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ON REGULAR ENDOMORPHISM RINGS OF TOPOLOGICAL ABELIAN GROUPS

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Abstract. We extend a result of Rangaswamy about regularity of endomorphism rings of Abelian groups to arbitrary topological Abelian groups. Regularity of discrete quasi-injective modules over compact rings modulo radical is proved. A characterization of torsion LCA groups A for which $\mathrm{End}_c(A)$ is regular is given.

Keywords: m-regular ring, discrete module, quasi-injective module, linearly compact group, LCA group, local product

MSC 2010: 16E50, 16S50

1. Introduction

Regular in the sense of von Neumann rings form an important subclass of the class of associative rings. Recall that a ring R is called *regular* in the sense of von Neumann if for every $a \in R$ there exists $b \in R$ such that $aba = a$. We study in this paper the following problems:

(i) What are the topological Abelian groups in which every endomorphic image is a direct summand?

(ii) What are the locally compact Abelian groups for which the ring $\mathrm{End}_c(A)$ is regular?

Rangaswamy studied analogous problems for abstract Abelian groups [8]. We note that Problem (i) has no complete answer in the class of abstract Abelian groups. We give a characterization of arbitrary topological Abelian groups whose rings $\mathrm{End}_c(A)$ are m-regular. A complete characterization of torsion (in abstract sense) LCA groups

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with regular $\mathrm{End}_c(A)$ is given (Theorem 3.13). We give examples of: (i) a LCA group $(A,\mathcal{T})$ for which $\mathrm{End}(A)$ is not regular but $\mathrm{End}_c(A)$ is regular; (ii) a LCA group $(A,\mathcal{T})$ for which $\mathrm{End}(A)$ is regular but $\mathrm{End}_c(A)$ is not.

We give a nontrivial example of a linearly compact Abelian group whose ring of continuous endomorphisms is regular. We indicate a natural ring topology for $\mathrm{End}_c(A)$ for any linearly compact Abelian group which is analogous to the compact-open topology.

2. Notation and conventions

$\mathbb{P}$ stands for the set of all positive prime numbers and $\mathbb{N}$ is the set $\{1,2,3,\ldots\}$. If $m,n\in\mathbb{N}$, $m\leqslant n$, then $[m,n]:=\{m,m+1,\ldots,n\}$. If $f\colon X\to Y$ is a mapping and $Z\subseteq X$, then by $f\restriction_Z$ we denote the restriction of f on Z. All topological rings are assumed to be Hausdorff, associative and with identity. Topological groups are assumed to be Hausdorff by default. If A is an internal direct sum of subgroups B and C, we write $A=B\oplus C$. $R_1\cong_{\mathrm{top}}R_2$ means that the topological rings R_1 and R_2 are isomorphic. We denote by $\mathrm{End}_c(A)$ the ring of all continuous endomorphisms of a topological Abelian group A. No topology on $\mathrm{End}_c(A)$ is assumed. When A is a locally compact Abelian group (briefly, LCA), the ring $\mathrm{End}_c(A)$ furnished with the compact-open topology is a topological ring. The additive group of a ring R is denoted by $R(+)$ and the center by $Z(R)$. Let $\{R_\alpha\}_{\alpha\in\Omega}$ be a family of topological rings. Fix for each $\alpha\in\Omega$ an open subring $S_\alpha\subseteq R_\alpha$. Consider the subring $A\subseteq\prod\limits_{\alpha\in\Omega}R_\alpha$ of the Cartesian product of R_α, $A=\{(x_\alpha)\in\prod\limits_{\alpha\in\Omega}R_\alpha\mid x_\alpha\in S_\alpha$ for almost all $\alpha\in\Omega\}$. Then the product $\prod\limits_{\alpha\in\Omega}S_\alpha$ of topological rings $S_\alpha,\alpha\in\Omega$, defines a ring topology on A. This ring A is called the *local direct product* of R_α with respect to S_α, $\alpha\in\Omega$ and is denoted by $\prod\limits_{\alpha\in\Omega}(R_\alpha\colon S_\alpha)$ (see [2], p. 46 and [10], p. 211). It is noteworthy that if all S_α are locally compact and almost all S_α are compact, then the ring $\prod\limits_{\alpha\in\Omega}(R_\alpha\colon S_\alpha)$ is locally compact.

3. Regular rings of continuous endomorphisms of LCA groups

Lemma 3.1. *Let A be a topological Abelian not necessarily Hausdorff group and α an idempotent of $\mathrm{End}_c(A)$. Then $\operatorname{im}\alpha$ and $\ker\alpha$ are direct summands of A.*

Proof. Since $a=\alpha a+a-\alpha a\in\operatorname{im}\alpha+\ker\alpha$ for each $a\in A$, we obtain that $A=\operatorname{im}\alpha+\ker\alpha$. If $x\in\operatorname{im}\alpha\cap\ker\alpha$, then there exists $y\in A$ such that $x=\alpha y$, hence $x=\alpha x=0=\alpha^2y=\alpha y$. Therefore $A=\operatorname{im}\alpha\oplus\ker\alpha$ (a direct sum in the algebraic sense).

We claim that this decomposition is topological. It suffices to show that $U\cap\alpha A+U\cap(1-\alpha)A$ is a neighborhood of 0_A for every neighborhood U of 0_A. Let V be a neighborhood of 0_A such that $\alpha V\subseteq U$, $(1-\alpha)V\subseteq U$. Then $v=\alpha v+(1-\alpha)v\in U\cap\alpha A+U\cap(1-\alpha)A$ for every $v\in V$, hence $V\subseteq(U\cap\alpha A)+(U\cap(1-\alpha)A)$, which implies that the last set is a 0-neighborhood for A. □

Recall that an element a of a ring is called *m-regular* if there exists a positive integer m such that a^m is regular. A ring is *π-regular* if each of its elements is m-regular. A ring is called *m-regular* if all its elements are m-regular for a fixed m (see [5], p. 239).

Theorem 3.2. *Let A be a topological Abelian group and $\alpha\in\operatorname{End}_c(A)$. Then α is an m-regular element if and only if $\operatorname{im}\alpha^m$ and $\ker\alpha^m$ are topological direct summands of A.*

P r o o f. ($\Rightarrow$) There exists $\beta\in\operatorname{End}_c(A)$ such that $\alpha^m=\alpha^m\beta\alpha^m$. Then

$$\operatorname{im}\alpha^m=\alpha^m A=\alpha^m\beta\alpha^m A\subseteq\operatorname{im}(\alpha^m\beta)\subseteq\operatorname{im}\alpha^m,$$

which implies $\operatorname{im}\alpha^m=\operatorname{im}(\alpha^m\beta)$. According to Lemma 3.1 $\operatorname{im}\alpha^m$ is a topological direct summand.

Furthermore, $\ker\alpha^m\subseteq\ker(\beta\alpha^m)\subseteq\ker\alpha^m$, therefore $\ker\alpha^m=\ker(\beta\alpha^m)$. Lemma 3.1 implies that $\ker(\beta\alpha^m)$ is a topological direct summand.

($\Leftarrow$) Let $B=\operatorname{im}\alpha^m$, $C=\ker\alpha^m$. There exists a subgroup D of A such that $A=C\oplus D$ is a direct topological sum of subgroups C and D. Then $\alpha^m\restriction_D$ is a continuous isomorphism of D on B:

$$\operatorname{im}\alpha^m=\alpha^m A=\alpha^m(C\oplus D)=\alpha^m D,$$

therefore $(\alpha^m\restriction_D)D=B$. If $x\in D$, $(\alpha^m\restriction_D)x=0$, then $\alpha^m x=0$, hence $x\in C\cap D=0$, and so $\alpha^m\restriction_D$ is an isomorphism of D on B.

The mapping $\alpha^m\restriction_D$ is open on its image: Indeed, the mapping

$$A\to\alpha^m A,$$
$$a\mapsto\alpha^m a,$$

is open since it coincides with the projection of A to $\alpha^m A$. Furthermore, if W is a neighborhood of 0_D, then $C+W$ is a neighborhood of 0_A. It follows that there exists a neighborhood U of 0_A such that $\alpha^m(C+W)\supseteq U\cap\alpha^m A=U\cap B$, which implies $\alpha^m W\supseteq U\cap B$, hence $(\alpha^m\restriction_B)W\supseteq U\cap B$ and so $\alpha^m\restriction_D$ is open on its image.

Define a topological isomorphism $\gamma\colon B \to D, b \mapsto (\alpha^m \restriction_D)^{-1} b$. Let $\theta \in \mathrm{End}_c(A)$ be a continuous extension of γ. Then $\alpha^m \theta \alpha^m = \alpha^m$.

Indeed, if $a \in A$, then $\alpha^m a \in B$; let $a = c + d, c \in C$ and $d \in D$. Then

$$\alpha^m a = \alpha^m d = (\alpha^m \restriction_D) d, \text{ hence } d = (\alpha^m \restriction_D)^{-1}(\alpha^m a) = \gamma(\alpha^m a) = \theta \alpha^m a,$$

therefore $\alpha^m a = \alpha^m \theta \alpha^m a$. Since a was arbitrary, $\alpha^m = \alpha^m \theta \alpha^m$. □

Corollary 3.3. *The ring* $\mathrm{End}_c(A)$ *of all continuous endomorphisms of a topological Abelian group A is π-regular iff for every $\alpha \in \mathrm{End}_c(A)$ there exists a positive integer m such that* $\operatorname{im} \alpha^m$ *and* $\ker \alpha^m$ *are topological direct summands of A.*

Corollary 3.4. *The ring* $\mathrm{End}_c(A)$ *of all continuous endomorphisms of a topological Abelian group A is regular if and only if the image and the kernel of every endomorphism are direct summands of the group.*

Lemma 3.5. *Let A be a topological Abelian group and let* $\mathrm{End}_c(A)$ *be regular. If $p \in \mathbb{P}$, $x \in A$ and $p^2 x = 0$ then $px = 0$.*

Proof. Indeed, consider $\varepsilon\colon A \to A, a \mapsto pa$. There exists $\beta \in \mathrm{End}_c(A)$ such that $\varepsilon = \varepsilon\beta\varepsilon$. Since ε is in the center of $\mathrm{End}_c(A)$, $\varepsilon = \beta\varepsilon^2$. Then $p^2 x = 0$ implies $\varepsilon^2 x = 0$, hence

$$\beta\varepsilon^2 x = \varepsilon x = px = 0.$$

□

Corollary 3.6. *If A is a topological Abelian group,* $\mathrm{End}_c(A)$ *is regular and A is a p-group then $px = 0$ for every $x \in A$.*

Theorem 3.7. *Let A be a torsion free LCA-group. If* $\mathrm{End}_c(A)$ *is regular then* $\mathrm{End}_c(A)(+)$ *is divisible.*

Proof. Let $0 \neq n \in \mathbb{N}$ and $\alpha \in \mathrm{End}_c(A)$. Let $\varepsilon_n\colon A \to A$, $a \mapsto na$. Then $\varepsilon_n \in Z(\mathrm{End}_c(A))$. There exists $\beta \in \mathrm{End}_c(A)$ such that $\varepsilon_n \alpha = \varepsilon_n \alpha \beta \varepsilon_n \alpha$. If $a \in A$, we obtain that $\varepsilon_n(\alpha - \varepsilon_n \alpha\beta\alpha)a = 0$, hence $n[(\alpha - \varepsilon_n\alpha\beta\alpha)a] = 0$. Since $n \neq 0$, we obtain $(\alpha - \varepsilon_n\alpha\beta\alpha)a = 0$. Then $\alpha = n(\alpha\beta\alpha)$, therefore $\mathrm{End}_c(A)$ is divisible. □

Corollary 3.8. *If* $\mathrm{End}_c(A)(+)$ *is divisible then* A *is a divisible group.*

Proof. For $1_A = 1 \in \mathrm{End}_c(A)$ and $n \in \mathbb{N}$ there exists $\beta \in \mathrm{End}_c(A)$ such that $1 = n\beta$. If $a \in A$, then $a = n(\beta a)$, hence $A(+)$ is divisible. □

Example 3.9. Let $p \in \mathbb{P}$ and $A = \prod\limits_{i\in\omega}(R_i\colon\ S_i)$ where $R_i = \mathbb{Q}_p, S_i = \mathbb{Z}_p$. Then $\mathrm{End}_c(A)$ is not regular.

According to Corollary 3.8 it suffices to show that A is not divisible. Let $a = (1) \in \prod\limits_{i\in\omega} S_i \subseteq A$. Assume that there exists $x \in A$ such that $px = a$. Then $x = (x_i)$ and there exists n_0 such that $x_i \in \mathbb{Z}_p$ for $i \geqslant n_0$. Then $px_i = 1$, a contradiction.

Recall that a subgroup B of a topological group A is called *fully invariant* provided $\alpha B \subseteq B$ for every continuous endomorphism α of A.

Lemma 3.10. *If the locally compact Abelian group* A *is a direct sum* $A = A_1 \oplus A_2$ *of fully invariant subgroups* A_1 *and* A_2 *then* $\mathrm{End}_c(A) \cong_{\mathrm{top}} \mathrm{End}_c(A_1) \times \mathrm{End}_c(A_2)$.

Proof. Define $f\colon\ \mathrm{End}_c(A) \to \mathrm{End}_c(A_1) \times \mathrm{End}_c(A_2)$, $\alpha \mapsto (\alpha\restriction_{A_1}, \alpha\restriction_{A_2})$.

f is a morphism:

$$\begin{aligned} f(\alpha+\beta) &= ((\alpha+\beta)\restriction_{A_1}, (\alpha+\beta)\restriction_{A_2}) \\ &= (\alpha\restriction_{A_1} + \beta\restriction_{A_1}, \alpha\restriction_{A_2} + \beta\restriction_{A_2}) \\ &= (\alpha\restriction_{A_1}, \alpha\restriction_{A_2}) + (\beta\restriction_{A_1}, \beta\restriction_{A_2}) \\ &= f\alpha + f\beta; \\ f(\alpha\beta) &= ((\alpha\beta)\restriction_{A_1}, (\alpha\beta)\restriction_{A_2}) \\ &= (\alpha\restriction_{A_1}\beta\restriction_{A_1}, \alpha\restriction_{A_2}\beta\restriction_{A_2}) \\ &= (\alpha\restriction_{A_1}, \alpha\restriction_{A_2})(\beta\restriction_{A_1}, \beta\restriction_{A_2}) \\ &= f\alpha f\beta. \end{aligned}$$

f is injective: Let $f\alpha = 0$. Then $(\alpha\restriction_{A_1}, \alpha\restriction_{A_2}) = 0$ which implies that $\alpha = 0$.

f is surjective: Obviously.

f is continuous: Let $T(K_1, V_1) \times T(K_2, V_2)$ be a canonical neighborhood of 0 of $\mathrm{End}_c(A_1) \times \mathrm{End}_c(A_2)$, where K_i is a compact subset of A_i and V_i is a neighborhood of 0 of A_i, $i \in [1,2]$. We can consider without loss of generality that $0 \in K_1 \cap K_2$. If $k_1 \in K_1$, then $\alpha\restriction_{A_1}(k_1) \in \alpha(K) \subseteq V_1$, hence $\alpha\restriction_{A_1}(k_1) \in T(K_1, V_1)$. We have proved that $fT(K,V) \subseteq T(K_1, V_1) \times T(K_2, V_2)$.

f is open on its image: Let $T(K,V)$ be a canonical neighborhood of 0 of $\mathrm{End}_c(A)$. We can assume that $0 \in K$. Let U be a neighborhood of 0 such that $U + U \subseteq V$. Then $fT(K,V) \supseteq T(K_1, U \cap A_1) \times T(K_2, U \cap A_2)$, where $K_i = \pi_i(K)$ and π_i is the projection of A to $A_i, i \in [1,2]$. Indeed, let $(\alpha_1, \alpha_2) \in T(K_1, U \cap A_1) \times T(K_2, U \cap A_2)$. We have $f\alpha = (\alpha_1, \alpha_2)$, where $\alpha(a_1, a_2) = (\alpha_1(a_1), \alpha_2(a_2))$. We claim that $\alpha \in T(K, V)$. Indeed, let $k = (k_1, k_2) \in K$. Then

$$\begin{aligned}\alpha(k) &= (\alpha_1(k_1), \alpha_2(k_2)) \\ &= (\alpha_1(k_1), 0) + (0, \alpha_2(k_2)) \in U + U \subseteq V, \text{ hence } \alpha \in T(K, V).\end{aligned}$$

$\square$

Rangaswamy [8] has proved that the ring $\mathrm{End}(A)$ where $A = \mathbb{R} \times \prod\limits_{p \in \mathbb{P}} \mathbb{Z}(p)$ is not regular. We claim that if A is viewed as a topological group with the product topology where $\mathbb{R}$ is furnished with the usual topology and each $\mathbb{Z}(p)$ with the discrete topology, then $\mathrm{End}_c(A)$ is regular. Indeed, the subgroups $\mathbb{R}$ and $\prod\limits_{p \in \mathbb{P}} \mathbb{Z}(p)$ are fully invariant. According to Lemma 3.10 $\mathrm{End}_c(A) \cong_{\mathrm{top}} \mathrm{End}_c(\mathbb{R}) \times \mathrm{End}_c\Big(\prod\limits_{p \in \mathbb{P}} \mathbb{Z}(p)\Big)$. The topological rings $\mathrm{End}_c\Big(\prod\limits_{p \in \mathbb{P}} \mathbb{Z}(p)\Big)$ and $\mathrm{End}\Big(\Big(\prod\limits_{p \in \mathbb{P}} \mathbb{Z}(p)\Big)^*\Big)$ are anti-isomorphic [9]. It follows from the duality theory that $\Big(\prod\limits_{p \in \mathbb{P}} \mathbb{Z}(p)\Big)^* \cong \bigoplus\limits_{p \in \mathbb{P}} \mathbb{Z}(p)^* \cong \bigoplus\limits_{p \in \mathbb{P}} \mathbb{Z}(p)$. Each subgroup $\mathbb{Z}(p)$ of $\bigoplus\limits_{p \in \mathbb{P}} \mathbb{Z}(p)$ is fully invariant. According to ([1], Proposition 1) $\mathrm{End}_c\Big(\bigoplus\limits_{p \in \mathbb{P}} \mathbb{Z}(p)\Big)$ is topologically isomorphic to $\prod\limits_{p \in \mathbb{P}} \mathrm{End}(\mathbb{Z}(p)) \cong_{\mathrm{top}} \prod\limits_{p \in \mathbb{P}} \mathbb{F}_p$. It is well-known that $\mathrm{End}_c(\mathbb{R}) \cong_{\mathrm{top}} \mathbb{R}$, therefore the ring $\mathrm{End}_c(A)$ is regular.

We will give now an example of a group whose ring of all endomorphisms is regular, but the ring of all continuous endomorphisms is not regular.

Proposition 3.11. *Let* $A = (\mathbb{Z}/p\mathbb{Z})^{\mathfrak{m}} \oplus \Big(\bigoplus\limits_{\mathfrak{n}} \mathbb{Z}/p\mathbb{Z}\Big)$ *where* $p \in \mathbb{P}$ *and* $\mathfrak{m}, \mathfrak{n}$ *are infinite cardinal numbers. Then the ring* $\mathrm{End}_c(A)$ *is not regular.*

P r o o f. We can assume without loss of generality that $\mathfrak{m} = \mathfrak{n} = \omega$. Since $\mathbb{Z}(p)^{\mathfrak{m}}$ is separable, there exists an element $\gamma \in \mathrm{End}_c(A)$ whose image is not closed. According to Corollary 3.3 $\mathrm{End}_c(A)$ is not regular. $\square$

Remark 3.12. If $A = (\mathbb{Z}/p\mathbb{Z})^{\mathfrak{m}} \oplus \Big(\bigoplus\limits_{\mathfrak{n}} \mathbb{Z}/p\mathbb{Z}\Big)$ where $p \in \mathbb{P}$ and $\mathfrak{m}, \mathfrak{n}$ are infinite cardinal numbers, then the ring $x\mathrm{End}_c(A)$ is prime.

Indeed, let $\alpha, \beta \in \mathrm{End}_c(A)$ be such that $\alpha\,\mathrm{End}_c(A)\beta = 0$. Assume that $\beta \neq 0$. Let $x \in A$ be such that $\beta x \neq 0$. We claim that $\alpha = 0$. Assume the contrary; there exists $y \in A$ such that $\alpha y \neq 0$. Let $\gamma \in \mathrm{End}_c(A)$ be such that $\gamma(\beta x) = y$. The existence of γ follows from the properties of the group A. Then $\alpha\gamma\beta(x) = \alpha(y) \neq 0$, a contradiction.

Open question: For which LCA groups the rings $\mathrm{End}(A)$ and $\mathrm{End}_c(A)$ are simultaneously regular?

Let A be an Abelian group and $p \in \mathbb{P}$. Then by A_p we denote the p-primary component of A.

Theorem 3.13. *Let A be a torsion in the abstract sense LCA group and $A = \bigoplus\limits_{p\in\mathbb{P}} A_p$ a decomposition in p-primary components. The following statements are equivalent:*

(i) $\mathrm{End}_c(A)$ *is regular;*

(ii) *every A_p is an elementary group and there exists a finite subset $P_0 \subset \mathbb{P}$ such that $A = \Big(\prod\limits_{p\in P_0} A_p\Big) \oplus \Big(\bigoplus\limits_{q\in\mathbb{P}\setminus P_0} A_q\Big)$ (a topological decomposition), where A_p, $p \in P_0$, are infinite compact groups and $\bigoplus\limits_{q\in\mathbb{P}\setminus P_0} A_q$ a discrete subgroup.*

P r o o f. (i $\Rightarrow$ ii) It is well-known that $A = \bigoplus\limits_{p\in\mathbb{P}} A_p$ (an algebraic direct sum). Since A is a torsion locally compact group, it is totally disconnected. Let V be a compact open subgroup of A. There exists $k \in \mathbb{N}$ such that $kx = 0$ for all $x \in V$. This implies that there exists $n \in \mathbb{N}$ such that $V \cap \Big(\bigoplus\limits_{i\geqslant n+1} A_{p_i}\Big) = 0$. Then $V = (V \cap A_{p_1}) \oplus \ldots \oplus (V \cap A_{p_n})$. We can assume without loss of generality that each $A_{p_i}, i \in [1, n]$ is non-discrete. This implies that the topological group A is a direct product of its compact open subgroup $\prod\limits_{p\in P_0} A_p$ and the discrete subgroup $\bigoplus\limits_{q\in\mathbb{P}\setminus P_0} A_q$. According to Corollary 3.6, every group A_p is an elementary group.

(ii $\Rightarrow$ i) Let $A = A_{p_1} \oplus \ldots \oplus A_{p_n} \oplus \Big(\bigoplus\limits_{i\geqslant n+1} A_{p_i}\Big)$ be a topological decomposition of A where $A_{p_1}, \ldots, A_{p_n}$ are non-discrete compact elementary groups. Then each of the subgroups $A_{p_1}, \ldots, A_{p_n}, \bigoplus\limits_{i\geqslant n+1} A_{p_i}$ is fully invariant. By Lemma 3.10 $\mathrm{End}_c(A)$ is topologically isomorphic to $\mathrm{End}_c(A_{p_1}) \times \ldots \times \mathrm{End}_c(A_{p_n}) \times \mathrm{End}_c\Big(\bigoplus\limits_{i\geqslant n+1} A_{p_i}\Big)$. By [9], $\mathrm{End}_c(A_{p_i})$ is topologically anti-isomorphic to $\mathrm{End}_c(A^*_{p_i})$, $i \in [1, n]$. Since $\mathrm{End}_c(A^*_{p_i})$ is regular, the ring $\mathrm{End}_c(A_{p_i})$ is regular for each $i \in [1, n]$. By Lemma 3.10, $\mathrm{End}_c\Big(\bigoplus\limits_{i\geqslant n+1} A_{p_i}\Big) = \mathrm{End}\Big(\bigoplus\limits_{i\geqslant n+1} A_{p_i}\Big) \cong_{\mathrm{top}} \prod\limits_{i\geqslant n+1} \mathrm{End}(A_{p_i})$. Since each $\mathrm{End}(A_{p_i})$ is regular, the ring $\mathrm{End}\Big(\bigoplus\limits_{i\geqslant n+1} A_{p_i}\Big)$ is regular. This implies that $\mathrm{End}_c(A)$ is regular.

$\square$

4. Regular rings of quasi-injective discrete modules over compact rings

The following concept for discrete topological modules is analogous to the concept of a quasi-injective module in the theory of modules. A left discrete topological R-module M is called *quasi-injective* provided every homomorphism $f\colon N \to M$, where N is a submodule of M, has an extension to an endomorphism of M.

We extend in this section the known result (see [3], [4], [6], [7], [11], [12]) about the ring of endomorphisms of quasi-injective modules to the case of discrete quasi-injective modules over compact rings. The proof of the next theorem is completely analogous to the proof of Theorem 19.27 from [3] and we omit it.

Theorem 4.1. *Let I_R be a quasi-injective R-module, $S = \mathrm{End}(\mathrm{I_R})$ and $N = \{\alpha \in S\colon \alpha$ annihilates a large submodule of $I_R\}$.*

Then

(i) *S/N is regular in the sense of von Neumann;*

(ii) *N is the Jacobson radical of S;*

(iii) *idempotents modulo N can be lifted in S.*

5. An example of a linearly compact Abelian group with regular ring of endomorphisms

The following fact is well-known; we recall it for convenience of the reader:

If A is a linearly compact group, B and C are two closed subgroups, then the decomposition $A = B \oplus C$ is topological.

Consider the mapping $B \times C \to A$, $(b, c) \mapsto b + c$. This mapping is a continuous isomorphism since $B \times C$ is linearly compact and open, hence it is a topological isomorphism.

Given any cardinal number α and a prime p, A will designate the group $\mathbb{Q}_p^{\alpha}$ where $\mathbb{Q}_p$ is the additive group of the locally compact field of p-adic numbers.

Lemma 5.1. *Every subspace $\mathbb{Q}_p a$, $a \in A$ is a topological direct summand.*

P r o o f. The group A carries a structure of a $\mathbb{Q}_p$ topological vector space, which will be used below. The mapping $\mathbb{Q}_p \to \mathbb{Q}_p^{\alpha}$, $\alpha \mapsto \alpha a$ is continuous, hence $\mathbb{Q}_p a$ is a linearly compact subgroup. Since every linearly compact Abelian group is complete (see [13], p. 233, Theorem 28.5), it follows that $\mathbb{Q}_p a$ is closed. Let V be a closed subspace of codimension 1 such that $a \notin V$. Then $\mathbb{Q}_p a \cap V = 0$ and $A = \mathbb{Q}_p a \oplus V$ (an algebraic direct sum). It follows from the above mentioned fact that it is a topological direct sum. $\square$

Lemma 5.2. *If B is a closed vector subspace of A then B is a topological direct summand.*

P r o o f. There exists at least one closed vector subspace C such that $A = B + C$; we can take $C = A$. We can assume by linear compactness of A that C is minimal with this property. We claim that $B \cap C = 0$. Assume the contrary. Let $0 \neq x \in B \cap C$. Then $\mathbb{Q}_p x$ is a direct summand of C. There exists a vector subspace C_1 such that $\mathbb{Q}_p x \oplus C_1 = C$ is a topological direct summand. Then

$$A = B + C = B + \mathbb{Q}_p x + C_1 = B + C_1,$$

a contradiction. $\square$

Lemma 5.3. *If $\alpha \in \mathrm{End_c(A)}$, then α is an endomorphism of a $\mathbb{Q}_p$-vector space.*

P r o o f. Since A is a divisible group, $\alpha(ra) = r\alpha(a)$ for every $r \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, this equality is true for every $r \in \mathbb{Q}_p$. $\square$

Lemma 5.4. *If $\alpha \in \mathrm{End_c(A)}$, then $im\alpha$ and $\ker \alpha$ are closed $\mathbb{Q}_p$-subspaces of A.*

P r o o f. Follows from Lemma 5.3. $\square$

Theorem 5.5. $\mathrm{End_c(A)}$ *is a regular ring.*

P r o o f. Follows from the above lemmas. $\square$

Theorem 5.6. *Let A be a locally linearly compact Abelian group. Then* $\mathrm{End}_c(A)$ *with the topology given by the family $\{T(K, V)\}$, where K runs over all linearly compact subgroups, and V runs over all open subgroups, is a topological ring.*

P r o o f. $T(K, V)$ is a subgroup and $\{T(K, V)\}$ gives a Haussdorf group topology on $\mathrm{End}_c(A)$.

i) Let both $T(K, V)$ and $f \in \mathrm{End}_c(A)$ be arbitrary. There exists an open linearly compact subgroup $V_1, f(V_1) \subseteq V$. Then $fT(K, V_1) \subseteq T(K, V)$: indeed, $f_1 \in T(K, V_1)$ implies that $ff_1(K) \subseteq f(V_1) \subseteq V$.

ii) $T(f(K), V)f \subseteq T(K, V)$.

Indeed, if $f_1 \in T(f(K), V)$, then $f_1 f(K) \subseteq V$ and so $f_1 f \in T(K, V)$.

iii) We can assume without loss of generality that $K \supseteq V$. Then $T(K, V)$ is a subring: if $f_1, f_2 \in T(K, V)$, then $f_1 f_2(K) \subseteq f_1(V) \subseteq f_1(K) \subseteq V$. $\square$

Open question: Classify all closed left (two-sided) ideals of $\mathrm{End}_c(A)$.

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