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DUAL SPACES OF LOCAL MORREY-TYPE SPACES

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Abstract. In this paper we show that associated spaces and dual spaces of the local Morrey-type spaces are so called complementary local Morrey-type spaces. Our method is based on an application of multidimensional reverse Hardy inequalities.

Keywords: local Morrey-type spaces, complementary local Morrey-type spaces, associated spaces, dual spaces, multidimensional reverse Hardy inequalities

MSC 2010: 46E30, 26D15

1. Introduction

In the theory of partial differential equations, along with the weighted Lebesgue spaces, generalized Morrey spaces also play an important role. These spaces appeared to be quite useful in the study of the local behavior of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of PDE.

In [1] local Morrey-type spaces $LM_{p\theta,\omega}$ were defined and some properties of these spaces were studied. The authors investigated the boundedness of the Hardy-Littlewood maximal operator in these spaces. After this paper there was an intensive study of boundedness of other classical operators such as the fractional maximal operator, Riesz potential and Calderón-Zygmund singular integral operator (see [1], [3]–[6], [9]–[12]).

Later in [2] so called complementary local Morrey-type spaces ${}^{\complement}LM_{p\theta,\omega}$ were introduced and the boundedness of fractional maximal operator from complementary local

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Morrey-type space ${}^{\complement}LM_{p\theta,\omega}$ to local Morrey-type space $LM_{p\theta,\omega}$ was investigated. As the definition of the space ${}^{\complement}LM_{p\theta,\omega}$ used the complement of the ball instead of the ball, it was named complementary local Morrey-type space and no relation between $LM_{p\theta,\omega}$ and ${}^{\complement}LM_{p\theta,\omega}$ was studied.

In this paper we characterize associated spaces and dual spaces of local Morrey-type spaces and complementary local Morrey-type spaces. More precisely, we show that associated spaces of local Morrey-type spaces $LM_{p\theta,\omega}$ are complementary local Morrey-type spaces ${}^{\complement}LM_{p'\theta',\widetilde{\omega}}$ (see Theorem 4.3). Moreover, for some values of parameters these associated spaces are duals of local Morrey-type spaces (see Theorem 6.1).

The paper is organized as follows. We start with notation and definitions of the local Morrey-type spaces and complementary local Morrey-type spaces in Section 2. In Section 3 we formulate known results about multidimensional reverse Hardy inequalities and give some corollaries of them, and in Section 4 we use these corollaries to characterize the associated spaces of local Morrey-type spaces and complementary local Morrey-type spaces. We prove completeness of the local Morrey-type spaces, as well as that of complementary local Morrey-type spaces in Section 5. Finally, we characterize the dual spaces in Section 6.

2. Notation and Definitions

Let E be a nonempty measurable subset on $\mathbb{R}^n$ and let f be a measurable function on E. We put

$$\|g\|_{L_p(E)} := \left(\int_E |f(y)|^p\,\mathrm{d}y\right)^{1/p}, \quad 0 < p < +\infty,$$
$$\|f\|_{L_\infty(E)} := \sup\{\alpha\colon\ |\{y \in E\colon\ |f(y)| \geqslant \alpha\}| > 0\}.$$

If I is a nonempty measurable subset on $(0, +\infty)$ and g is a measurable function on I, then we define $\|g\|_{L_p(I)}$ and $\|g\|_{L_\infty(I)}$ correspondingly.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball centered at x of radius r and ${}^{\complement}B(x,r) := \mathbb{R}^n \setminus B(x,r)$.

Let us recall the definitions of local Morrey-type space and complementary local Morrey-type space.

Definition 2.1 ([1]). Let $0 < p, \theta \leqslant \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,\omega}$ the local Morrey-type space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}} \equiv \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)}.$$

Definition 2.2 ([2]). Let $0 < p, \theta \leqslant \infty$ and let w be a non-negative measurable function on $(0, \infty)$. We denote by ${}^{\complement}LM_{p\theta,\omega}$ the complementary local Morrey-type space, the space of all functions $f \in L_p({}^{\complement}B(0,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{L_p({}^{\complement}(B(0,r))}\big\|_{L_\theta(0,\infty)}.$$

Definition 2.3. Let $0 < p, \theta \leqslant \infty$. We denote by Ω_θ the set all non-negative measurable functions ω on $(0, \infty)$ such that

$$0 < \|\omega\|_{L_\theta(t,\infty)} < \infty, \quad t > 0,$$

and by ${}^{\complement}\Omega_\theta$ the set all non-negative measurable functions ω on $(0, \infty)$ such that

$$0 < \|\omega\|_{L_\theta(0,t)} < \infty, \quad t > 0.$$

Now we make some conventions. Throughout the paper, we always denote by c a positive constant which is independent of main parameters, but it may vary from line to line. By $A \lesssim B$ we mean that $A \leqslant cB$ with some positive constant c independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Constant with subscript such as c_1, does not change in different occurrences. For a measurable set E, χ_E denotes the characteristic function of E.

For a fixed p with $p \in [1, \infty)$, p' denotes the conjugate exponent of p, namely,

$$p' := \begin{cases} \dfrac{p}{1-p} & \text{if } \ 0 < p < 1, \\ +\infty & \text{if } \ p = 1, \\ \dfrac{p}{p-1} & \text{if } \ 1 < p < +\infty, \\ 1 & \text{if } \ p = +\infty, \end{cases}$$

and $1/(+\infty) = 0$, $0/0 = 0$, $0 \cdot (\pm\infty) = 0$.

3. The multidimensional reverse Hardy inequality

In this section we formulate known results for multidimensional reverse Hardy inequalities and give some corollaries which will be used in the next section.

Theorem 3.1 ([8], Theorem 4.1)**.** *Assume that $0 < q \leqslant p \leqslant 1$. Let ω and u be weight functions on $\mathbb{R}^n$ and $(0, \infty)$, respectively. Let $\|u\|_{L_q(0,t)} < +\infty$ for all $t \in (0, \infty)$. Then the inequality*

$$\|gw\|_{L_p(\mathbb{R}^n)} \leqslant c\bigg\|u(t)\int_{{}^{\complement}B(0,t)} g(y)\,\mathrm{d}y\bigg\|_{L_q(0,\infty)} \tag{3.1}$$

holds for all non-negative measurable g if and only if

$$A_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}(B(0,t))} \|u\|^{-1}_{L_q(0,t)} < +\infty.$$

The best possible constant c in (3.1) satisfies $c \approx A_1$.

From Theorem 3.1 we get the following Corollary.

Corollary 3.2. *Assume $1 \leqslant p < \infty$ and $0 < \theta \leqslant 1$. Let $\omega \in {}^{\complement}\Omega_\theta$. Then*

$$\sup_{g\in {}^{\complement}LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x}{\|g\|_{{}^{\complement}LM_{p\theta,\omega}}} \approx \sup_{t\in(0,\infty)} \frac{\|f\|_{L_{p'}(B(0,t))}}{\|\omega\|_{L_\theta(0,t)}}.$$

P r o o f. At first note that

$$\sup_{g\in {}^{\complement}LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x}{\|g\|_{{}^{\complement}LM_{p\theta,\omega}}} = \sup_g \left(\frac{\|g^p f^p\|_{L_{1/p}(\mathbb{R}^n)}}{\left\|\omega^p(t) \int_{{}^{\complement}B(0,t)} g^p(y)\,\mathrm{d}y\right\|_{L_{\theta/p}(0,\infty)}} \right)^{1/p}.$$

On the other hand, since $\theta/p \leqslant 1/p \leqslant 1$, by Theorem 3.1 the inequality

$$\|g^p f^p\|_{L_{1/p}(\mathbb{R}^n)} \leqslant c \left\|\omega^p(t) \int_{{}^{\complement}B(0,t)} g^p(y)\,\mathrm{d}y\right\|_{L_{\theta/p}(0,\infty)} \tag{3.2}$$

holds for all non-negative measurable g on $\mathbb{R}^n$ if and only if

$$c_1 := \sup_{t\in(0,\infty)} \|f^p\|_{L_{(1/p)'}(B(0,t))} \|\omega^p\|^{-1}_{L_{\theta/p}(0,t)} = \sup_{t\in(0,\infty)} \left(\frac{\|f\|_{L_{p'}(B(0,t))}}{\|\omega\|_{L_\theta(0,t)}} \right)^p < +\infty.$$

The best possible constant c in (3.2) satisfies $c \approx c_1$. That is,

$$\sup_g \frac{\|g^p f^p\|_{L_{1/p}(\mathbb{R}^n)}}{\left\|\omega^p(t) \int_{{}^{\complement}B(0,t)} g^p(y)\,\mathrm{d}y\right\|_{L_{\theta/p}(0,\infty)}} \approx \sup_{t\in(0,\infty)} \frac{\|f\|_{L_{p'}(B(0,t))}}{\|\omega\|_{L_\theta(0,t)}}.$$

□

Consider now the inequality (3.1) in the case when $0 < p \leqslant 1$, $p < q \leqslant +\infty$ and define r by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{3.3}$$

In such a case we shall write a condition characterizing the validity of the inequality (3.1) in a compact form involving $\int_{(0,\infty)} f\,\mathrm{d}h$, where $f(t) = \|w\|^r_{L_{p'}(B(0,t))}$ and

$h(t) = -\|u\|^{-r}_{L_q(0,t+)}$, $t \in (0,\infty)$. (Here $\|u\|_{L_q(0,t+)} := \lim_{s\to t+} \|u\|_{L_q(0,s)}$.) Hence, the Lebesgue-Stieltjes integral $\int_{(0,\infty)} f\,\mathrm{d}h$ is defined by the non-decreasing and right-continuous function h on $(0,\infty)$.

However, it can happen that $\|u\|_{L_q(0,t+)} = 0$ for all $t \in (0,c)$ with a convenient $c \in (0,\infty)$ (provided that we omit the trivial case when $u = 0$ a.e. on $(0,\infty)$). Then we have to explain what is the meaning of the Lebesgue-Stieltjes integral since in such a case the function $h = -\infty$ on $(0,c)$. To this end, we adopt the following convention.

Convention 3.3. Let $I = (a,b) \subseteq \mathbb{R}$, $f\colon I \to [0,+\infty]$ and $h\colon I \to [-\infty,0]$. Assume that h is non-decreasing and right-continuous on I. If $h\colon I \to (-\infty,0]$, then the symbol $\int_I f\,\mathrm{d}h$ means the usual Lebesgue-Stieltjes integral. However, if $h = -\infty$ on some subinterval (a,c) with $c \in I$, then we define $\int_I f\,\mathrm{d}h$ only if $f = 0$ on $(a,c]$ and we put

$$\int_I f\,\mathrm{d}h = \int_{(c,b)} f\,\mathrm{d}h.$$

Theorem 3.4 ([8], Theorem 4.4). *Assume that* $0 < p \leqslant 1$, $p < q \leqslant +\infty$ *and* r *is given by* (3.3). *Let* ω *and* u *be weight functions on* $\mathbb{R}^n$ *and* $(0,\infty)$, *respectively. Let* u *satisfy* $\|u\|_{L_q(0,t)} < +\infty$ *for all* $t \in (0,\infty)$ *and* $u \neq 0$ *a.e. on* $(0,\infty)$. *Then the inequality* (3.1) *holds for all non-negative measurable* g *on* $\mathbb{R}^n$ *if and only if*

$$A_2 := \Bigl(\int_{(0,\infty)} \|w\|^r_{L_{p'}(B(0,t))}\,\mathrm{d}\bigl(-\|u\|^{-r}_{L_q(0,t+)}\bigr)\Bigr)^{1/r} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}} < +\infty.$$

The best possible constant c *in* (3.1) *satisfies* $c \approx A_2$.

Corollary 3.5. *Assume* $1 \leqslant p < \infty$, $1 < \theta \leqslant \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$. *Then*

$$\sup_{g \in {}^{\complement}LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x}{\|g\|_{{}^{\complement}LM_{p\theta,\omega}}} \approx \Bigl(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(0,t))}\,\mathrm{d}\bigl(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\bigr)\Bigr)^{1/\theta'} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}}.$$

P r o o f. The proof follows from Theorem 3.4 in a similar way as in the proof of Corollary 3.2. □

Remark 3.6. Let $q < +\infty$ in Theorem 3.4. Then

$$\|u\|_{L_q(0,t+)} = \|u\|_{L_q(0,t)} \quad \text{for all } t \in (0,\infty),$$

which implies that

$$A_2 = \left(\int_{(0,\infty)} \|w\|^r_{L_{p'}(B(0,t))} \, \mathrm{d}\bigl(-\|u\|^{-r}_{L_q(0,t)}\bigr) \right)^{1/r} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}}.$$

Our next assertion is a counterpart of Theorem 3.1.

Theorem 3.7 ([8], Theorem 5.1)**.** *Assume that* $0 < q \leqslant p \leqslant 1$. *Let* ω *and* u *be weight functions on* $\mathbb{R}^n$ *and* $(0,\infty)$, *respectively. Let* $\|u\|_{L_q(t,\infty)} < +\infty$ *for all* $t \in (0,\infty)$. *Then the inequality*

$$\|gw\|_{L_p(\mathbb{R}^n)} \leqslant c \left\| u(t) \int_{B(0,t)} g(y) \, \mathrm{d}y \right\|_{L_q(0,\infty)} \tag{3.4}$$

holds for all non-negative measurable g *on* $\mathbb{R}^n$ *if and only if*

$$B_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}(\complement B(0,t))} \|u\|^{-1}_{L_q(t,\infty)} < +\infty. \tag{3.5}$$

The best possible constant c *in* (3.4) *satisfies* $c \approx B_1$.

From Theorem 3.7 we conclude the following statement.

Corollary 3.8. *Assume* $1 \leqslant p < \infty$, $\theta \leqslant 1$. *Let* $\omega \in \Omega_\theta$. *Then*

$$\sup_{g\in LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} |f(x)g(x)| \, \mathrm{d}x}{\|g\|_{LM_{p\theta,\omega}}} \approx \sup_{t\in(0,\infty)} \frac{\|f\|_{L_{p'}(\complement B(0,t))}}{\|\omega\|_{L_\theta(t,\infty)}}.$$

P r o o f. The proof is similar to the proof of Corollary 3.2. We just need to use Theorem 3.7 instead of Theorem 3.1. □

Let us denote by $\|u\|_{L_q(t-,\infty)} := \lim\limits_{s\to t-} \|u\|_{L_q[s,\infty)}$, $t \in (0,\infty)$. The following theorem is true.

Theorem 3.9 ([8], Theorem 5.4)**.** *Assume that* $0 < p \leqslant 1$, $p < q \leqslant +\infty$ *and* r *is given by* (3.3). *Let* ω *and* u *be weight functions on* $\mathbb{R}^n$ *and* $(0,\infty)$, *respectively. Let* u *satisfy* $\|u\|_{L_q(t,\infty)} < +\infty$ *for all* $t \in (0,\infty)$ *and* $u \neq 0$ *a.e. on* $(0,\infty)$. *Then the inequality* (3.4) *holds for all non-negative measurable* g *if and only if*

$$B_2 := \left(\int_{(0,\infty)} \|w\|^r_{L_{p'}(\complement B(0,t))} \, \mathrm{d}\bigl(\|u\|^{-r}_{L_q(t-,\infty)}\bigr) \right)^{1/r} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}} < +\infty.$$

The best possible constant c *in* (3.4) *satisfies* $c \approx B_2$.

Remark 3.10. Let $q < +\infty$ in Theorem 3.9. Then

$$\|u\|_{L_q(t-,\infty)} = \|u\|_{L_q(t,\infty)} \quad \text{for all } t \in (0,\infty),$$

which implies that

$$B_2 = \left(\int_{(a,b)} \|w\|^r_{L_{p'}(^{\complement}B(0,t))} \, \mathrm{d}\big(\|u\|^{-r}_{L_q(t,\infty)}\big)\right)^{1/r} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}}.$$

From Theorem 3.9 we get the following corollary.

Corollary 3.11. *Assume* $1 \leqslant p < \infty$, $1 < \theta \leqslant \infty$. *Let* $\omega \in \Omega_\theta$. *Then*

$$\sup_{g \in LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} |f(x)g(x)| \, \mathrm{d}x}{\|g\|_{LM_{p\theta,\omega}}} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(^{\complement}B(0,t))} \, \mathrm{d}\big(\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\big)\right)^{1/\theta'} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}}.$$

P r o o f. Using Theorem 3.9 we can get the statement as in the proof of Corollary 3.2. $\square$

4. Associated spaces of local Morrey-type spaces

In this section by using results of the previous section we calculate the associated spaces of local Morrey-type spaces and complementary local Morrey-type spaces.

Let $(\mathcal{R}, \mu)$ be a totally σ-finite non-atomic measure space. Let $\mathfrak{M}(\mathcal{R}, \mu)$ be the set of all μ-measurable a.e. finite real functions on $\mathcal{R}$.

Definition 4.1. Let X be a set of functions from $\mathfrak{M}(\mathcal{R}, \mu)$, endowed with a positively homogeneous functional $\|\cdot\|_X$, defined for every $f \in \mathfrak{M}(\mathcal{R}, \mu)$ and such that $f \in X$ if and only if $\|f\|_X < \infty$. We define the associate space X' of X as the set of all functions $f \in \mathfrak{M}(\mathcal{R}, \mu)$ such that $\|f\|_{X'} < \infty$, where

$$\|f\|_{X'} = \sup\left\{\int_{\mathcal{R}} |fg| \, \mathrm{d}\mu \colon \ \|g\|_X \leqslant 1\right\}.$$

In what follows we assume $\mathcal{R} = \mathbb{R}^n$ and $\mathrm{d}\mu = \mathrm{d}x$.

The following theorem is true.

Theorem 4.2. *Assume* $1 \leqslant p < \infty$, $0 < \theta \leqslant \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$. *Set* $X = {}^{\complement}LM_{p\theta,\omega}$.

(i) *Let* $0 < \theta \leqslant 1$. *Then*

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(B(0,t))} \|\omega\|^{-1}_{L_\theta(0,t)}.$$

(ii) *Let* $1 < \theta \leqslant \infty$. *Then*

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(0,t))}\, \mathrm{d}\big(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\big)\right)^{1/\theta'} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}}.$$

P r o o f. (i) Let $0 < \theta \leqslant 1$. Since

$$\|f\|_{X'} = \sup_{g\in {}^{\complement}LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x}{\|g\|_{{}^{\complement}LM_{p\theta,\omega}}}, \tag{4.1}$$

it remains to apply Corollary 3.2.

(ii) Let $1 < \theta \leqslant \infty$. The statement follows from (4.1) and Corollary 3.5. □

Theorem 4.3. *Assume* $1 \leqslant p < \infty$, $0 < \theta \leqslant \infty$. *Let* $\omega \in \Omega_\theta$. *Set* $X = LM_{p\theta,\omega}$.

(i) *Let* $0 < \theta \leqslant 1$. *Then*

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}({}^{\complement}B(0,t))} \|\omega\|^{-1}_{L_\theta(t,\infty)}.$$

(ii) *Let* $1 < \theta \leqslant \infty$. *Then*

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}({}^{\complement}B(0,t))}\, \mathrm{d}\big(\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\big)\right)^{1/\theta'} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}}.$$

P r o o f. In view of fact that

$$\|f\|_{X'} = \sup_{g\in LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x}{\|g\|_{LM_{p\theta,\omega}}},$$

it remains to apply Corollary 3.8 (when $0 < \theta \leqslant 1$) or Corollary 3.11 (when $1 < \theta \leqslant \infty$). □

5. Completeness of local Morrey-type spaces

In this section we prove completeness of the local Morrey-type spaces and complementary local Morrey-type spaces.

The following theorem is true.

Theorem 5.1. *Let $1 \leqslant p < \infty$, $1 \leqslant \theta \leqslant \infty$ and $\omega \in \Omega_\theta$. Suppose $f_n \in LM_{p\theta,\omega}$ $(n = 1, 2, \ldots)$ and*

$$\sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}} < \infty. \tag{5.1}$$

Then $\sum\limits_{n=1}^{\infty} f_n$ converges in $LM_{p\theta,\omega}$ to a function f in $LM_{p\theta,\omega}$ and

$$\|f\|_{LM_{p\theta,\omega}} \leqslant \sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}}. \tag{5.2}$$

In particular, $LM_{p\theta,\omega}$ is complete.

Proof. It is easy to see that for any $R > 0$

$$\|\omega\|_{L_\theta(R,\infty)} \|f\|_{L_p(B(0,R))} \leqslant \|f\|_{LM_{p\theta,\omega}}.$$

Thus

$$\sum_{n=1}^{\infty} \|f_n\|_{L_p(B(0,R))} \leqslant c \sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}}.$$

In view of the completeness of $L_p(B(0,R))$, we get that $\sum\limits_{n=1}^{\infty} f_n$ converges a.e. to some $f \in L_p(B(0,R))$ and

$$\|f\|_{L_p(B(0,R))} \leqslant \sum_{n=1}^{\infty} \|f_n\|_{L_p(B(0,R))}. \tag{5.3}$$

Since R is arbitrary, the function $f = \sum\limits_{n=1}^{\infty} f_n$ is well-defined a.e. on $\mathbb{R}^n$. Moreover,

$$\begin{aligned}
\|f\|_{LM_{p\theta,\omega}} &= \|\omega(r)\|f\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)} \\
&\leqslant \Big\|\omega(r) \sum_{n=1}^{\infty} \|f_n\|_{L_p(B(0,r))}\Big\|_{L_\theta(0,\infty)} \\
&\leqslant \sum_{n=1}^{\infty} \|\omega(r)\|f_n\|_{L_p(B(0,r))}\|_{L_\theta(0,\infty)} = \sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}}.
\end{aligned}$$

Therefore, $f \in LM_{p\theta,\omega}$. Since

$$\Big\| f - \sum_{k=1}^{m} f_k \Big\|_{LM_{p\theta,\omega}} = \Big\| \sum_{k=m+1}^{\infty} f_k \Big\|_{LM_{p\theta,\omega}} \leqslant \sum_{k=m+1}^{\infty} \|f_k\|_{LM_{p\theta,\omega}} \to 0,$$

when $m \to \infty$, we get that $\sum_{n=1}^{\infty} f_n$ converges in $LM_{p\theta,\omega}$ to a function f. □

The following theorem can be proved in analogous way.

Theorem 5.2. *Let $1 \leqslant p < \infty$, $1 \leqslant \theta \leqslant \infty$ and $\omega \in {}^{\complement}\Omega_\theta$. Suppose $f_n \in {}^{\complement}LM_{p\theta,\omega}$ $(n = 1, 2, \ldots)$ and*

$$\sum_{n=1}^{\infty} \|f_n\|_{{}^{\complement}LM_{p\theta,\omega}} < \infty. \tag{5.4}$$

Then $\sum_{n=1}^{\infty} f_n$ converges in ${}^{\complement}LM_{p\theta,\omega}$ to a function f in ${}^{\complement}LM_{p\theta,\omega}$ and

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}} \leqslant \sum_{n=1}^{\infty} \|f_n\|_{{}^{\complement}LM_{p\theta,\omega}}. \tag{5.5}$$

In particular, ${}^{\complement}LM_{p\theta,\omega}$ is complete.

6. Dual spaces of local Morrey-type spaces

In this section we show that for some values of the parameters the dual spaces coincide with the asssociated spaces.

The following theorem is true.

Theorem 6.1. *Assume $1 \leqslant p < \infty$ and $1 \leqslant \theta < \infty$. Let $\omega \in \Omega_\theta$ and $\|\omega\|_{L_\theta(0,\infty)} = \infty$. Then*

$$(LM_{p\theta,\omega})^* = {}^{\complement}LM_{p'\theta',\widetilde{\omega}}, \tag{6.1}$$

where $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\big(\int_t^\infty \omega^\theta(s)\,\mathrm{d}s\big)^{-1}$, under the following pairing:

$$\langle f, g\rangle = \int_{\mathbb{R}^n} fg.$$

Moreover, $\|f\|_{{}^{\complement}LM_{p'\theta',\widetilde{\omega}}} = \sup_g \big|\int_{\mathbb{R}^n} fg\big|$, where the supremum is taken over all functions $g \in LM_{p\theta,\omega}$ with $\|g\|_{LM_{p\theta,\omega}} \leqslant 1$.

P r o o f. Assume $1 < \theta < \infty$. If $f \in {}^{\complement}LM_{p'\theta',\widetilde{\omega}}$ and $g \in LM_{p\theta,\omega}$, then by Corollary 3.11 we have

$$|\langle f,g\rangle| \leqslant \int_{\mathbb{R}^n} |fg| \leqslant c\|f\|_{{}^{\complement}LM_{p'\theta',\widetilde{\omega}}}\|g\|_{LM_{p\theta,\omega}}.$$

In particular, every function $f \in {}^{\complement}LM_{p'\theta',\widetilde{\omega}}$ induces a bounded linear functional on $LM_{p\theta,\omega}$.

Conversely, suppose L is a bounded linear functional on $LM_{p\theta,\omega}$ with the norm $\|L\| < \infty$. If g is supported in $D_0 = {}^{\complement}B(0,r_0)$ for some $r_0 > 0$, then

$$\|g\|_{LM_{p\theta,\omega}} \leqslant \|\omega\|_{L_\theta(r_0,\infty)}\|g\|_{L_p({}^{\complement}B(0,r_0))},$$

and

$$|L(g)| \leqslant \|L\|\|\omega\|_{L_\theta(r_0,\infty)}\|g\|_{L_p({}^{\complement}B(0,r_0))}.$$

Hence L induces a bounded linear functional on $L_p({}^{\complement}B(0,r_0))$ and acts with some function $f^0 \in L_{p'}({}^{\complement}B(0,r_0))$. By taking $D_j = {}^{\complement}B(0,r_0/j)$, $j = 1,2,3,\ldots$, we have $f^j = f^{j+1}$ on D_j, so we get a single function f on $\mathbb{R}^n$ that $f \in L_{p'}({}^{\complement}B(0,r))$ for any $r > 0$, and such that $L(g) = \int_{\mathbb{R}^n} fg$ when $g \in L_p({}^{\complement}B(0,t))$ with support in ${}^{\complement}B(0,t)$ for any $t > 0$.

Now we show that $f \in {}^{\complement}LM_{p'\theta',\widetilde{\omega}}$. For $g \in LM_{p\theta,\omega}$ and any $n \in \mathbb{N}$, denote by $g_n(x) = |g(x)|\chi_{B(0,n)\setminus B(0,1/n)}$. Obviously,

$$\int |g_n f| = \int g_n \operatorname{sgn}(f) f = L(g_n \operatorname{sgn}(f)).$$

But L bounded on $LM_{p\theta,\omega}$, so

$$\int |g_n f| \leqslant c\|L\|\|g_n \operatorname{sgn}(f)\|_{LM_{p\theta,\omega}} \leqslant c\|L\|\|g_n\|_{LM_{p\theta,\omega}}. \tag{6.2}$$

It is evident that $0 \leqslant g_n \nearrow |g|$, $n \to \infty$ a.e. in $\mathbb{R}^n$. By the monotone convergence theorem we see that the left-hand side and the right-hand side of (6.2) converge to $\int |gf|$ and $c\|L\|\|g\|_{LM_{p\theta,\omega}}$, correspondingly. Hence

$$\int |gf| \leqslant c\|L\|\|g\|_{LM_{p\theta,\omega}}, \quad g \in LM_{p\theta,\omega}. \tag{6.3}$$

This, together with Corollary 3.11, implies that $f \in {}^{\complement}LM_{p'\theta',\widetilde{\omega}}$.

The argument at the beginning of the proof shows that the linear functional $L_f(g) = \int fg$ induced by f belongs to $(LM_{p\theta,\omega})^*$. Now we need to prove that L_f and L coincide on the whole $LM_{p\theta,\omega}$. For $g \in LM_{p\theta,\omega}$ and any $n \in \mathbb{N}$, denote by $g_n(x) = g(x)\chi_{B(0,n)\setminus B(0,1/n)}$. It is evident that $g_n \to g$, $n \to \infty$ a.e. in $\mathbb{R}^n$. By Lebesgue's dominated convergence theorem, we get that $\|g - g_n\|_{LM_{p\theta,\omega}} \to 0$, $n \to \infty$. Therefore

$$L(g_n) \to L(g), \quad n \to \infty. \tag{6.4}$$

On the other hand, by Corollary 3.11

$$\left|\int fg - \int fg_n\right| \leqslant \int |f(g-g_n)| \leqslant c\|f\|_{{}^{\complement}LM_{p'\theta',\tilde{\omega}}}\|g-g_n\|_{LM_{p\theta,\omega}} \to 0. \tag{6.5}$$

Since $g_n \in L_p({}^{\complement}B(0,1/n))$, we get

$$L(g_n) = \int fg_n.$$

Consequently, from (6.4) and (6.5) we obtain

$$L(g) = \int fg = L_f(g).$$

When $\theta = 1$ the statement can be proved analogously by applying Corollary 3.8 instead of Corollary 3.11. □

In a similar manner the following theorem is proved.

Theorem 6.2. *Assume* $1 \leqslant p < \infty$ *and* $1 \leqslant \theta < \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then*

$$\left({}^{\complement}LM_{p\theta,\omega}\right)^* = LM_{p'\theta',\overline{\omega}}, \tag{6.6}$$

where $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)\,\mathrm{d}s\right)^{-1}$, *under the following pairing:*

$$\langle f, g\rangle = \int_{\mathbb{R}^n} fg.$$

Moreover, $\|f\|_{LM_{p'\theta',\overline{\omega}}} = \sup\limits_g\left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in {}^{\complement}LM_{p\theta,\omega}\colon\ \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \leqslant 1$.

Remark 6.3. Set

$$LM^\circ_{p\infty,\omega} := \{f \in LM_{p\infty,\omega}\colon\ \|f - f_n\|_{LM_{p\infty,\omega}} \to 0,\ n \to \infty\},$$

where $f_n = f\chi_{\{x\in\mathbb{R}^n\colon 1/n<|x|<n\}}$. Assume $1 \leqslant p < \infty$. Then from the proof of Theorem 6.1 it follows that

$$(LM^{\circ}_{p\infty,\omega})^* = {}^{\complement}LM_{p'1,\widetilde{\omega}}. \tag{6.7}$$

Remark 6.4. Let us show that

$$LM^{\circ}_{p\infty,\omega}(\mathbb{R}^n) \subsetneqq LM_{p\infty,\omega}(\mathbb{R}^n).$$

For simplicity we consider one-dimensional case and $\omega(t) = t^{-\lambda}$, $0 < \lambda < 1$. Denote by $LM_{p\infty,\lambda} := LM_{p\infty,\omega}$. Let us define the following function on $\mathbb{R}$

$$f(x) = \sum_{k=-\infty}^{+\infty} 2^{k\lambda - k/p}\chi_{[2^k,32^{k-1}]}(|x|).$$

Since

$$\sup_{t>0} t^{-\lambda}\biggl(\int_{\{x\in\mathbb{R}\colon |x|\leqslant t\}} |f(x)|^p \,\mathrm{d}x\biggr)^{1/p} \approx \sup_{m\in\mathbb{Z}} 2^{-(m+1)\lambda}\biggl(\int_{2^m}^{2^{m+1}} |f(x)|^p \,\mathrm{d}x\biggr)^{1/p} \tag{6.8}$$

(see [7] and [8], for instance), we have

$$\|f\|_{LM_{p\infty,\lambda}(\mathbb{R})} \approx \sup_{m\in\mathbb{Z}} 2^{-(m+1)\lambda}2^{m\lambda-m/p}(3\cdot 2^{m-1} - 2^m)^{1/p} = 2^{\lambda-1/p}.$$

On the other hand, using (6.8), we get

$$\begin{aligned}
\|f - f_{2^n}\|_{LM_{p\infty,\lambda}} &\geqslant \|f\chi_{\{x\in\mathbb{R}^n\colon 2^n\leqslant|x|<2^{n+1}\}}\|_{LM_{p\infty,\lambda}} \\
&= \sup_{t>0} t^{-\lambda}\biggl(\int_{\{x\in\mathbb{R}\colon |x|\leqslant t\}} |f(x)\chi_{\{x\in\mathbb{R}\colon 2^n<|x|<2^{n+1}\}}(x)|^p \,\mathrm{d}x\biggr)^{1/p} \\
&= 2^{-(n+1)\lambda}\biggl(\int_{2^n}^{2^{n+1}} |f(x)|^p \,\mathrm{d}x\biggr)^{1/p} \\
&= 2^{-(n+1)\lambda}2^{n\lambda-n/p}(3\cdot 2^{n-1} - 2^n)^{1/p} = 2^{\lambda-1/p},
\end{aligned}$$

that is,

$$\|f - f_{2^n}\|_{LM_{p\infty,\lambda}} \not\to 0.$$

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References

[1] *V. I. Burenkov, H. V. Guliyev*: Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. Stud. Math. *163* (2004), 157–176.

[2] *V. I. Burenkov, H. V. Guliyev, V. S. Guliyev*: On boundedness of the fractional maximal operator from complementary Morrey-type spaces to Morrey-type spaces. The Interaction of Analysis and Geometry. International School-Conference on Analysis and Geometry, Novosibirsk, Russia, August 23–September 3, 2004. American Mathematical Society (AMS), Providence; Contemporary Mathematics *424* (2007), 17–32.

[3] *V. I. Burenkov, H. V. Guliyev, V. S. Guliyev*: Necessary and sufficient conditions for boundedness of fractional maximal operators in local Morrey-type spaces. J. Comput. Appl. Math. *208* (2007), 280–301.

[4] *V. I. Burenkov, H. V. Guliyev, T. V. Tararykova, A. Serbetci*: Necessary and sufficient conditions for the boundedness of genuine singular integral operators in the local Morrey-type spaces. Dokl. Math. *78* (2008), 651–654; Translation from Dokl. Akad. Nauk, Ross. Akad. Nauk *422* (2008), 11–14.

[5] *V. I. Burenkov, V. S. Guliyev*: Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces. Potential Anal. *30* (2009), 211–249.

[6] *V. I. Burenkov, A. Gogatishvili, V. S. Guliyev, R. Ch. Mustafayev*: Boundedness of the fractional maximal operator in Morrey-type spaces. Complex Var. Elliptic Equ. *55* (2010), 739–758.

[7] *W. D. Evans, A. Gogatishvili, B. Opic*: The reverse Hardy inequality with measures. Math. Inequal. Appl. *11* (2008), 43–74.

[8] *A. Gogatishvili, R. Mustafayev*: The multidimensional reverse Hardy inequalities. Math. Inequal. & Appl. 14 (2011). Preprint, Institute of Mathematics, AS CR, Prague 2009-5-27. Available at `http://www.math.cas.cz/preprint/pre-179.pdf`.

[9] *V. S. Guliyev*: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. Doctor's degree dissertation. Mat. Inst. Steklov, Moscow, 1994. (In Russian.)

[10] *V. S. Guliyev*: Function Spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications. Baku, 1999. (In Russian.)

[11] *V. S. Guliyev, R. Ch. Mustafayev*: Integral operators of potential type in spaces of homogeneous type. Dokl. Math. *55* (1997), 427–429; Translation from Dokl. Akad. Nauk, Ross. Akad. Nauk *354* (1997), 730–732.

[12] *V. S. Guliyev, R. Ch. Mustafayev*: Fractional integrals on spaces of homogeneous type. Anal. Math. *24* (1998), 1810–200. (In Russian.)

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