{aligned}$$

So

$$\begin{aligned}
r_n(\delta) - r_n(\delta_n) &= E_T\left[\left(\Psi_{T/a}(\nu+1/n) - \Psi_{T/a}(\nu)\right)^2\right]\\
&= \int_0^\infty \left(\Psi_{T/a}(\nu+1/n) - \Psi_{T/a}(\nu)\right)^2 \frac{a^{1/n}\Gamma_{T/a}(\nu+1/n)}{n\Gamma(\nu)t^{1/n+1}} \mathrm{d}t.
\end{aligned}$$

Then

$$n(r_n(\delta) - r_n(\delta_n)) = \int_0^\infty \left(\Psi_{T/a}(\nu+1/n) - \Psi_{T/a}(\nu)\right)^2 \frac{a^{1/n}\Gamma_{T/a}(\nu+1/n)}{\Gamma(\nu)t^{1/n+1}} \mathrm{d}t.$$

Note that $\Gamma_{T/a}(\nu+1/n)$ and $\Gamma'_{T/a}(\nu+1/n)$ tends to $\Gamma_{T/a}(\nu)$ and $\Gamma'_{T/a}(\nu)$, respectively as $n \to \infty$, since ν is only a function of m (sample size) and is completely free from n. Therefore Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} [n(r_n(\delta) - r_n(\delta_n))] = 0,$$

or

$$r_n(\delta) - r_n(\delta_n) = o(\frac{1}{n}).$$

The proof is completed. □

Minimaxity of the limiting Bayes estimator δ in (11) as an estimator of $\eta(\geq a)$ under squared-log error loss function is equivalent to the minimaxity of the estimator $\delta' = \ln\delta = -\Psi_{T/a}(\nu)\ln T$, as an estimator of $\eta' = \ln\eta(\geq \ln a)$ under squared error loss function.

In the next section, according to the linear estimation problem of the scale parameter in the classical (untruncated) case, we introduce the class of truncated linear estimators of $\eta \geq a$ and compare the estimators in this class with the admissible estimator of the lower-bounded scale parameter obtained in Section 3.

4. THE CLASS OF TRUNCATED LINEAR ESTIMATORS

It often happens in truncated problems that the most natural estimator to consider first is a truncated version of a classical estimator in the original problem. The MLE in the truncated normal problem is one such example. In exponential families a class of reasonable estimators of the expectation parameter in the untruncated problem are the linear estimators which arise as (proper or generalized) Bayes estimators for conjugate families. Hence it is natural to consider a truncated version of such linear estimators.

A theme which runs throughout much of the literature on such truncated procedures is that while they improve on the untruncated estimator, they are themselves inadmissible because they are not generalized Bayes. For example, in estimating the lower bounded scale parameter in exponential and gamma distributions, van Eeden and Zidek [22, 23], van Eeden [19] and Shao and Strawderman [17] consider the class of truncated linear estimators and show the inadmissibility of the estimators in this class. Also, they obtained the minimax estimators in this class.

In this section we consider the class of truncated linear estimators for $\eta(\geq a)$, which is given by

$$\mathcal{C} = \{\hat{\eta}_c | \hat{\eta}_c = \max(cT, a), \quad c > 0\}. \tag{16}$$

This class is studied by van Eeden and Zidek [21, 22] for the case when the random variable T has an F distribution and by van Eeden [19] for gamma distribution, under the scale-invariant squared error loss function. In this paper, under squared-log error loss function, we establish several interesting properties of the estimators in $\mathcal{C}$. It is shown that each estimator $\hat{\eta}_c$ in class $\mathcal{C}$ is inadmissible. Also it is shown that the subclass $\mathcal{C}'$ of $\mathcal{C}$ defined by

$$\mathcal{C}' = \{\hat{\eta}_c \in \mathcal{C} |\ 0 < c \leq c^*\}, \tag{17}$$

where $c^* = \exp(-\Psi(\nu))$, consists of all those estimators which are admissible with respect to $\mathcal{C}$. On the other hand each estimator in class $\mathcal{C} - \mathcal{C}'$ is dominated by the estimators in class $\mathcal{C}'$. In the end of this section we compare the admissible estimator of $\eta(\geq a)$, introduced in Section 3, and the truncated linear estimators introduced in this section.

The change of loss function in this paper helped us to find better estimators with some optimal properties and can increase our subjective information about the statistical problem.

In this section two lemmas, needed for the proof of Theorems 4.3 and 4.4, will be given as well as the proofs of these theorems.

Lemma 4.1. For each $c > 0$ and $\eta \geq a$ we have

$$\frac{\mathrm{d}}{\mathrm{d}c} R(\hat{\eta}_c, \eta) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \Longleftrightarrow c \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \exp \left\{ -\frac{\displaystyle\int_{\frac{a}{c\eta}}^{\infty} (\ln y) y^{\nu-1} e^{-y} dy}{\displaystyle\int_{\frac{a}{c\eta}}^{\infty} y^{\nu-1} e^{-y} \mathrm{d}y} \right\}.$$

P r o o f. The risk function of $\hat{\eta}_c$ is given by

$$\begin{aligned} R(\hat{\eta}_c, \eta) &= E\left[\ln \tfrac{\hat{\eta}_c}{\eta}\right]^2 = \int_0^{\frac{a}{c}} (\ln \frac{a}{\eta})^2 f(t|\eta) \mathrm{d}t + \int_{\frac{a}{c}}^{\infty} (\ln \frac{ct}{\eta})^2 f(t|\eta) \mathrm{d}t \\ &= (\ln \tfrac{a}{\eta})^2 + \int_{\frac{a}{c}}^{\infty} \left[(\ln \frac{ct}{\eta})^2 - (\ln \frac{a}{\eta})^2 \right] f(t|\eta) \mathrm{d}t. \end{aligned} \tag{18}$$

Differentiating both sides of (18) w.r.t. c gives

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}c} R(\hat{\eta}_c, \eta) &= \tfrac{2}{c} \int_{\frac{a}{c}}^{\infty} (\ln \frac{ct}{\eta}) f(t|\eta) \mathrm{d}t \\ &= \tfrac{2}{c\Gamma(\nu)} \int_{\frac{a}{c\eta}}^{\infty} (\ln cy) y^{\nu-1} e^{-y} \mathrm{d}y, \end{aligned} \tag{19}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}c} R(\hat{\eta}_c, \eta) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \Longleftrightarrow c \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} \exp \left\{ -\frac{\displaystyle\int_{\frac{a}{c\eta}}^{\infty} (\ln y) y^{\nu-1} e^{-y} \mathrm{d}y}{\displaystyle\int_{\frac{a}{c\eta}}^{\infty} y^{\nu-1} e^{-y} \mathrm{d}y} \right\}.$$

The proof is completed. □

The next lemma involves H, defined by

$$H(x) = \exp \left\{ -\frac{\displaystyle\int_{x}^{\infty} (\ln y) y^{\nu-1} e^{-y} \mathrm{d}y}{\displaystyle\int_{x}^{\infty} y^{\nu-1} e^{-y} \mathrm{d}y} \right\}, \quad x \geq 0, \tag{20}$$

gives some mathematical properties of $H(x)$.

Lemma 4.2. The function $H(x)$ in (20) satisfies the following properties:

(i) $H(0) = \exp\{-\Psi(\nu)\} = c^*$,

(ii) $\lim_{x\to\infty} H(x) = 0$, $\lim_{x\to\infty} xH(x) = 1$,

(iii) $H(x)$ is strictly decreasing and $xH(x)$ is strictly increasing in x.

Proof. The results follow by straightforward calculation with the use of the l'Hôpital's rule. □

The next Theorem, which studies the risk function of the estimators in the class $\mathcal{C}$ and gives some their properties as functions of c and η, is used in the proof of Theorem 4.4 below.

Theorem 4.3. For every $\eta > a$, there exists a $g(c, \eta)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}c} R(\hat{\eta}_c, \eta) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \Longleftrightarrow c \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} g(c, \eta). \tag{21}$$

This $g(c, \eta)$ satisfies the following properties:

(i) $0 < g(c, \eta) < c^*$,

(ii) $g(c, \eta)$ is strictly increasing in η, for each $c > 0$,

(iii) $\lim_{\eta\to a} g(c, \eta) = 0$ and $\lim_{\eta\to\infty} g(c, \eta) = c^*$,

(iv) $R(\hat{\eta}_c, a)$ is strictly increasing in c,

(v) For each $c \in (0, c^*)$, there exists $\eta\,(c) > a$ such that

$$R(\hat{\eta}_{c^*}, \eta(c)) < R(\hat{\eta}_c, \eta(c)).$$

Proof. Lemma 4.1 implies

$$\frac{\mathrm{d}}{\mathrm{d}c} R(\hat{\eta}_c, \eta) >, =, < 0 \quad \text{according as} \quad c >, =, < H(\frac{a}{c\eta}), \tag{22}$$

and Lemma 4.2 (part (ii))

$$H(\frac{a}{c\eta}) < \frac{c\eta}{a} = c \;\; \text{for} \;\; \eta = a, \;\; \text{where } H \text{ is defined in (20).}$$

This proves (iv).Now let $\eta > a$, using Lemma 4.2 part (ii),

$$H(\frac{a}{c\eta})/c \to \eta/a \;\; \text{as } c \to 0 \;\; \text{for fixed } \eta. \tag{23}$$

Further, by Lemma 4.1, $H(x) < c^*$ for $x > 0$, so

$$H(\frac{a}{c\eta})/c < 1 \quad \text{for} \quad c = c^*. \tag{24}$$

From (22), (23) and (24) and the fact that $xH(x)$ is continuous as well as strictly increasing, it then follows that there exists a $g(c, \eta)$ satisfying (21) and (i).

To prove (ii) it is sufficient to show that, for each $c \in (0, c^*)$ there exists a $\eta(c) > a$ such that

$$\frac{\mathrm{d}}{\mathrm{d}c} R(\hat{\eta}_c, \eta) >, =, < 0 \quad \text{according as} \quad \eta <, =, > \eta(c), \tag{25}$$

or, equivalently, such that

$$c >, =, < H(\frac{a}{c\eta}) \quad \text{according as} \quad \eta <, =, > \eta(c).$$

To establish the existence of such a $\eta(c)$, fixed a $c \in (0, c^*)$ and note that, using Lemma 4.2 (part (iii)), $H(\frac{a}{c\eta})$ is strictly increasing in η. Further part (ii) implies

$$H(\frac{a}{c\eta}) < \frac{c\eta}{a} = c \quad \text{for} \quad \eta = a,$$

and that $H(\frac{a}{c\eta}) \to c^*$ as $\eta \to \infty$. The result then follows from the fact that $H(x)$ is strictly decreasing and continuous.

For the prove of (iii), first note that the monotonicity and boundedness of $g(c, \eta)$ imply the existence of the required limits. That $g(c, \eta) \to 0$ as $\eta \to a$, then follows from the existence, for each $\eta > a$, of a $g(c, \eta)$ satisfying (21). That $g(c, \eta) \to c^*$ as $\eta \to \infty$, follows from the existence, for each $c \in (0, c^*)$, of $g(c, \eta)$ satisfying (25).

For the proof of (v), note that

$$R(\hat{\eta}_c, \eta) = \left(\ln \frac{a}{\eta}\right)^2 + \int_{\frac{a}{c\eta}}^{\infty} \left[(\ln cs)^2 - \left(\ln \frac{a}{\eta}\right)^2\right] f_T(s|1)\mathrm{d}s.$$

So

$$\begin{aligned} R(\hat{\eta}_{c^*}, \eta) - R(\hat{\eta}_c, \eta) &= \int_{\frac{a}{c^*\eta}}^{\infty} \left[(\ln c^* s)^2 - (\ln \frac{a}{\eta})^2\right] f_T(s|1)\mathrm{d}s \\ &\quad - \int_{\frac{a}{c\eta}}^{\infty} \left[(\ln cs)^2 - (\ln \frac{a}{\eta})^2\right] f_T(s|1)\mathrm{d}s. \end{aligned} \tag{26}$$

It is now sufficient to prove that the limit, as $\eta \to \infty$, of the right hand side of (26) is negative. But this limit is

$$\int_0^{\infty} \left[(\ln c^* s)^2 - (\ln cs)^2\right] f_T(s|1)\mathrm{d}s < 0,$$

for all $c \neq c^*$. The proof is completed. □

The following theorem shows that the class $\mathcal{C}'$ defined in (17) consists of all those δ_c, $c > 0$, which are admissible in $\mathcal{C}$. It also identifies the estimators in $\mathcal{C}$ which dominate the ones in $\mathcal{C} - \mathcal{C}'$.

Theorem 4.4. If $\eta \geq a > 0$, then

(i) $\hat{\eta}_c$ dominates $\hat{\eta}_{c'}$ for $c^* \leq c < c'$,

(ii) Each estimator in the class $\mathcal{C}'$ is admissible in $\mathcal{C}$.

Proof. Part (i) follows from the parts (i), (iv) and (21) of Theorem 4.3. For the proof of part (ii) note that $\hat{\eta}_c$ admissible in $\mathcal{C}'$ implies $\hat{\eta}_c$ admissible in $\mathcal{C}$. This can be seen as follows. Suppose this is not true, i. e., suppose there exists an estimator $\hat{\eta}_c$ which is admissible in $\mathcal{C}'$ and an estimator $\eta^* \in \mathcal{C} - \mathcal{C}'$ which dominates $\hat{\eta}_c$. Then because $\hat{\eta}_{c^*}$ dominates η^*, $\hat{\eta}_{c^*}$ dominates $\hat{\eta}_c$. But this contradicts the admissibility of $\hat{\eta}_c$ in $\mathcal{C}'$. So now it is sufficient to show that every estimator in $\mathcal{C}'$ is admissible in $\mathcal{C}'$. To prove this, let $0 < c' \neq c'' \leq c^*$ and consider the following cases:

(i) $0 < c', c'' < c^*$. By Theorem 4.3 parts (i)-(iii) and Equation (21), there exists $\eta_0(c') > a$ such that $R(\hat{\eta}_{c'}, \eta_0(c')) < R(\hat{\eta}_{c''}, \eta_0(c'))$, which shows that $\hat{\eta}_{c''}$ does not dominate $\hat{\eta}_{c'}$.

(ii) $0 < c' < c^*$, $c'' = c^*$. Then $\hat{\eta}_{c''}$ does not dominate $\hat{\eta}_{c'}$ because, by Theorem 4.3 part (iv), $R(\hat{\eta}_{c'}, a) < R(\hat{\eta}_{c^*}, a)$. Nor does $\hat{\eta}_{c'}$ dominate $\hat{\eta}_{c''}$ because, by Theorem 4.3, part (v), there exists $\eta(c') > a$ such that $R(\hat{\eta}_{c^*}, \eta(c')) < R(\hat{\eta}_{c'}, \eta(c'))$.

Proof of this theorem is completed. □

In this part, some other properties and comparisons of the truncated linear estimators in class $\mathcal{C}$ and the admissible estimator of the lower-bounded scale parameter in (11) are presented.

Proposition 4.5. For each $c > 0$, $\hat{\eta}_c(\mathbf{x}) = \max(cT(\mathbf{x}), a)$ is nondecreasing in $T(\mathbf{x})$. Further, the estimator $\delta(\mathbf{x})$ in (11) is strictly increasing in $T(\mathbf{x})$. Further,

(i) $\lim_{T(\mathbf{x}) \to \infty} \delta(\mathbf{x}) = \infty$,

(ii) $\lim_{T(\mathbf{x}) \to 0} \delta(\mathbf{x}) \geq a$.

Proof. The proof is easy using some mathematical calculations with the use of the l'Hôpital's rule. □

The class $\mathcal{C}$ of truncated linear estimators is the class of inadmissible estimators. By choosing $c = c^*$, the inadmissible estimator $\hat{\eta}_{c^*}(\mathbf{x})$ in class $\mathcal{C}$ dominates all the estimators in the class $\mathcal{C} - \mathcal{C}'$ (Theorem 4.4). For values η/a close to 1 and $c < c'$, $\hat{\eta}_c$ dominates $\hat{\eta}_{c'}$ because by Theorem 4.3, $R(\hat{\eta}_c, a)$ is strictly increasing in c. $\delta(\mathbf{x})$ in (11) as an estimator of a lower-bounded scale parameter under squared-log error loss function, is admissible.

Using the above contents, as for choice between the estimators in class $\mathcal{C}$ and the admissible estimator $\delta(\mathbf{x})$, considering the following points are essential:

1. $\delta(\mathbf{x})$ has the advantage of being admissible, the disadvantages of poor performance for values of η close to a and the complexity of computations.

2. The other estimators $\hat{\eta}_c(\mathbf{x})$ in class $\mathcal{C}$ have the disadvantage of being inadmissible. They have the advantage of good performance for values of η close to a. Another plus of these estimators is their simplicity for compute.

5. CONCLUSION

All problems in practice have a truncated parameter space and it is most impossible to argue in practice that a parameter is not bounded. In such problems the frequently used criterion of unbiasedness is useless. So, other optimality criteria such as admissibility and minimaxity are useful to resolving this captivity. In this article, under the squared-log error loss function, we derive the Bayes estimator of the lower-bounded scale parameter. Also the admissibility of the limiting Bayes estimator is proved. In the end, a class of truncated linear estimation is considered and its properties are given. It is shown that each estimator in this class is an inadmissible.

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Eisa Mahmoudi, Department of Statistics, Yazd University, P.O. Box 89175-741, Yazd. Iran.

e-mail: emahmoudi@yazduni.ac.ir

Hojatollah Zakerzadeh, Department of Statistics, Yazd University, P.O. Box 89175-741, Yazd. Iran.

e-mail: hzaker@yazduni.ac.ir