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NEUTRAL WRENCHES OF 3-PARAMETRIC ROBOT-MANIPULATORS OF THE SPHERICAL RANK 1

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Abstract. Let $SE(3)$ be the Lie group of all Euclidean motions in the Euclidean space E_3, let $se(3)$ be its Lie algebra and $se^*(3)$ the space dual to $se(3)$. This paper deals with structures of the subspaces of $se^*(3)$ which are formed by all the forces whose power exerted on the robot effector is zero.

Keywords: robotics, Lie algebra, twist, wrench

MSC 2010: 70E60

1. Introduction

We will confine ourselves to the 3-parametric serial robot-manipulators (shortly "robots") with parallel axes of their revolute joints, i.e. robot-manipulators of the spherical rank 1, see for example [1]. We use the mathematical description of robots by the exponential map $\exp\colon se(3) \to SE(3)$ of the Lie algebra $se(3)$ into its Lie group $SE(3)$ of all Euclidean motions in the Euclidean space E_3, see [4]. We prefer the notions "twist" for the elements of $se(3)$ and "wrench" or "force" for the elements of the space $se^*(3)$ dual to $se(3)$. Following Selig [5], we say that a force $F \in se^*(3)$ exerts on a twist $Y \in se(3)$ the power $F(Y)$, where $F(Y)$ is the value of F at Y. Every robot determines at instant t a subspace $A_n(t) \subset se(3)$, see the relation (1). Our main purpose is to give the geometrical description of the subspaces of such forces $F \in se^*(3)$ (called "neutral forces of robot"), for which $F(A_n(t)) = 0$.

The structure of this paper is as follows. The first two chapters are devoted to recalling basic notions and properties of twists and wrenches. In the third chapter the wrenches neutral to a robot-manipulator and their characterizations are introduced. In the fourth chapter we describe the structure of the spaces of R-neutral wrenches at the instant t in the case of 3-parametric robot-manipulators of the spherical rank 1. First of all we concentrate on the structures of the lines of these wrenches.

2. Basic notions of twists

We will recall some known notions and propositions from robotics.

The significant instrument for solutions of kinematic and dynamic problems in robotics is the Lie group $SE(3)$ of Euclidean transformations in the Euclidean space E_3 and its Lie algebra $se(3)$. The coordinate system in E_3 determines the isomorphism of the group $SE(3)$ and the group of matrices $H = \begin{pmatrix} A & \bar{p}^\top \\ 0 & 1 \end{pmatrix}$, where $A = (a_{ij})$, $i, j = 1, 2, 3$, is an orthogonal matrix, $\det A = 1$ and $\bar{p}^\top$ is the transposed positional vector $\bar{p} = (p_1, p_2, p_3)$ of the point P into which the coordinate system origin O is moved by the Euclidean transformation. In homogeneous coordinates this transformation is expressed by the relation

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{pmatrix} = \begin{pmatrix} A & \bar{p}^\top \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix},$$

where $AA^\top = E$, $\det A = 1$, $A^\top$ denotes the transposed matrix of the matrix A and E is the unit matrix. Let $H(t)$ be a curve in $SE(3)$. Then $H(0) = \begin{pmatrix} E & \bar{0}^T \\ 0 & 1 \end{pmatrix}$ is the unit matrix and $\dot{H}(0) = \begin{pmatrix} \dot{A}(0) & \dot{\bar{p}}^\top(0) \\ 0 & 0 \end{pmatrix} \in se(3)$. Since $\dot{A}A^\top + A\dot{A}^\top = 0$ we have $\dot{A} + \dot{A}^\top = 0$ for $t = 0$ and therefore $\dot{A}(0)$ is a skew-symmetric matrix. We can identify this matrix $\dot{A} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$ with the vector $\overline{\omega} = (\omega_1, \omega_2, \omega_3)$. Denoting $\dot{\bar{p}}(0) = \bar{b}$, the Lie algebra $se(3)$ can be identified with the vector space of vector couples $(\overline{\omega}, \bar{b})^\top$ in which the Lie bracket is defined by the formula $[(\overline{\omega}_1, \bar{b}_1)^\top, (\overline{\omega}_2, \bar{b}_2)^\top] = ((\overline{\omega}_1 \times \overline{\omega}_2)^\top, (\overline{\omega}_1 \times \bar{b}_2 - \overline{\omega}_2 \times \bar{b}_1)^\top)$, where the sign "$\times$" stands for the cross product of vectors in E_3. The elements of the Lie algebra $se(3)$ can be called the operators of the velocity. By the robotic literature we use the notion "twist", see Fig. 1. The twist $(\overline{\omega}, \bar{b})^\top$ is rotational if $\overline{\omega} \neq \bar{0}$ and $\bar{b} \cdot \overline{\omega} = 0$, where the sign "$\cdot$" denotes the scalar product of vectors in E_3. The twist is helical if $\overline{\omega} \neq \bar{0} \wedge \bar{b} \cdot \overline{\omega} \neq 0$, and the twist is translational if $\overline{\omega} = \bar{0}$.

A line of a twist $(\overline{\omega}, \bar{b})^\top$, $\overline{\omega} \neq \bar{0}$, is the line with the direction vector $\overline{\omega}$, which is incident with the point C, $\overline{OC} = (\overline{\omega} \times \bar{b})/\overline{\omega}^2$. Conversely, the line p incident with C with the directional vector $\overline{v}$ determines the rotational twist $(\overline{v}, \overline{OC} \times \overline{v})$, which some

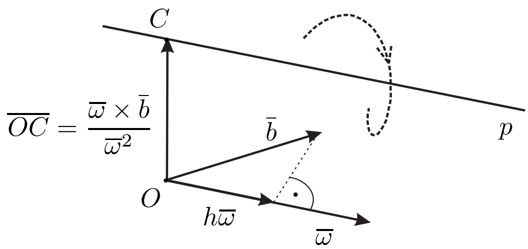

Figure 1. A twist (taken from [1]).

authors call the Plücker coordinates of the line p. Every twist $(\overline{\omega}, \bar{b})^\top$, the line of which is p, is of the form $(t\overline{v}, t\overline{OC} \times \overline{v} + k\overline{v})$, $t, k \in \mathbb{R}$.

There is an exponential map $\exp\colon g \to G$ between the Lie algebra g and its group G which is a local diffeomorphism. In case of the group $SE(3)$ and its Lie algebra $se(3)$ we use its matrix form $\exp \dot{H} = E + \frac{1}{2!}\dot{H}^2 + \frac{1}{3!}\dot{H}^3 + \ldots$. We obtain: If $X = (\overline{\omega}, \bar{b})^\top$ is rotational, then $\exp tX$ is a rotational motion around the line of the twist X with the angular velocity $\overline{\omega}$. If X is helical, then $\exp tX$ is a helical motion around its line with the angular velocity $\overline{\omega}$ and with the translation velocity $h\overline{\omega}$, where $h = (\bar{b} \cdot \overline{\omega})/\overline{\omega}^2$. If $X = (\bar{0}, \bar{b})^\top$ is translational, then $\exp tX$ is a translation motion with the velocity $\bar{b}$. For these motions $\bar{b}$ is the instantaneous velocity of the origin O of the coordinate system.

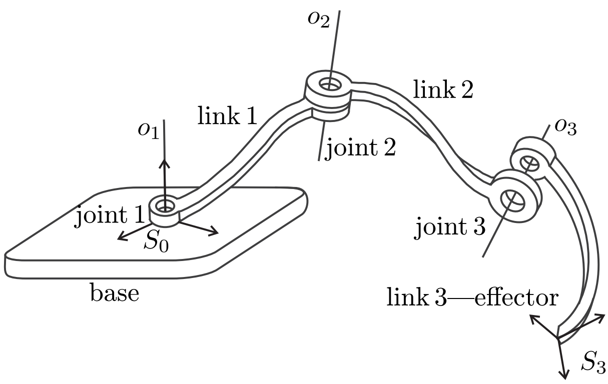

Figure 2. 3-parametric robot-manipulator (taken from [1]).

An n-parametric robot-manipulator is a sequence of n joints and links, see Fig. 2. Each joint has an axis and makes a rotational, or helical, or translational motion of the next link possible. This link is ended with the other joint. Each axis of the joint determines the twist of this joint $X_i = (\overline{\omega}_i, \bar{b}_i)^\top$, $i = 1, \ldots, n$, such that $\exp u_i(t) X_i$ is a Euclidean motion which is made possible by this joint. The link which starts from the last joint is an effector. The point L_0 of the effector by an action of the robot performs the motion determined by the equation $L(t) = \exp u_1(t)X_1 \exp u_2(t)X_2 \ldots \exp u_n(t)X_n L_0$. Therefore, we can theoretically identify the robot with the map $R\colon U \to SE(3)$, $U \subset \mathbb{R}_n$, $(u_1, u_2, \ldots, u_n) \mapsto \exp u_1 X_1 \ldots \exp u_n X_n$, where U is an open neighborhood of admissible parameters. A control curve $u(t) = (u_1(t), u_2(t), \ldots, u_n(t)) \subset U$ determines the curve $\gamma(t) = R(u(t)) = \exp u_1(t)X_1 \exp u_2(t)X_2 \ldots \exp u_n(t)X_n$ in $SE(3)$. Its tangential vector $\dot\gamma(t)$ is transferred to $Y(t) = \dot\gamma(t)\gamma^{-1}(t) \in se(3)$ by the right translation of the element $\gamma^{-1}(t)$. It is easy to see that $\dot\gamma(t) = \sum\limits_{i=1}^{n} \exp u_1(t)X_1 \ldots \dot u_i(t)X_i \exp u_i(t)X_i \ldots \exp u_n(t)X_n$. Then

$$Y(t) = \dot u_1(t)Y_1(t) + \dot u_2(t)Y_2(t) + \ldots + \dot u_n(t)Y_n(t), \quad Y_i = H_{i-1}X_iH_{i-1}^{-1}, \tag{1}$$

where $H_{i-1} = \exp u_1(t)X_1 \ldots \exp u_{i-1}(t)X_{i-1}$. The tangential vectors $\dot u(t_0)$, which we can call the vectors of joint velocities of all the curves $u(t)$ incident with the same point $u_0 = u(t_0)$ at the instant t_0, generate the vector space $\mathbb{R}_n(u_0)$. Denote $A_n(t_0) = \mathrm{span}(Y_1(t_0), Y_2(t_0), \ldots Y_n(t_0))$. The relation (1) determines the Jacobian map J of the robot-manipulator $J\colon \mathbb{R}_n(u_0) \mapsto se(3)$, $J(\dot u_1(t_0), \ldots, \dot u_n(t_0)) = Y(t_0)$. Evidently $J(\mathbb{R}_n(u_0)) = A_n(t_0)$.

2.1. Kl-orthogonal twists. The Klein form Kl is the bilinear form defined on $se(3)$ by formula $\mathrm{Kl}((\bar\varpi_1, \bar b_1)^\top, (\bar\varpi_2, \bar b_2)^\top) = \bar\varpi_1 \cdot \bar b_2 + \bar\varpi_2 \cdot \bar b_1$. Two twists X_1, X_2 are called *Kl-orthogonal* if $\mathrm{Kl}(X_1, X_2) = 0$. This easily implies

Proposition 2.1. *A translational twist X_1 is* Kl-*orthogonal to a nontranslational twist X_2 iff its direction is orthogonal to the axis of the twist X_2.*

The following property is well known (see for example [2]).

Proposition 2.2. *Two rotational twists are Kl-orthogonal iff their axes are complanar.*

Definition 2.1. Denote by $A_n^{\mathrm{Kl}} \subset se(3)$ the subspace which is Kl-orthogonal to A_n, i.e. $A_n^{\mathrm{Kl}} = \{X \in se(3),\ \mathrm{Kl}(X, Y) = 0\ \forall Y \in A_n\}$. The subspace $A_n^{\mathrm{Kl}} \cap A_n = K_n$ is said to be the *Klein subspace* of the robot.

According to Proposition 2.1, the translational twist X is an element of K_n iff its direction is orthogonal to the line of each twist which is not translational from A_n.

3. Wrenches. External forces. Canonical isomorphism

3.1. Wrenches. Let $se^*(3)$ denote the dual space to the vector space $se(3)$, i.e. $se^*(3)$ is the space of linear forms (linear maps) $\alpha\colon se(3) \to \mathbb{R}$ on $se(3)$. Each vector couple $\alpha = (\overline{m}, \overline{f})$ determines the linear form on $se(3)$ by the formula: if $X = (\overline{\omega}, \bar{b})^\top \in se(3)$ then $\alpha(X) = (\overline{m}, \overline{f})(\overline{\omega}, \bar{b})^\top = \overline{m} \cdot \overline{\omega} + \overline{f} \cdot \bar{b}$. Seeing that $\dim se^*(3) = \dim se(3) = 6$ and the vector space of vector couples is 6-dimensional, each element from $se^*(3)$ can be identified with a vector couple. Elements of $se^*(3)$ are called *wrenches* (or general forces).

We can interpret $\alpha(X) = \overline{m} \cdot \overline{\omega} + \overline{f} \cdot \bar{b}$ (the internal product) as the power of the wrench α applied on the twist X.

3.2. Forces and torques. The line of the wrench $(\overline{m}, \overline{f})$ with the direction vector $\overline{f}$ is the line incident with the point C, $\overline{OC} = (\overline{f} \times \overline{m})/\overline{f}^2$. The wrench $(\overline{m}, \overline{f})$ is called a pure force if $\overline{m} \cdot \overline{f} = 0$, $\overline{f} \neq \bar{0}$ or a pure torque if $\overline{f} = \bar{0}$ or a general wrench (a general force) if $\overline{f} \neq \bar{0}$, $\overline{m} \cdot \overline{f} \neq 0$. Each line p with a direction vector $\overline{f}$ incident with the point C determines the pure force $(\overline{OC} \times \overline{f}, \overline{f})$ and each wrench with the line p has the form $(c_1 \overline{OC} \times \overline{f} + c_2 \overline{f}, c_1 \overline{f})$, $c_1, c_2 \in \mathbb{R}$, i.e. it is a sum of a pure force and a pure torque. In the case of a torque $(\overline{m}, \bar{0})$ the vector $\overline{m}$ will be called the moment of $(\overline{m}, \bar{0})$ and in the case of a pure force $F = (\overline{OC} \times \overline{f}, \overline{f})$, $\overline{OC} \times \overline{f}$ is referred to as the moment of F at O.

3.3. Canonical isomorphism. With the Klein bilinear form the canonical isomorphism $i^{\mathrm{Kl}}\colon se(3) \to se^*(3)$, $i^{\mathrm{Kl}}(X) = \alpha_X \equiv \mathrm{Kl}(X, \circ)$, i.e. $\alpha_X(Y) = \mathrm{Kl}(X, Y)$ is connected. From the above it follows that if $X = (\overline{\omega}, \bar{b})^\top$ then $i^{\mathrm{Kl}}(X) = \alpha_X = (\bar{b}, \overline{\omega})$ and then the map i^{Kl} is determined by the block matrix $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, $i^{\mathrm{Kl}} \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \begin{pmatrix} \overline{\omega} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} \bar{b} \\ \overline{\omega} \end{pmatrix} \approx (\bar{b}, \overline{\omega})$.

4. External forces which do not act on a robot

Let R be an n-parametric robot. The subspace $A_n(t) \subset se(3)$ at the instant t is the space of velocity operators of the robot at the time instant t.

Definition 4.1. We say that a wrench α does not act on R at the instant t or shortly that α is *R-neutral* if $\alpha(Y) = 0$ for each twist $Y \in A_n(t)$.

The space of all the R-neutral wrenches at the instant t is denoted by Ω_t. The dual map to Jacobian map J is the map $J^*\colon se^*(3) \mapsto \mathbb{R}^*_n(u_0)$ defined by the formula

$$J^*(\alpha) = \varepsilon, \quad \varepsilon(\dot{u}(t_0)) = \alpha(J(\dot{u}(t_0))). \tag{2}$$

Proposition 4.1. *A wrench α is R-neutral iff $\alpha \in \operatorname{Ker} J^*$. Therefore, $\Omega_t = \operatorname{Ker} J^*$.*

P r o o f. According to Definition 4.1, a wrench α is R-neutral iff $\alpha|_{A_n(t_0)} = 0$. If $\alpha|_{A_n(t_0)} = 0$ then $J^*(\alpha)(\dot{u}(t_0)) = \alpha(J(\dot{u}(t_0))) = 0$, because $J(\dot{u}(t_0)) \in A_n(t_0)$. Therefore, $\alpha \in \operatorname{Ker} J^*$. Conversely, if $\alpha \in \operatorname{Ker} J^*$ then $J^*(\alpha) = \varepsilon = 0$ and consequently, it results from the relation (2): If $X \in A_n(t)$ then $X = J(\dot{u}(t_0))$. Hence, $\alpha(X) = \varepsilon(\dot{u}(t_0)) = 0$. $\square$

Proposition 4.2. *A wrench α is R-neutral iff $\alpha \in i^{\mathrm{Kl}}(A_n^{\mathrm{Kl}})$, and then $\Omega_t = i^{\mathrm{Kl}}(A_n^{\mathrm{Kl}})$.*

P r o o f. Since Kl is a symmetric regular bilinear form, i^{Kl} is an isomorphism. If $X \in A_n^{\mathrm{Kl}}$, i.e. $\mathrm{Kl}(X,Y) = 0$ for all $Y \in A_n(t_0)$ then $i^{\mathrm{Kl}}(X)(Y) = \mathrm{Kl}(X,Y) = 0$ and then $\alpha_X = i^{\mathrm{Kl}}(X) \in \Omega_{t_0}$. Conversely, if $\alpha \in \Omega_t$ then $0 = \alpha(Y) = \mathrm{Kl}((i^{\mathrm{Kl}})^{-1}(\alpha), Y)$ for all $Y \in A_n(t_0)$. Therefore, $(i^{\mathrm{Kl}})^{-1}(\alpha) \in A_n^{\mathrm{Kl}}$. $\square$

In the next part we will deal with the 3-parametric robots. We will describe what properties wrenches which do not act on the robot have.

5. Neutral wrenches of 3-parametric robots of the spherical rank 1

We will consider only robots with prismatic and revolute joints. The revolute (prismatic) joint of the robot will be designated by the capital letter R(T). Consequently, for example a symbol RTR denotes the 3-parametric robot whose first joint is revolute, the second is prismatic and the third is a revolute one. There is a base $Y_i = (\overline{\omega}_i, \bar{b}_i)^\top$, $i = 1,2,3$, in A_3 in each regular position of the robot. We will deal only with regular positions when $\dim A_3 = 3$. According to Definition 2.1, the twist $Y = (\overline{\omega}, \bar{b})^\top = t_1 Y_1 + t_2 Y_2 + t_3 Y_3 \in A_3$, $t_1, t_2, t_3 \in \mathbb{R}$, is the Klein twist iff

$$\mathrm{Kl}(Y, Y_i) = t_1 \mathrm{Kl}(Y_1, Y_i) + t_2 \mathrm{Kl}(Y_2, Y_i) + t_3 \mathrm{Kl}(Y_3, Y_i) = 0, \quad i = 1,2,3. \tag{3}$$

We will describe the set of neutral wrenches for individual types of the robots.

5.1. TTT—the robots without revolute joints. Now $Y_i = (\bar{0}, \bar{b}_i)^\top$ and $\mathrm{Kl}(Y_i, Y_j) = 0$, $i,j = 1,2,3$. Relation (3) results in $K_3 = A_3 = A_3^{\mathrm{Kl}} \cong R_3$. According to Proposition 4.2, we obtain

Proposition 5.1. *A wrench α is TTT-neutral iff it is a torque.*

5.2. RTT, TRT, TTR—the robots $R_{(1,2)}$ with one revolute joint. We can distinguish several situations that can occur in these robots. We will analyze them on a cases-by-case basis.

Remark. The core of our considerations are neutral wrenches of robots. We study mainly their geometric properties such as mutual positions of wrench axes and their positions with respect to joint axes. These properties are independent of the choice of a coordinate system. We use the coordinate system $\mathcal{S}$ whose origin O lies on the axis of the revolute joint in the centre of the joint construction. Under this condition we have the following base: $Y_1 = (\overline{\omega}_1, \bar{0})^\top$, $Y_2 = (\bar{0}, \bar{b}_2)^\top$, $Y_3 = (\bar{0}, \bar{b}_3)^\top$ in A_3.

Let $\tau = \operatorname{span}(\bar{b}_2, \bar{b}_3)$ denote the space of all translation directions. Now

$$\mathrm{Kl}(Y_1, Y_1) = 0, \quad \mathrm{Kl}(Y_1, Y_2) = \overline{\omega}_1 \cdot \bar{b}_2, \quad \mathrm{Kl}(Y_1, Y_3) = \overline{\omega}_1 \cdot \bar{b}_3, \quad \mathrm{Kl}(Y_i, Y_j) = 0$$

for $i, j = 2, 3$. There are two cases:

Case a) $\overline{\omega}_1 \cdot \bar{b}_2 = 0 = \overline{\omega}_1 \cdot \bar{b}_3$, i.e. the axis of the revolute joint is orthogonal to the space τ in each position $R(u_1(t), u_2(t), u_3(t))$. The explored robot is planar (the points of the effector do planar motions) and we denote it by the symbol $R_{(1,2)p}$. From the relation (3) we have $K_3 = A_3 = A_3^{\mathrm{Kl}}$. Therefore, according to Proposition 4.2 and Subsection 3.3 it follows that $\Omega = \operatorname{span}((\bar{0}, \overline{f} = \overline{\omega}_1), (\overline{m}_2 = \bar{b}_2, \bar{0}), (\overline{m}_3 = \bar{b}_3, \bar{0}))$. Each wrench $\alpha \in \Omega$ is of the form $\alpha = (t_2\bar{b}_2 + t_3\bar{b}_3, t_1\overline{\omega}_1)$ and therefore, α is either a torque ($t_1 = 0$), whose moment is orthogonal to the axis of the revolute joint, or the pure force ($t_1 \neq 0$) whose line is parallel to the axis of the revolute joint.

Proposition 5.2. *In the case of the 3-parametric robot $R_{(1,2)p}$ whose axis of the revolute joint is orthogonal to the axes of the prismatic joints, the wrench α is R-neutral iff it is either a torque whose moment is orthogonal to the revolute joint axis or a pure force whose line is parallel to the axis of the revolute joint.*

Proof. From the above mentioned consideration it follows that if the wrench α is neutral on the planar 3-parametric robot then it has the property described in Proposition 5.2. Conversely, if α is the torque whose moment is orthogonal to the axis of the revolute joint then it is of the form $\alpha = (t_2\bar{b}_2 + t_3\bar{b}_3, \bar{0})$ and then $\alpha \in \Omega$. If α is the pure force, whose line p is parallel to the axis of the revolute joint then according to Subsection 3.2 it is of the form $t_1(\overline{OC} \times \overline{\omega}_1, \overline{\omega}_1) = \alpha$, where C is the foot of the perpendicular to the line p from origin of the coordinate system, i.e. $\overline{OC} = c_2\bar{b}_2 + c_3\bar{b}_3$, where $c_2, c_3 \in \mathbb{R}$. Then $\overline{OC} \times \overline{\omega}_1 = c_2\bar{b}_2 \times \overline{\omega}_1 + c_3\bar{b}_3 \times \omega_1 = k_2\bar{b}_2 + k_3\bar{b}_3$. Therefore, $\alpha = (t_2\bar{b}_2 + t_3\bar{b}_3, t_1\overline{\omega}_1)$ and then $\alpha \in \Omega$. □

Corollary 5.1. *The wrench whose line is parallel to the axis of the revolute joint but which is not a pure force is not neutral on the planar 3-parametric robot.*

Now we turn to the second case.

Case b) The axis of the revolute joint is not orthogonal to the subspace $\tau = \operatorname{span}(\bar{b}_2, \bar{b}_3)$. Then $\overline{\omega}_1 \cdot \bar{b}_2 \neq 0$ or $\overline{\omega}_1 \cdot \bar{b}_3 \neq 0$. Now the system (3) has the form $t_2(\overline{\omega}_1 \cdot \bar{b}_2) + t_3(\overline{\omega}_1 \cdot \bar{b}_3) = 0$, $t_1(\overline{\omega}_1 \cdot \bar{b}_2) = 0$, $t_1(\overline{\omega}_1 \cdot \bar{b}_3) = 0$ and it has the solution $t_1 = 0$, $t_2 = k(\overline{\omega}_1 \cdot \bar{b}_3)$, $t_3 = -k(\overline{\omega}_1 \cdot \bar{b}_2)$, $k \in \mathbb{R}$. Therefore, the Klein subspace is

$$K_3 = \operatorname{span}(\bar{0}, (\overline{\omega}_1 \cdot \bar{b}_3)\bar{b}_2 - (\overline{\omega}_1 \cdot \bar{b}_2)\bar{b}_3). \tag{4}$$

The wrench $F = (\overline{m}, \overline{f})$ is $R_{(1,2)}$-neutral at the position $R(u_1(t), u_2(t), u_3(t))$ iff $F(Y_i) = 0$, $i = 1, 2, 3$, i.e. iff $\overline{\omega}_1 \cdot \overline{m} = 0$, $\bar{b}_2 \cdot \overline{f} = 0$, $\bar{b}_3 \cdot \overline{f} = 0$. Therefore, in the space $\Omega \subset se^*(3)$ of R-neutral wrenches we have the base $F_1 = (\bar{0}, \overline{f}_1 = \bar{b}_2 \times \bar{b}_3)$, $F_2 = (\overline{m}_2, \bar{0})$, $F_3 = (\overline{m}_3, \bar{0})$, where $\operatorname{span}(\overline{m}_2, \overline{m}_3)$ is the subspace orthogonal to $\overline{\omega}_1$, and then $\overline{m}_2$, $\overline{m}_3$ can be chosen so that $\overline{\omega}_1 = \overline{m}_2 \times \overline{m}_3$.

The pure force $F_1 = t(\bar{0}, \overline{f}_1)$, $t \in \mathbb{R}$, which is $R_{(1,2)}$-neutral is called the base force. Its line crosses the centre O of the revolute joint and is orthogonal to $\operatorname{span}(\bar{b}_2, \bar{b}_3)$. As according to Proposition 4.2, $\Omega = i^{\mathrm{Kl}}(A_3^{\mathrm{Kl}})$, and therefore $\{(\overline{f}_1, \bar{0}), (\bar{0}, \overline{m}_2), (\bar{0}, \overline{m}_3)\}$ is a base β in the space A_3^{Kl}. Since the Klein twist is Kl-orthogonal to the elements of the base β, by the procedure by which we deduced the relation (4), we get

$$K_3 = \operatorname{span}(\bar{0}, (\overline{f}_1 \cdot \overline{m}_3)\overline{m}_2 - (\overline{f}_1 \cdot \overline{m}_2)\overline{m}_3) \equiv \operatorname{span}(\bar{0}, \overline{m}_k). \tag{5}$$

The direction $\overline{m}_k = (\overline{f}_1 \cdot \overline{m}_3)\overline{m}_2 - (\overline{f}_1 \cdot \overline{m}_2)\overline{m}_3$ is called the *Klein direction.* We see that $\overline{m}_k \cdot \overline{f}_1 = \bar{0}$. The torque $i^{\mathrm{Kl}}(Y_k) = (\overline{m}_k, \bar{0}) = F_k$ is called the *Klein torque* of the the robot $R_{(1,2)}$. A wrench is $R_{(1,2)}$-neutral iff it is of the form

$$F = (t_2\overline{m}_2 + t_3\overline{m}_3, t_1\overline{f}_1), \quad t_1, t_2, t_3 \in \mathbb{R}. \tag{6}$$

Let us denote $\tau' :\equiv \operatorname{span}(\overline{m}_2, \overline{m}_3)$. Then a wrench $(\overline{m}, \overline{f}_1)$ is $R_{(1,2)}$-neutral iff $\overline{m} \in \tau'$. Evidently $\overline{f}_1 \in \tau'$ iff $\overline{f}_1 \cdot \overline{\omega}_1 = 0$, i.e. iff $\overline{\omega}_1 \in \tau$.

Now let us characterize the individual types of the $R_{(1,2)}$-neutral wrenches.

Case b 1) The following propositions result directly from the relation (6).

Proposition 5.3. *A torque is $R_{(1,2)}$-neutral iff its moment is orthogonal to the axis of the revolute joint. Generally, the wrench W is $R_{(1,2)}$-neutral if it is the sum of the base force F and the $R_{(1,2)}$-neutral torque.*

Case b 2) The case of pure forces. Let ζ be the plane incident with the point O and orthogonal to $\overline{f}_1$. Let λ be a plane incident with the axis of the revolute joint and orthogonal to $\operatorname{span}(\bar{b}_2, \bar{b}_3)$, i.e. parallel to $\overline{f}_1$. Evidently, the Klein direction $\overline{m}_k$ is its normal vector. Let us denote $s = \lambda \cap \zeta$.

Proposition 5.4. *If F is an $R_{(1,2)}$-neutral pure force, then its line is orthogonal to τ and lies in the plane λ. Conversely, if the robot $R_{(1,2)}$ has the property $\overline{\omega}_1 \notin \tau$ ($\overline{\omega}_1 \in \tau$) and a line p lies in the plane λ and is orthogonal to τ, then only pure forces (all forces) whose line is the line p are $R_{(1,2)}$-neutral.*

Proof. According to Subsection 3.2 from the relation (6) it follows that a neutral wrench is a pure force ($t_1 \neq 0$) iff $t_2(\overline{m}_2 \cdot \overline{f}_1) + t_3(\overline{m}_3 \cdot \overline{f}_1) = 0$, i.e. iff it is of the form $F = (t\overline{m}_k, t_1\overline{f}_1)$, where $\overline{m}_k$ is the Klein direction. The line p of such a wrench is parallel to $\overline{f}_1$ and is incident with the foot C of the perpendicular from O to p:

$$
\begin{aligned}
\text{(7)} \qquad \overline{OC} &= -\frac{t_1 t}{\overline{f}_1^2}\,\overline{m}_k \times \overline{f}_1 = -\frac{t_1 t}{\overline{f}_1^2}\overline{m}_k \times (\bar{b}_2 \times \bar{b}_3) \\
&= -\frac{t_1 t}{\overline{f}_1^2}\,[(\overline{m}_k \cdot \bar{b}_3)\bar{b}_2 - (\overline{m}_k \cdot \bar{b}_2)\bar{b}_3] = k\bar{b}_C, \quad k \in \mathbb{R},
\end{aligned}
$$

where $\bar{b}_C = (\overline{m}_k \cdot \bar{b}_3)\bar{b}_2 - (\overline{m}_k \cdot \bar{b}_2)\bar{b}_3$. Obviously $\bar{b}_C \cdot \overline{m}_k = 0$, $\bar{b}_C \cdot \overline{f}_1 = 0$. Therefore, $\bar{b}_C$ is a direction vector of the line $s = \lambda \cap \zeta$. Therefore, the point C lies on s and then the line p lies in λ. From the relation (7) it follows that each point of the line s is the foot C of the perpendicular from O to the line of an $R_{(1,2)}$-neutral pure force. Conversely, let a line p lie in the plane λ and let it be orthogonal to τ. Let C be the foot of the perpendicular from O to p. Evidently, $C \in s$ and therefore, there is such k that $\overline{OC} = k\bar{b}_C$. All wrenches with the line p are of the form $W = (\overline{OC} \times t_1\overline{f}_1 + t_2\overline{f}_1, t_1\overline{f}_1)$. The relation (7) yields $\overline{OC} \times \overline{f}_1 = -\overline{f}_1 \times (k\bar{b}_C) = -k\overline{f}_1 \times (\overline{m}_k \times (\bar{b}_2 \times \bar{b}_3)) = -kf_1^2\overline{m}_k$ and then $W = (-kt_1f_1^2\overline{m}_k + t_2\overline{f}_1, t_1\overline{f}_1)$. As $-kt_1f_1^2\overline{m}_k \in \tau'$, we have $-kt_1f_1^2\overline{m}_k + t_2\overline{f}_1 \in \tau'$ iff either $t_2 = 0$ when $\overline{f}_1 \notin \tau'$, or $\overline{f}_1 \in \tau'$, i.e. $\overline{\omega}_1 \in \tau$. In the former case W is a pure force, in the latter every $W \in \Omega_t$. The proof is complete. □

Remark. The case $\overline{f}_1 \cdot \overline{\omega}_1 = 0$, i.e. $\overline{\omega}_1 \in \tau = \operatorname{span}(\bar{b}_2, \bar{b}_3)$ will be dealt with in detail in Proposition 5.6.

Case b 3) We will investigate when a general wrench $W_g = (\overline{m}, \overline{f})$, $\overline{f} \neq \bar{0}$, $\overline{m}\cdot\overline{f} \neq 0$ is $R_{(1,2)}$-neutral. We will describe the set of all foots C of the perpendiculars from the origin O to the lines of neutral wrenches W_g. By the relation (6), W_g has the vectors of the form $\overline{m} = t_2\overline{m}_2 + t_3\overline{m}_3$, $\overline{f} = t_1\overline{f}_1$, $t_1 \neq 0$ and therefore, according to Subsection 3.2,

$$
\text{(8)} \qquad \overline{OC} = \frac{\overline{f} \times \overline{m}}{f^2} = \frac{t_1\overline{f}_1 \times (t_2\overline{m}_2 + t_3\overline{m}_3)}{t_1^2 f_1^2} = \frac{1}{t_1 f_1^2}(t_2\overline{f}_1 \times \overline{m}_2 + t_3\overline{f}_1 \times \overline{m}_3).
$$

There are two cases:

Case b 3.1) $\overline{f}_1 \times \overline{m}_2$, $\overline{f}_1 \times \overline{m}_3$ are linearly independent. This case occurs iff $\overline{f}_1 \in \mathrm{span}(\overline{m}_2, \overline{m}_3)$, i.e. iff $\overline{\omega}_1 \in \mathrm{span}(\bar{b}_2, \bar{b}_3)$, because $\overline{f}_1$ is orthogonal to $\mathrm{span}(\bar{b}_2, \bar{b}_3)$ and the vector $\overline{\omega}_1$ is orthogonal to $\mathrm{span}(\overline{m}_2, \overline{m}_3)$. Now $\overline{OC} = k\overline{v}$, $\overline{v} = \overline{f}_1 \times \overline{m}_2$ and $\overline{v} \cdot \overline{f}_1 = 0$, $\overline{v} \cdot \overline{m}_2 = 0$, $\overline{v} \cdot \overline{m}_3 = 0$. As also $\overline{\omega}_1 \cdot \overline{f}_1 = 0$, $\overline{\omega}_1 \cdot \overline{m}_2 = 0$, $\overline{\omega}_1 \cdot \overline{m}_3 = 0$, therefore the vectors $\overline{v}$, $\overline{\omega}_1$ are linearly independent. Hence, the set of points C is identified with the axis o of the revolute joint and thus $o = s = \lambda \cap \zeta$. The lines of wrenches W_g are lying in λ. Denote the robot with the property $\overline{\omega}_1 \in \mathrm{span}(\bar{b}_2, \bar{b}_3)$ by the symbol $R_{(1,2)_\kappa}$. We get

Proposition 5.5. *A general wrench W_g is $R_{(1,2)_\kappa}$-neutral iff its line is orthogonal to* $\mathrm{span}(\bar{b}_2, \bar{b}_3)$ *and lies in λ.*

P r o o f. In the text above Proposition 5.5 it is shown that if W_g is $R_{(1,2)_\kappa}$-neutral then its line lies in the plane λ. We will show that each wrench with the line p is $R_{(1,2)_\kappa}$-neutral. The line p crosses the axis of the revolute joint at the point C which is the foot of the perpendicular from O to p. Therefore, $\overline{OC} = k\overline{\omega}_1$. Each wrench W which has the line p is of the form $(t_1\overline{OC} \times \overline{f}_1 + t_2\overline{f}_1, t_1\overline{f}_1) = (t_1\overline{\omega}_1 \times \overline{f}_1 + t_2\overline{f}_1, t_1\overline{f}_1)$. But $t_2\overline{f}_1 \in \mathrm{span}(\overline{m}_2, \overline{m}_3)$ and $\overline{\omega}_1 \times \overline{f}_1 \in \mathrm{span}(\overline{m}_2, \overline{m}_3)$ because $\overline{\omega}_1 \cdot \overline{m}_2 = 0 = \overline{\omega}_1 \cdot \overline{m}_3$ and $\overline{f}_1 \in \mathrm{span}(\overline{m}_2, \overline{m}_3)$. Consequently, $W_g \in \Omega$. The proof is completed. □

Corollary 5.2 (of Propositions 5.4, 5.5). *The wrench W which is not a torque is $R_{(1,2)_\kappa}$-neutral iff its line is orthogonal to* $\mathrm{span}(\bar{b}_2, \bar{b}_3)$ *and lies in the plane λ. Let us recall that in this case the lines of the neutral wrenches cross the axis of the revolute joint orthogonally (because $\overline{\omega}_1 \in \mathrm{span}(\bar{b}_2, \bar{b}_3) = \tau$).*

Case b 3.2) Let $\overline{f}_1 \times \overline{m}_2$, $\overline{f}_1 \times \overline{m}_3$ be linearly independent, i.e. $\overline{\omega}_1 \notin \mathrm{span}(\bar{b}_2, \bar{b}_3)$. Then $\overline{OC} = k_1\overline{f}_1 \times \overline{m}_2 + k_2\overline{f}_1 \times \overline{m}_3$ and then the foots C of the perpendiculars from O to the axes of the wrenches W_g which are $R_{(1,2)}$-neutral fill the whole plane π. For each line p which is perpendicular to $\mathrm{span}(\bar{b}_2, \bar{b}_3)$ there are $R_{(1,2)}$-neutral wrenches whose lines are identical with p. If the line p lies in the plane λ then just pure forces with the line p are neutral. If the line p does not lie in the plane λ then wrenches W_g which are $R_{(1,2}$-neutral cannot be pure forces. We will try to characterize them.

Let the line p be orthogonal to $\mathrm{span}(\bar{b}_2, \bar{b}_3)$ and let it cross π at the point C, which is the foot of the perpendicular from O to p. Then $F = (\overline{OC} \times \overline{f}_1, \overline{f}_1)$ is the pure force which has the line p. This is $R_{(1,2)}$-neutral iff $C \in \lambda \cap \pi$. Each wrench which has the line p is of the form $W = (t\overline{OC} \times \overline{f}_1 + h\overline{f}_1, t\overline{f}_1)$, $t, h \in \mathbb{R}$. This is $R_{(1,2)}$-neutral iff $t\overline{OC} \times \overline{f}_1 + k\overline{f}_1 \in \mathrm{span}(\overline{m}_2, \overline{m}_3)$, i.e. iff $(t\overline{OC} \times \overline{f}_1 + h\overline{f}_1) \cdot \overline{\omega}_1 = 0$, i.e. $h(\overline{f}_1 \cdot \overline{\omega}_1) = t(\overline{f}_1 \times \overline{OC}) \cdot \overline{\omega}_1$, i.e. $h = t((\overline{f}_1 \times \overline{OC}) \cdot \overline{\omega}_1)/\overline{f}_1 \cdot \overline{\omega}_1$. We have proved

Proposition 5.6. *Let the robot $R_{(1,2)}$ have the rotational axis neither parallel to the subspace* $\operatorname{span}(\bar{b}_2, \bar{b}_3)$ *nor orthogonal to this space. Let W be a wrench which is neither a torque nor a pure force. Then W is $R_{(1,2)}$-neutral just when it is the sum of a pure force F whose line is not incident with the plane λ and is orthogonal to $\tau = \operatorname{span}(\bar{b}_2, \bar{b}_3)$, and of such a torque whose moment $\overline{m}$ is orthogonal to τ that the sum of the moment of the force F at the centre O of the revolute joint and the moment of the torque is orthogonal to the axis of the revolute joint.*

5.3. 3-parametric robots $R_{(2,1)r}$ whose axes of two revolute joints are parallel. According to the remark in Subsection 5.2, the joints in $R_{(2,1)r}$ determine the twists $Y_1 = (\overline{\omega}_1, \bar{0})^\top$, $Y_2 = (\overline{\omega}_1, \bar{b}_2)^\top$, $\overline{\omega}_1 \cdot \bar{b}_2 = 0$, $Y_3 = (\bar{0}, \bar{b}_3)^\top$ which generate the base in A_3. Let us denote by ξ the plane determined by the parallel axes o_1, o_2 of the revolute joints. The twist Y_1 is determined by the revolute joint whose centre is chosen as the origin O of the coordinate system. Now $\bar{b}_2$ is the velocity of the point O (and hence of all points of the rotational axis o_1) of the rotation around the axis o_2 of the second revolute joint. Therefore, $\bar{b}_2$ is the normal vector of the plane ξ. We put $B_2 = Y_2 - Y_1 = (\bar{0}, \bar{b}_2)^\top$. In A_3 we have the base Y_1, B_2, Y_3 which is the same as in the case of robots $R_{(1,2)}$. Consequently, the structure of the set Ω of the $R_{(2,1)r}$-neutral wrenches is the same as in the case of robots $R_{(1,2)}$. Now the subspace $\tau = \operatorname{span}(\bar{b}_2, \bar{b}_3)$ is orthogonal to the plane ξ. Hence, $\bar{b}_2 \times \bar{b}_3 = \overline{f}_1$ is parallel to ξ and therefore $\xi = \lambda$. Therefore, for the Klein direction we have $\bar{b}_k = \overline{m}_k = \bar{b}_2$. Propositions 5.2, 5.3, 5.4, 5.5, 5.6 which describe the properties of wrenches from the set Ω are valid also for the set Ω of the robot $R_{(2,1)r}$ if instead of symbols $R_{(1,2)}$, $R_{(1,2)p}$, $R_{(1,2)\kappa}$ we use the corresponding symbols $R_{(2,1)}$, $R_{(2,1)r}$, $R_{(2,1)\kappa}$.

5.4. 3-parametric robots $R_{(3,0)r}$ of the spherical rank 1 with three revolute joints. All three axes of the revolute joints are parallel. The twists determined by the joints have the coordinates $Y_1 = (\overline{\omega}, \bar{0})^\top$, $Y_2 = (\overline{\omega}, \bar{b}_2)^\top$, $\overline{\omega} \cdot \bar{b}_2 = 0$, $Y_3 = (\overline{\omega}, \bar{b}_3)^\top$, $\overline{\omega} \cdot \bar{b}_3 = 0$. Let o_1, o_2, o_3 be the axes of these twists. They are parallel and the origin O of the coordinate system lies on o_1 in the centre of the joint which has the axis o_1. Let us denote the plane determined by the parallel lines o_1, o_2 as ξ_1. And let us denote the plane determined by the parallel lines o_1, o_3 as ξ_2. Since $\bar{b}_2$ or $\bar{b}_3$ is the velocity of the point O of the rotation around the axis o_1 or o_2, therefore $\bar{b}_2$ or $\bar{b}_3$ is orthogonal to ξ_1 or ξ_2, respectively. Let us denote $B_2 :\equiv Y_2 - Y_1 = (\bar{0}, \bar{b}_2)^\top$, $B_3 :\equiv Y_3 - Y_1 = (\bar{0}, \bar{b}_3)^\top$. We assume that $\xi_1 \neq \xi_2$. Then Y_1, B_2, B_3 generate a base in A_3. Now the subspace $\tau = \operatorname{span}(\bar{b}_2, \bar{b}_3)$ is orthogonal to $\overline{\omega}_1 = \overline{\omega}$ and then the structure of the space Ω of the $R_{(3,0)r}$-neutral wrenches is the same as in the case of the robot $R_{(1,2)p}$, i.e. Proposition 5.2 is valid if instead of the symbol $R_{(1,2)}$ we use $R_{(3,0)r}$.

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