Czechoslovak Mathematical Journal, 61 (136) (2011), 169–198

LIOUVILLE THEOREMS, A PRIORI ESTIMATES, AND BLOW-UP RATES FOR SOLUTIONS OF INDEFINITE SUPERLINEAR PARABOLIC PROBLEMS

Juraj Földes, Nashville

(Received Oktober 31, 2009)

Abstract. In this paper we establish new nonlinear Liouville theorems for parabolic problems on half spaces. Based on the Liouville theorems, we derive estimates for the blow-up of positive solutions of indefinite parabolic problems and investigate the complete blow-up of these solutions. We also discuss a priori estimates for indefinite elliptic problems.

Keywords: a priori estimates, Liouville theorems, blow-up rate, positive solution, indefinite parabolic problem

MSC 2010: 35B09, 35B44, 35B45, 35B53, 35J61, 35K59

1. Introduction

In this paper we consider the problem

$$\begin{aligned} u_t &= \Delta u + a(x)|u|^{p-1}u, && (x,t) \in \Omega \times (0,T), \\ u &= 0, && (x,t) \in \partial\Omega \times (0,T), \end{aligned} \tag{1.1}$$

which, if needed, is completed with an initial condition

$$u(\cdot,0) = u_0(\cdot) \in L^\infty(\Omega). \tag{1.2}$$

We assume that Ω is a smooth domain in $\mathbb{R}^N$ and $p > 1$. Furthermore, we suppose that $a\colon \bar{\Omega} \to \mathbb{R}$ belongs to $C^2(\bar{\Omega})$ and

$$\text{if} \quad \lim_{k\to\infty} a(x_k) = 0, \quad \text{then} \quad \limsup_{k\to\infty} |\nabla a(x_k)| > 0. \tag{1.3}$$

Here $C^k(D)$ denotes the space of k-times differentiable, bounded functions on $D \subset \mathbb{R}^N$, with bounded, continuous derivatives up to the kth order.

If Ω is bounded and if we denote

$$\Gamma := \{x \in \bar{\Omega}\colon\ a(x) = 0\}, \tag{1.4}$$

$$\Omega^+ := \{x \in \Omega\colon\ a(x) > 0\}, \tag{1.5}$$

$$\Omega^- := \{x \in \Omega\colon\ a(x) < 0\}, \tag{1.6}$$

then (1.3) is equivalent to

$$|\nabla a(x)| \neq 0 \qquad (x \in \Gamma), \tag{1.7}$$

that is, a has nondegenerate zeros in $\bar{\Omega}$. Since u_0 and a are bounded, standard results [21] yield the unique, strong solution of the problem (1.1), (1.2), with the maximal existence time $T_{\max} \in (0, \infty]$. Moreover, by regularity results, if $T_{\max} < \infty$, then $\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to T_{\max}$. We do not indicate the dependence of $T_{\max}$ on u_0 if no confusion seems possible. Here and in the rest of the paper we assume $T \in (0, T_{\max}]$.

As the main result of this paper, we derive an upper bound for the blow-up rate of nonnegative solutions of (1.1). The blow-up rates and related a priori estimates were studied under various assumptions on a, Ω and u in [1], [10], [11], [17], [13], [14], [15], [22], [26], [27], [28], [36], [34], [35], see also references therein. We just briefly describe the results directly connected to our results. First, Friedman and McLeod [11] studied blowing up solutions ($T_{\max} < \infty$) of the problem

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1} u, \qquad && (x, t) \in \Omega \times (0, T), \\ u &= 0, && (x, t) \in \partial\Omega \times (0, T), \end{aligned} \tag{1.8}$$

with $T = T_{\max}$, and the initial condition (1.2). They proved

$$|u(x, t)| \leqslant C\bigl(1 + (T_{\max} - t)^{-1/(p-1)}\bigr) \qquad (x \in \Omega), \tag{1.9}$$

where Ω is a bounded convex domain, $p > 1$, and u is a positive, increasing (in time) solution of (1.8). These results were generalized by Giga and Kohn [13] and later by Giga et al. [14], [15]. With help of localized energy estimates and iterative arguments, they proved that (1.9) holds true if Ω is a bounded convex domain or $\Omega = \mathbb{R}^N$, u is, a not necessarily positive, solution of (1.8), (1.2), and $1 < p < p_S$, where

$$p_S = p_S(N) := \begin{cases} \infty, & N \leqslant 2, \\ \dfrac{N+2}{N-2}, & N \geqslant 3. \end{cases}$$

In [9] Fila and Souplet employed scaling and Fujita type results to remove the assumption on convexity of Ω and established (1.9) for all positive solutions of (1.8), (1.2), and $1 < p \leqslant 1 + 2/(N+1)$.

Finally, Poláčik et al. [26] investigated positive solutions of (1.8) with a sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ and $1 < p < p_B$, where

$$p_B = p_B(N) := \begin{cases} \infty, & N \leqslant 1, \\ \dfrac{N(N+2)}{(N-1)^2}, & N \geqslant 2. \end{cases} \tag{1.10}$$

Using scaling, doubling lemma and Liouville theorems they obtained

$$u(x,t) \leqslant C(1 + t^{-1/(p-1)} + (T-t)^{-1/(p-1)}) \qquad ((x,t) \in \Omega \times (0,T)), \tag{1.11}$$

where C is a universal constant depending only on p, N and Ω. We remark that the estimates for the initial blow-up rate had been previously established by Bidaut-Véron [5] (see also [3]) for $1 < p < p_B$ and $\Omega = \mathbb{R}^N$. Some estimates on the initial blow-up rates for bounded Ω were proved by Quittner et al. [29].

The first a priori estimates for positive solutions of (1.1), (1.2) with sign-changing a were derived in the form (see [27] and references therein)

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C(\|u_0\|_{L^\infty(\Omega)}, \delta, N, p, \Omega, a) \qquad (t \in [0, T_{\max} - \delta],\ \delta > 0,\ T_{\max} < \infty). \tag{1.12}$$

Later, Xing [36] obtained an upper estimate for the blow-up rate of positive solutions of (1.1), (1.2)

$$u(x,t) \leqslant C\bigl(1 + (T_{\max} - t)^{-3/(2(p-1))}\bigr) \qquad ((x,t) \in \Omega \times (0, T_{\max}),\ T_{\max} < \infty)$$

when Ω is bounded, $1 < p < p_B$ and $\Gamma \subset \Omega$, that is, when a does not vanish on $\partial\Omega$. Here C depends on $\|u_0\|_{L^\infty(\Omega)}$, N, p, Ω, a.

The next theorem refines the results in [36] in various directions. It includes unbounded domains and it allows for a very general behavior of a on $\partial\Omega$. In addition, it also yields an estimate for the initial blow-up rate. Denote by $\nu_\Omega(x)$ the unit outward normal vector to $\partial\Omega$ at x.

Theorem 1.1. *Let Ω be a uniformly regular domain of class C^2 in $\mathbb{R}^N$ (cf. [2]) and let $1 < p < p_B$. Suppose that $a \in C^2(\bar\Omega)$ satisfies (1.3) and*

$$\left| \frac{\nabla a(x_0)}{|\nabla a(x_0)|} - \nu_\Omega(x_0) \right| \geqslant \tilde{c} > 0 \qquad (x_0 \in \Gamma \cap \partial\Omega). \tag{1.13}$$

Then every nonnegative solution u of (1.1) *satisfies*

$$u(x,t) \leqslant C\big(1 + t^{-3/(2(p-1))} + (T-t)^{-3/(2(p-1))}\big) \qquad ((x,t) \in \Omega \times (0,T)), \tag{1.14}$$

where C depends on N, p, Ω and a.

Remark 1.2. (a) The nonlinearity $|u|^{p-1}u$ in (1.1) can be replaced by $f(u)$ with

$$\lim_{v\to\infty} \frac{f(v)}{v^p} = l > 0.$$

Then (1.14) holds with C depending on N, f, Ω and a. Also, we can add lower order terms to the right hand side, that is, we can add a function $g\colon\ \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{u\to\infty} \sup_{(x,t)\in\Omega\times(0,T)} \frac{g(x,t,u)}{u^p} = 0.$$

Then (1.14) holds with C depending on N, p, Ω, a and g.

(b) For the blowing-up solutions ($T_{\max} < \infty$) of (1.8) one has (cf. [28, Proposition 23.1]) $\sup\limits_{x\in\mathbb{R}^N} u(x,t) \geqslant C(T_{\max} - t)^{-1/(p-1)}$. This shows the optimality of the final blow up estimate in (1.11) for $a \equiv 1$. However, it is not known whether or not the weaker estimate (1.14) is optimal for sign changing a. Below, we show that under additional assumptions the stronger estimate (1.11) holds true even if a changes sign.

(c) If a also depends on t and $p > (N+2)/N$, the initial blow-up estimate in (1.14) does not hold even if $0 \leqslant a \leqslant 1$ (see e.g. [32], [33]). If Ω is bounded, then (1.13) is equivalent to $\nabla a(x_0)/|\nabla a(x_0)| \neq \nu_\Omega(x_0)$ for any $x_0 \in \Gamma \cap \partial\Omega$. It is not known if this assumption is technical or not.

(d) Universal estimates of the form (1.11) or (1.14) are not true for $p \geqslant p_S$, $N \geqslant 3$, $\Omega = \mathbb{R}^N$, due to the existence of arbitrarily large stationary radial solutions of (1.1). We require $p < p_B < p_S$ mainly because the Liouville theorem for the problem

$$u_t = \Delta u + u^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.15}$$

with $p_B \leqslant p < p_S$ is not known. If one proved such a Liouville theorem for some $p \in [p_B, p_S)$, then the conclusion of Theorem 1.1 would hold for the same p as well.

(e) If we restrict ourselves to the class of radial solutions (of course now Ω and a are radially symmetric), then similarly to [26], one can prove Theorem 1.1 for each $1 < p < p_S$. This is possible, since the Liouville theorem is known for nonnegative radial solutions of (1.15) for any $1 < p < p_S$ (see [24]).

(f) If a nonnegative solution u of (1.1) is global ($T_{\max} = \infty$), then after letting $T \to \infty$ in (1.14) we obtain

$$u(x,t) \leqslant C\big(1 + t^{-3/(2(p-1))}\big) \qquad ((x,t) \in \Omega \times (0,\infty)). \tag{1.16}$$

In particular, u is bounded on $\Omega \times (1,\infty)$. For previous results, see [5], [26].

Remark 1.3. Observe that (1.14) is equivalent to

$$M(x,t) \leqslant C(1+d^{-1}(t)) \qquad ((x,t)\in\Omega\times(0,T)), \tag{1.17}$$

where

$$M := u^{(p-1)/3} \quad \text{and} \quad d(t) := \min\{t, T-t\}^{1/2}.$$

Also, for each $x \in \Omega$, one has $d(t) = d_P[(x,t),\Theta]$, where $\Theta := \Omega\times\{0,T\}$ and d_P denotes the parabolic distance:

$$d_P[(x,t),(y,s)] = |x-y| + |t-s|^{\frac{1}{2}} \qquad ((x,t),(y,s)\in\Omega\times(0,T)). \tag{1.18}$$

In this notation we obtain yet another form of (1.14):

$$u(x,t) \leqslant C\big(1 + d_P^{-3/(p-1)}[(x,t),\Theta]\big) \qquad ((x,t)\in\Omega\times(0,T)).$$

If u is a stationary solution of (1.1), that is, if u solves

$$\begin{aligned} 0 &= \Delta u + a(x)|u|^{p-1}u, && x\in\Omega,\\ u &= 0, && x\in\partial\Omega, \end{aligned} \tag{1.19}$$

we obtain the following corollary.

Corollary 1.4. *Let $\Omega\subset\mathbb{R}^N$ be a uniformly regular domain of class C^2 (cf. [2]), $1<p<p_S$, and let $a\in C^2(\bar\Omega)$ satisfy (1.3) and (1.13). If u is a nonnegative solution of (1.19), then $u \leqslant C(p,N,\Omega,a)$.*

This corollary extends the results of Du and Li [7] (see also references therein), as it allows a to vanish on $\partial\Omega$. If $1<p<p_B(N)$, then since $T_{\max}=\infty$, Corollary 1.4 follows from (1.16). If we merely assume $1<p<p_S(N)$, then one has to reprove Theorem 1.1 for solutions of (1.19). The only difference is the application of elliptic Liouville theorems [12], instead of parabolic ones, whenever $p<p_B$ is required.

The next propositions shows that final blow-up rates in Theorem 1.1 (and the main results in [36]) can be improved if $a>0$ and Ω is a convex bounded set. Notice that a is allowed to vanish on $\partial\Omega$. In this case, the universal bounds (1.12) were already obtained in [27].

Proposition 1.5. *Let $\Omega\subset\mathbb{R}^N$ be a bounded, smooth, convex set and let $1<p<p_B$. Assume $a\in C^2(\bar\Omega)$ satisfies (1.7) and $a(x)>0$ for $x\in\Omega$. Then a nonnegative solution u of (1.1), (1.2) satisfies*

$$u(x,t) \leqslant C\big(1+(T-t)^{-1/(p-1)}\big) \qquad ((x,t)\in\Omega\times(0,T)), \tag{1.20}$$

where C depends on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.

If a changes sign in Ω, we formulate sufficient conditions for (1.20) only in the one-dimensional case. However, one can generalize the following propositions to the higher dimensional case if Ω is convex and certain monotonicity of a and u_0 near $\partial\Omega$ is assumed.

Proposition 1.6. *Let $N = 1$ and $\Omega = (0, 1)$. Suppose that $a \in C^2([0, 1])$ and has exactly one nondegenerate zero $\mu \in [0, 1]$, that is, $a(\mu) = 0$ and $a'(\mu) \neq 0$. If*

$$\operatorname{sign}[a(x)](u_0(2\mu - x) - u_0(x)) \leqslant 0 \qquad (x \in (\max\{0, 2\mu - 1\}, \mu)),$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

Proposition 1.7. *Let $N = 1$ and $\Omega = (0, 1)$. Suppose that $a \in C^2([0, 1])$ and has exactly two nondegenerate zero $\mu_1 < \mu_2$ in $[0, 1]$, that is, $a(\mu_i) = 0$ and $a'(\mu_i) \neq 0$ for $i = 1, 2$. If $\max\{\mu_1, 1 - \mu_2\} < \mu_2 - \mu_1$ and*

$$\begin{aligned} a(x) < 0, \quad u_0(2\mu_1 - x) &\geqslant u_0(x) \qquad (x \in (0, \mu_1)), \\ u_0(2\mu_2 - x) &\geqslant u_0(x) \qquad (x \in (\mu_2, 1)), \end{aligned}$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

One can also employ Liouville theorems and universal estimates in the investigation of the complete blow-up and the continuity of the blow-up time. Let us recall these notions and explain the results.

Let u be a nonnegative solution of (1.1), (1.2) with $T_{\max} < \infty$. Let u_k ($k \in \mathbb{N}$) be the solution of the approximation problem

$$\begin{aligned} (u_k)_t - \Delta u_k &= f_k(x, u_k), \qquad && (x, t) \in \Omega \times (0, \infty), \\ u_k &= 0, && (x, t) \in \partial\Omega \times (0, \infty), \\ u_k(x, 0) &= u_0(x) \geqslant 0, && x \in \Omega, \end{aligned}$$

where

$$f_k(x, v) := \begin{cases} a(x) \min\{v^p, k\} & \text{if } a(x) \geqslant 0,\ v \in \mathbb{R}, \\ a(x) v^p & \text{if } a(x) < 0,\ v \in \mathbb{R}. \end{cases}$$

Since f_k is bounded from above, the nonnegative solution u_k exists globally (for all positive times). Since $f_k \leqslant f_{k+1}$, the maximum principle implies $u_{k+1}(x, t) \geqslant u_k(x, t)$ for any $(x, t) \in \Omega \times (0, \infty)$. Thus

$$\bar{u}(x, t) := \lim_{k \to \infty} u_k(x, t) \in [0, \infty] \qquad ((x, t) \in \Omega \times [0, \infty))$$

is well defined. Moreover, $\bar{u}(x,t) = u(x,t)$ for any $(x,t) \in \bar{\Omega} \times [0, T_{\max})$. We say that u *blows-up completely* in $D \subset \Omega$ at T, if $\bar{u}(x,t) = \infty$ for any $x \in D$ and $t > T$.

Theorem 1.8. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $1 < p < p_B$. Suppose that $a \in C^2(\bar{\Omega})$ satisfies (1.7) and (1.13). If $T_{\max} < \infty$ for a nonnegative solution u of (1.1), (1.2), then u blows-up completely in Ω^+ at $T_{\max}$. In addition, the function*

$$T\colon \{u_0 \in L^\infty(\Omega)\colon u_0 \geqslant 0\} \to (0, \infty], \qquad T\colon u_0 \mapsto T_{\max}(u_0)$$

is continuous.

If $a \equiv 1$, Baras and Cohen [4] proved complete blow-up of nonnegative solutions of (1.8), (1.2) in Ω at $T_{\max} < \infty$ for each $1 < p < p_S$ (see also [28]). However, for $p > p_S$, $N \leqslant 10$, and Ω being a ball, there exist radial solutions of (1.8) that do not blow-up completely in Ω at $T_{\max}$. For further discussion see [28] and references therein.

If a changes sign, then one cannot expect the complete blow-up in the whole Ω, since $\bar{u}$ stays bounded in Ω^- for any $t > 0$ (see [20]). Quittner and Simondon [27] proved the complete blow-up of u in Ω^+ at $T_{\max} < \infty$ for $1 < p \leqslant 1 + 3/(N+1)$ and $\Gamma \subset \Omega$. Later Poláčik and Quittner [23] replaced the former assumption by $1 < p < p_B$ and proved Theorem 1.8 under an additional assumption $\Gamma \subset \Omega$.

The rest of the paper is organized as follows. In Section 2 we state and prove parabolic Liouville theorems. In Section 3 we formulate the doubling lemma and prove our main results.

2. Liouville theorems

Since some results in this section can be of independent interest, we formulate them in a more general setting than that required for the proofs of the main results. Let us define

$$\mathbb{R}^N_\lambda := \{x = (x_1, x') \in \mathbb{R}^N\colon\ x_1 > \lambda\} \qquad (\lambda \in \mathbb{R}), \tag{2.1}$$

$$H_\lambda := \partial\mathbb{R}^N_\lambda = \{x = (x_1, x') \in \mathbb{R}^N\colon\ x_1 = \lambda\} \qquad (\lambda \in \mathbb{R})\,. \tag{2.2}$$

The following two lemmas were proved in [36] for increasing functions f. Here we propose simpler proofs that remove this unnecessary assumption. The elliptic counterparts can be found in [8], [30], [31], see also references therein.

Lemma 2.1. *Let f be a continuous function with $f(v) > 0$ for any $v > 0$. If $u\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative bounded solution of*

$$u_t - \Delta u = -f(u), \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

then $u \equiv 0$.

Proof. We proceed by way of contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}^N \times \mathbb{R}$ such that

$$u(x^*, t^*) \geqslant C^* := \frac{1}{2} \sup_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} u(x,t) > 0.$$

For each $\varepsilon > 0$ denote

$$v_\varepsilon(x,t) := u(x,t) - \varepsilon|x - x^*|^2 - \varepsilon\bigl(\sqrt{(t-t^*)^2+1} - 1\bigr) \qquad ((x,t) \in \mathbb{R}^N \times \mathbb{R}).$$

Since $v_\varepsilon(x,t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$, there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^N \times \mathbb{R}$ with

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} v_\varepsilon(x,t).$$

Then for each $\varepsilon > 0$

$$2C^* \geqslant u(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x^*, t^*) = u(x^*, t^*) \geqslant C^* > 0,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0, \qquad \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \leqslant 0.$$

Consequently,

$$\begin{aligned}
0 &\leqslant (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \\
&= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon \frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N \\
&= -f(u(x_\varepsilon, t_\varepsilon)) - \varepsilon \frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N \\
&\leqslant - \inf_{2C^* \geqslant v \geqslant C^*} f(v) + \varepsilon + 2\varepsilon N \qquad (\varepsilon > 0).
\end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. □

Lemma 2.2. *Suppose $f \in C^1$ satisfies $f(0) = 0$ and $f(v) > 0$ for any $v > 0$. Let h be a continuous function with $h(x_1) < 0$ for each $x_1 > 0$, and let $\limsup_{x_1 \to \infty} h(x_1) < 0$. If u is a nonnegative bounded solution of the problem*

$$\begin{aligned} u_t - \Delta u = h(x_1) f(u), \qquad & (x,t) \in \mathbb{R}_0^N \times \mathbb{R}, \\ u = 0, \qquad & (x,t) \in H_0 \times \mathbb{R}, \end{aligned}$$

then $u \equiv 0$.

P r o o f. The proof is similar to that of Lemma 2.1. We again proceed by a contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}_0^N \times \mathbb{R}$ such that

$$u(x^*, t^*) \geqslant C^* := \frac{1}{2} \sup_{(x,t) \in \mathbb{R}_0^N \times \mathbb{R}} u(x,t) > 0.$$

It is easy to see that there exists a function $\varphi \in C^2(\mathbb{R}^N \times \mathbb{R})$ with

$$\begin{gathered} \varphi(x,t) \geqslant 0, \quad |\nabla \varphi(x,t)| \leqslant 1, \quad |\varphi_t - \Delta \varphi| \leqslant 1 \qquad ((x,t) \in \mathbb{R}^N \times \mathbb{R}), \\ \varphi(0,0) = 0, \quad \varphi(x,t) \to \infty \quad \text{if} \quad |x| \to \infty \quad \text{or} \quad t \to \pm\infty. \end{gathered}$$

For each $\varepsilon \in (0,1)$ denote

$$v_\varepsilon(x,t) := u(x,t) - \varepsilon \varphi(x - x^*, t - t^*) \qquad ((x,t) \in \mathbb{R}_0^N \times \mathbb{R}).$$

Since u is bounded, $v_\varepsilon(x,t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$. Moreover, $v_\varepsilon(x,t) \leqslant 0 < v_\varepsilon(x^*, t^*)$ for any $(x,t) \in H_0 \times \mathbb{R}$, and therefore there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}_0^N \times \mathbb{R}$ such that

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t) \in \mathbb{R}_0^N \times \mathbb{R}} v_\varepsilon(x,t).$$

Consequently,

$$2C^* \geqslant u(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x^*, t^*) = u(x^*, t^*) \geqslant C^* > 0,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0, \qquad (\Delta v_\varepsilon)(x_\varepsilon, t_\varepsilon) \leqslant 0.$$

Observe that u satisfies

$$u_t = \Delta u + h(x_1) \frac{f(u)}{u} u = \Delta u + c(x,t) u.$$

Since $f \in C^1$, $f(0) = 0$, and u is bounded, c is a bounded function in $\{(x,t) \in \mathbb{R}_0^N \times \mathbb{R} : x_1 < 2\}$. Hence, standard parabolic regularity (see for example [19, Theorem 1.15]) implies

$$|\nabla u(x,t)| \leqslant C \qquad ((x,t) \in \bar{\mathbb{R}}_0^N \times \mathbb{R},\ x_1 < 1),$$

and consequently,

$$|\nabla v_\varepsilon(x,t)| \leqslant C + 1 \qquad ((x,t) \in \bar{\mathbb{R}}_0^N \times \mathbb{R},\ x_1 < 1),$$

where C is independent of $\varepsilon \in (0,1)$. Furthermore, $v_\varepsilon(x_\varepsilon, t_\varepsilon) \geqslant C^* > 0$ and $v_\varepsilon(x,t) \leqslant 0$ for all $(x,t) \in H_0 \times \mathbb{R}$ yield $\operatorname{dist}(x_\varepsilon, H_0) = (x_\varepsilon)_1 \geqslant c_0$, where c_0 is a constant independent of ε. Finally,

$$\begin{aligned}
0 &\leqslant (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \\
&= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon[\varphi_t(x_\varepsilon, t_\varepsilon) - \Delta\varphi(x_\varepsilon, t_\varepsilon)] \\
&\leqslant h((x_\varepsilon)_1) f(u(x_\varepsilon, t_\varepsilon)) + \varepsilon \\
&\leqslant \sup_{y \geqslant c_0} h(y) \inf_{2C^* \geqslant v \geqslant C^*} f(v) + \varepsilon.
\end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. $\square$

Next, consider the problem

$$\text{(2.3)} \qquad \begin{aligned}
u_t - \Delta u &= h(x \cdot v) f(u), && (x,t) \in \Omega \times \mathbb{R}, \\
u &= 0, && (x,t) \in \partial\Omega \times \mathbb{R},
\end{aligned}$$

where

(v1) $v = (v_1, v_2, \ldots, v_N) \in \mathbb{R}^N$ is a unit vector with $v_1 > 0$ and $v_i = 0$ for $i \geqslant 3$.

About Ω, we assume that

(d1) Ω is a subset of $\mathbb{R}^N$, convex and unbounded in x_1, that is, $x + \xi e_1 \in \Omega$ for any $x \in \Omega$ and $\xi > 0$;

(d2) there is a constant d^* such that $x_2 v_2 \leqslant d^*$ for any $x = (x_1, x_2, \ldots, x_N) \in \Omega$.

Next, the function $h \colon \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis.

(h1) h is continuous, nondecreasing, and strictly increasing on $(0, \infty)$;

(h2) $h(0) = 0$ and $\lim_{y \to \infty} h(y) = \infty$.

About f we assume

(f1) $f \in C^1([0, \infty))$, with $f(0) = f'(0) = 0$, and $f(v) > 0$, $f'(v) \geqslant 0$ for each $v > 0$.

The following theorem is a generalization of elliptic [7] and parabolic [23] results proved for $v = e_1$ and $\Omega = \mathbb{R}^N$. The general framework of the proof is similar to one used in [7], [23].

Theorem 2.3. *If* (v1), (d1), (d2), (h1), (h2), *and* (f1) *hold true, then the only nonnegative, bounded solution* u *of* (2.3) *is* $u \equiv 0$.

As a corollary we obtain the Liouville theorem for indefinite problems on half spaces.

Corollary 2.4. *Given unit vectors* $b, v \in \mathbb{R}^N$ *and a constant* c^*, *let* $\Omega := \{x \in \mathbb{R}^N \colon\ x \cdot b > c^*\}$. *Consider functions* h *and* f *that satisfy* (h1), (h2), *and* (f1), *respectively. Let* u *be a nonnegative, bounded solution of* (2.3). *If* $v \neq -b$, *then* $u \equiv 0$.

Remark 2.5. The statement of Corollary 2.4 still holds true if $v = -b$, $c^* \geqslant 0$, and h in addition to (h1), (h2) satisfies $h(y) < 0$ for $y < 0$. This follows after suitable rotation and translation from Lemma 2.2. However, if $v = -b$ and $c^* < 0$, there are nontrivial, nonnegative solutions of (2.3). This result will be published elsewhere.

Proof of Corollary 2.4. We rotate the coordinates so that $b = e_2$, $v_1 \geqslant 0$, and $v_i = 0$ for $i \geqslant 3$. Then $\Omega = \{x \in \mathbb{R}^N \colon\ x_2 > c^*\}$ and (d1) holds true. Notice that (2.3), (h1), (h2), and (f1) are invariant under rotations.

If $v_1 > 0$ and $v_2 \leqslant 0$, then (v1) and (d2) are satisfied with $d^* = c^* v_2$, and the corollary follows from Theorem 2.3.

If $v_2 > 0$, consider another rotation that maps v to e_1 and fixes the space spanned by $\{e_3, \ldots, e_N\}$. Then (v1) and (d2) are clearly satisfied with $d^* = 0$. Also, Ω is transformed to $\Omega' := \{x \in \mathbb{R}^N \colon\ x \cdot b' > c^*\}$, where $b' = (v_2, v_1, 0, \ldots, 0)$. In particular, $b'_1 > 0$ and (d1) holds. Now, the corollary follows from Theorem 2.3.

If $v_1 = 0$ and $v_2 \leqslant 0$, then $v = -e_2 = -b$, a contradiction to our assumptions. □

Before we proceed, define $Lu := u_t - \Delta u$ and $M := \sup_{\Omega} u$. Furthermore, given $\lambda \in \mathbb{R}$ set

$$
\begin{aligned}
\Sigma_\lambda &:= \{x \in \Omega \colon\ x_1 < \lambda\}, \\
x^\lambda &:= (2\lambda - x_1, x_2, \ldots, x_N) \qquad (x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N), \\
w_\lambda(x,t) &:= u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \bar{\Sigma}_\lambda \times \mathbb{R}), \\
\lambda(t) &:= \sup\{\mu \colon\ w_\lambda(x,t) \geqslant 0 \ \text{ for all } \ x \in \Sigma_\lambda \ \text{ and } \ \lambda < \mu\}, \\
\lambda^* &:= \inf\{\lambda(t) \colon\ t \in \mathbb{R}\}.
\end{aligned} \tag{2.4}
$$

Observe that (d1) implies $x^\lambda \in \bar{\Omega}$ for any $x \in \bar{\Sigma}_\lambda$, and therefore w_λ is well defined. Moreover, since u is nonnegative in Ω and vanishes on $\partial\Omega$,

$$w_\lambda(x,t) = u(x^\lambda,t) - u(x,t) = u(x^\lambda,t) \geqslant 0 \qquad ((x,t) \in (\partial\Omega \cap \bar{\Sigma}_\lambda) \times \mathbb{R}).$$

Clearly $w_\lambda(x,t) = 0$ if $(x,t) \in (\Omega \cap \partial\Sigma_\lambda) \times \mathbb{R}$, and therefore

$$w_\lambda(x,t) \geqslant 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times \mathbb{R}). \tag{2.5}$$

We divide the proof of Theorem 2.3 into several lemmas, in which we implicitly suppose the assumptions of the theorem.

First, notice that $v_1 > 0$ implies

$$x^\lambda \cdot v - x \cdot v = 2(\lambda - x_1)v_1 \geqslant 0 \qquad (x \in \Sigma_\lambda), \tag{2.6}$$

and consequently by (h1)

$$h(x \cdot v) \leqslant h(x^\lambda \cdot v) \qquad (x \in \Sigma_\lambda). \tag{2.7}$$

Lemma 2.6. *If there are $\lambda \in \mathbb{R}$, $\tilde{x} \in \Sigma_\lambda$ and $\tilde{t} \in \mathbb{R}$ with $h(\tilde{x} \cdot v) \leqslant 0$ and $w_\lambda(\tilde{x},\tilde{t}) \leqslant 0$, then $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$. Moreover, if $\tilde{x}_1 \leqslant -d^*/v_1$, then $w_\lambda(\tilde{x},\tilde{t}) \leqslant 0$ implies $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$.*

P r o o f. The positivity and monotonicity of f, together with (2.7) yields

$$\begin{aligned} Lw_\lambda(\tilde{x},\tilde{t}) &= h(\tilde{x}^\lambda \cdot v) f(u(\tilde{x}^\lambda,\tilde{t})) - h(\tilde{x} \cdot v) f(u(\tilde{x},\tilde{t})) \\ &\geqslant h(\tilde{x} \cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x},\tilde{t}))] \geqslant 0, \end{aligned}$$

and the first statement follows. Next, assume $\tilde{x}_1 \leqslant -d^*/v_1$. Then $v_1 > 0$ and (d2) imply

$$\tilde{x} \cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leqslant \tilde{x}_1 v_1 + d^* \leqslant 0,$$

and by (h1) and (h2) one has $h(\tilde{x} \cdot v) \leqslant 0$. Now, the second statement follows from the first one. □

Lemma 2.7. $\lambda(t) \geqslant -d^*/v_1$ *for all* $t \in \mathbb{R}$.

P r o o f. We proceed by a contradiction, that is, we assume the existence of $\lambda < -d^*/v_1$ and $(\tilde{x},\tilde{t}) \in \Sigma_\lambda \times \mathbb{R}$ with $w_\lambda(\tilde{x},\tilde{t}) < 0$. Then $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$ by the second statement of Lemma 2.6. One can easily verify that for any sufficiently smooth function $g\colon (-\infty,\lambda] \to (0,\infty)$

$$g(x_1) L\overline{w}_\lambda(x,t) = Lw_\lambda(x,t) + 2(\partial_{x_1}\overline{w}_\lambda(x,t))g'(x_1) + \overline{w}_\lambda(x,t)g''(x_1) \qquad ((x,t) \in \Sigma_\lambda \times (0,\infty)), \tag{2.8}$$

where $\overline{w}_\lambda(x,t) := w_\lambda(x,t)/g(x_1)$. If we set

$$g(y) := \ln(\lambda + 1 - y) + 1 \qquad (y \in (-\infty, \lambda]),$$

then $g > 0$ and for already fixed $\tilde{x}$ and $\tilde{t}$ we have

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)} + \overline{w}_\lambda(\tilde{x},\tilde{t})\frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)}. \tag{2.9}$$

Consider the solution of the problem

$$\begin{aligned} z_t - z_{yy} &= F(y, z, z_y), \qquad && (y,t) \in \mathbb{R} \times (0,\infty),\\ z(y,0) &= -M, && y \in \mathbb{R}, \end{aligned} \tag{2.10}$$

where

$$F(y,z,z_y) = \begin{cases} 2z_y g'/g & y < \lambda - 1,\\ 2z_y g'/g - az & y \in [\lambda-1, \lambda],\\ 0 & y > \lambda, \end{cases}$$

and $a := -g''(\lambda-1)/g(\lambda-1) > 0$. Then the maximum principle implies $z(y,t) < 0$ for all $(y,t) \in \mathbb{R} \times (0,\infty)$, and since $F(y,-M,0) \geqslant 0$, z is increasing in t. Also, for any $T \geqslant 0$ the function $Z\colon\ (x,t) \mapsto z(x_1, t+T)$ satisfies

$$L[Z] \leqslant 2\frac{g'(x_1)}{g(x_1)}\partial_{x_1} Z + \frac{g''(x_1)}{g(x_1)} Z \qquad ((x,t) \in \mathbb{R}^N \times (0,\infty),\ x_1 < \lambda).$$

Then the maximum principle on the set where $\overline{w}_\lambda \leqslant 0$ yields $\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant Z(\tilde{x},\tilde{t}) = z(\tilde{x}_1, \tilde{t}+T)$ for any $T > 0$.

Since z is increasing in t, $\tilde{z}(y) := \lim_{t\to\infty} z(y,t)$ is well defined for each $y \in \mathbb{R}$ and

$$-\tilde{z}_{yy} = F(y, \tilde{z}, \tilde{z}_y), \qquad y \in \mathbb{R}.$$

An analysis of this problem (for details see [23, Claim 2]) implies $\tilde{z} \equiv 0$. Thus, $\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant z(\tilde{x}_1, \tilde{t}+T) \to 0$ as $T \to \infty$, a contradiction. □

Lemma 2.8. *The mapping $t \mapsto \lambda(t)$ is nondecreasing. If $\lambda(t_1) = \infty$, this means that $\lambda(t_2) = \infty$ for all $t_2 \geqslant t_1$.*

Proof. Fix $t_0 \in \mathbb{R}$ and $\lambda < \lambda(t_0)$. Then

$$w_\lambda(x,t_0) \geqslant 0 \qquad (x \in \Sigma_\lambda),$$

and by (2.5)

$$w_\lambda(x,t) \geqslant 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times [t_0,\infty)).$$

Next, (2.7) and the mean value theorem imply

$$\begin{aligned} Lw_\lambda(x,t) &= h(x^\lambda\cdot v)f(u(x^\lambda,t)) - h(x\cdot v)f(u(x,t)) \\ &\geqslant h(x\cdot v)[f(u(x^\lambda,t)) - f(u(x,t))] \\ &= h(x\cdot v)f'(\theta(x,t))w_\lambda(x,t), \qquad (x,t)\in\Sigma_\lambda\times[t_0,\infty), \end{aligned}$$

where $\theta(x,t)$ is a number between $u(x,t)$ and $u(x^\lambda,t)$. In particular, $\theta\colon (x,t)\mapsto [0,\infty)$ is a bounded function. Since by (d2)

$$x\cdot v = x_1v_1 + x_2v_2 \leqslant x_1v_1 + d^* \leqslant \lambda + d^* \qquad (x\in\Sigma_\lambda),$$

one has $h(x\cdot v)\leqslant h(\lambda+d^*)$ for each $x\in\Sigma_\lambda$. Now, the maximum principle, with the coefficient $c(x,t) := h(x\cdot v)f'(\theta(x,t))$ being possibly unbounded from below (see [6], [18]), gives $w_\lambda(x,t)\geqslant 0$ for all $(x,t)\in\Sigma_\lambda\times[t_0,\infty)$. Since $\lambda<\lambda(t_0)$ was chosen arbitrary, $\lambda(t)\geqslant\lambda(t_0)$ for each $t\geqslant t_0$. □

Lemma 2.9. *$\lambda^* = \infty$, or equivalently, u is nondecreasing in x_1.*

Proof. We proceed by contradiction, that is, we suppose $\lambda^*<\infty$. Lemma 2.7 guarantees $\lambda^*\geqslant -d^*/v_1$. By the definition of λ^* and by Lemma 2.8, there exist $\lambda_k\searrow\lambda^*$ and $t_k\searrow-\infty$ with

$$\inf_{x\in\Sigma_{\lambda_k}} w_{\lambda_k}(x,t_k) < 0.$$

Since u is bounded there is $M>0$ with $u\leqslant M$. Consequently, by (f1), there exists C_f such that $f'\leqslant C_f$ on $[0,M]$. Set $b_2 := h(\lambda^*v_1 + d^* + 1)C_f > 0$ and choose $1>\delta>0$ with

$$2\delta^{-2}\geqslant 3^3(2b_2+1). \tag{2.11}$$

Since $f'(0)=0$, we can fix $\eta>0$ with

$$f'(z)\leqslant \frac{\delta}{h(\lambda^*+d^*+1)(\lambda^*+1+d^*/v_1)^3} \qquad (z\in[0,\eta]). \tag{2.12}$$

Let ε with $0<\varepsilon<\delta$ be sufficiently small (as specified below), and fix k such that $\lambda_k<\lambda^*+\varepsilon$. To simplify the notation set $\lambda:=\lambda_k$ and denote

$$\begin{aligned} g(y) &:= 2 - \frac{\delta}{\delta+\lambda-y} \qquad (y\in(-\infty,\lambda]), \\ \overline{w}_\lambda(x,t) &:= \frac{w_\lambda(x,t)}{g(x_1)} \qquad ((x,t)\in\Sigma_\lambda\times\mathbb{R}). \end{aligned}$$

Observe that $g''(y) \leqslant 0$ and $g(y) > 0$ for any $y \leqslant \lambda$. For λ already fixed, define

$$S := \{(x,t) \in \Sigma_\lambda \times \mathbb{R}\colon\ w_\lambda(x,t) \leqslant 0\}.$$

Case 1. If $(\tilde{x},\tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$, then (2.8) and the concavity of g yield

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}.$$

Case 2. If $(\tilde{x},\tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x},\tilde{t}) < 0$, then Lemma 2.6 yields $h(\tilde{x}\cdot v) > 0$. Consequently, (h1) and (d2) yield

$$0 \leqslant \tilde{x}\cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leqslant \tilde{x}_1 v_1 + d^* \leqslant \lambda^* + d^* + 1. \tag{2.13}$$

Also, Lemma 2.6 implies $\tilde{x}_1 > -d^*/v_1$, and therefore

$$\tilde{x}^\lambda \cdot v = (2\lambda - \tilde{x}_1)v_1 + \tilde{x}_2 v_2 \leqslant 2\lambda v_1 + 2d^* \leqslant 2\lambda^* + 2d^* + 1. \tag{2.14}$$

Now, (2.7) implies $h(\tilde{x}^\lambda\cdot v) \geqslant h(\tilde{x}\cdot v) > 0$ and (h1), (2.13), (2.14) yield

$$h(-1) \leqslant h(x\cdot v) \leqslant h(2(\lambda^* + d^*) + 2) \qquad ((x,t) \in \mathbb{R}^{N+1}, d_P[(x,t),S^*] < 1),$$

where d_P was defined in (1.18) and S^* is the convex hull of S and the set $\{(x^\lambda,t)\colon (x,t)\in S\}$. Next, the boundedness of u and standard local parabolic estimates give

$$|\nabla u(x,t)| \leqslant C_\lambda \qquad ((x,t) \in S^*).$$

Furthermore,

$$u(\tilde{x}^{\lambda^*},\tilde{t}) \geqslant u(\tilde{x},\tilde{t}) \geqslant u(\tilde{x}^\lambda,\tilde{t}) \tag{2.15}$$

and

$$|\tilde{x}^{\lambda^*} - \tilde{x}^\lambda| = |\tilde{x}_1^{\lambda^*} - \tilde{x}_1^\lambda| = 2(\lambda - \lambda^*) \leqslant 2\varepsilon.$$

Also, by (f1) and $h(\tilde{x}\cdot v) \geqslant 0$

$$\begin{aligned} 0 > Lw_\lambda(\tilde{x},\tilde{t}) &= h(\tilde{x}^\lambda\cdot v)f(u(\tilde{x}^\lambda,\tilde{t})) - h(\tilde{x}\cdot v)f(u(\tilde{x},\tilde{t})) \\ &\geqslant h(\tilde{x}^\lambda\cdot v)f(u(\tilde{x}^\lambda,\tilde{t})) - h(\tilde{x}\cdot v)f(u(\tilde{x}^{\lambda^*},\tilde{t})) \\ &= h(\tilde{x}^\lambda\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x}^{\lambda^*},\tilde{t}))] + [h(\tilde{x}^\lambda\cdot v) - h(\tilde{x}\cdot v)]f(u(\tilde{x}^{\lambda^*},\tilde{t})). \end{aligned} \tag{2.16}$$

Let us estimate each term separately. Since the segment connecting $\tilde{x}$ and $\tilde{x}^{\lambda^*}$ belongs to S^*, one has by (2.14), (2.15) and the definition of C_f and C_λ

$$(2.17)\quad \begin{aligned} h(\tilde{x}^\lambda \cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x}^{\lambda^*},\tilde{t}))] \\ \geqslant h(2(\lambda^* + d^*) + 1)C_f(u(\tilde{x}^\lambda,\tilde{t}) - u(\tilde{x}^{\lambda^*},\tilde{t})) \\ \geqslant -2h(2(\lambda^* + d^*) + 1)C_f C_\lambda \varepsilon. \end{aligned}$$

To estimate the second term, notice that $\tilde{x}_1 \leqslant \lambda^* - \delta$ implies

$$\tilde{x}^\lambda \cdot v - \tilde{x} \cdot v = 2(\lambda - \tilde{x}_1)v_1 \geqslant 2(\lambda - \lambda^* + \delta)v_1 \geqslant 2\delta v_1.$$

Thus by the monotonicity of h and (2.13) we have

$$(2.18)\qquad h(\tilde{x}^\lambda \cdot v) - h(\tilde{x} \cdot v) \geqslant \inf_{y\in[0,\lambda^*+d^*+1]} (h(y + 2\delta v_1) - h(y)) > 0\,.$$

A substitution of (2.17) and (2.18) into (2.16) yields

$$0 > -2h(2(\lambda^* + d^*) + 1)C_f C_\lambda \varepsilon + \Big[\inf_{y\in[0,\lambda^*+d^*+1]} \big(h(y + 2\delta v_1) - h(y)\big)\Big] f(u(\tilde{x}^{\lambda^*},\tilde{t})),$$

or equivalently,

$$f(u(\tilde{x}^{\lambda^*},\tilde{t})) < \frac{2h(2(\lambda^* + d^*) + 1)C_f C_\lambda}{\inf\limits_{y\in[0,\lambda^*+d^*+1]} (h(y + 2\delta v_1) - h(y))}\,\varepsilon.$$

Hence, by (f1) it follows that for sufficiently small $\varepsilon > 0$ one has $u(\tilde{x}^{\lambda^*},\tilde{t}) \leqslant \eta$, and for such ε, (2.12) holds true for any $z \in [0, u(\tilde{x}^{\lambda^*},\tilde{t})]$. Then (2.12), (2.13) and (2.15) imply

$$\begin{aligned} Lw_\lambda(\tilde{x},\tilde{t}) &\geqslant h(\tilde{x} \cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x},\tilde{t}))] \\ &\geqslant h(\lambda^* + d^* + 1)\frac{\delta}{h(\lambda^* + d^* + 1)(\lambda^* + 1 + d^*/v_1)^3}\, w_\lambda(\tilde{x},\tilde{t}) \\ &= \frac{\delta}{(\lambda^* + 1 + d^*/v_1)^3}\, w_\lambda(\tilde{x},\tilde{t}). \end{aligned}$$

Easy calculations show that

$$\frac{\delta}{(\lambda^* + 1 + d^*/v_1)^3} \leqslant \frac{\delta}{(\delta + \lambda - y)^3} = -\frac{g''(y)}{2} \leqslant -\frac{g''(y)}{g(y)} \qquad \Big(y \in \Big[\frac{-d^*}{v_1}, \lambda^*\Big]\Big),$$

and since $\tilde{x}_1 \geqslant -d^*/v_1$,

$$Lw_\lambda(\tilde{x},\tilde{t}) \geqslant \frac{\delta}{(\lambda^* + 1 + d^*/v_1)^3}\, w_\lambda(\tilde{x},\tilde{t}) \geqslant -\frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)}\, w_\lambda(\tilde{x},\tilde{t}) = -g''(\tilde{x}_1)\overline{w}_\lambda(\tilde{x},\tilde{t}).$$

Consequently, (2.8) implies

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}.$$

Case 3. Consider $(\tilde{x},\tilde{t}) \in S$ with $\tilde{x}_1 \in [\lambda^* - \delta, \lambda]$. Then by (d2)

$$\tilde{x}\cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leqslant \lambda v_1 + d^* \leqslant \lambda^* v_1 + d^* + 1,$$

and therefore for b_2 and C_f already fixed we have

$$\begin{aligned} Lw_\lambda(\tilde{x},\tilde{t}) &\geqslant h(\tilde{x}\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x},\tilde{t}))] \geqslant h(\lambda^* v_1 + d^* + 1)C_f w_\lambda(\tilde{x},\tilde{t}) \\ &= b_2 w_\lambda(\tilde{x},\tilde{t}). \end{aligned}$$

Moreover, (2.11) implies

$$-g''(y) = \frac{2\delta}{(\delta+\lambda-y)^3} \geqslant 2b_2 + 1 \geqslant g(y)b_2 + 1 \qquad (y \in [\lambda^* - \delta, \lambda]).$$

After a substitution into the previous estimate and then into (2.8), we obtain

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)} - \frac{\overline{w}_\lambda(\tilde{x},\tilde{t})}{g(\tilde{x}_1)}.$$

The rest of the proof uses the comparison principle similarly to Lemma 2.7, for more details see [23, Proof of Claim 4]. □

P r o o f of Theorem 2.3. We proceed by a contradiction, that is, we assume $M := \|u\|_{L^\infty(\Omega\times\mathbb{R})} > 0$. Then by the continuity of u, there are $t_0 \in \mathbb{R}$ and a smooth bounded domain $K_0 \subset \Omega$ with $|K_0| \leqslant 1$ (here $|K_0|$ denotes the Lebesgue measure of K_0) such that $u(x,t_0) > 0$ for all $x \in K_0$. Define

$$K_\sigma := \{x + \sigma e_1 \colon x \in K_0\} \qquad (\sigma \geqslant 0).$$

Since Ω is convex and unbounded in x_1, one has $K_\sigma \subset \Omega$ for all $\sigma \geqslant 0$. Let $\mu > 0$ be the first eigenvalue of the problem

$$\begin{aligned} -\Delta\varphi_0 &= \mu\varphi_0, & x &\in K_0, \\ \varphi_0 &= 0, & x &\in \partial K_0, \end{aligned}$$

where the eigenfunction φ_0 is normalized so that $\max\limits_{K_0}\varphi_0 = 1$. Set

$$\varphi_\sigma(x) := \varphi_0(x_1 - \sigma, x') \qquad (x = (x_1, x') \in K_\sigma)$$

and

$$\psi_\sigma(t) := \int_{K_\sigma} u(x,t)\varphi_\sigma(x)\,\mathrm{d}x \qquad (t \in \mathbb{R}).$$

Since by Lemma 2.9 u is nondecreasing in x_1 and $u > 0$ in $K_0 \times \{t_0\}$,

$$\psi_\sigma(t_0) \geqslant \psi_0(t_0) =: c_0 > 0 \qquad (\sigma \geqslant 0).$$

Denote

$$K_\sigma^*(t) := \{x \in K_\sigma\colon\ u(x,t)\varphi_\sigma(x) \geqslant c_0/2\} \qquad (t \geqslant t_0).$$

If $\psi_\sigma(t^*) \geqslant c_0$ for some $t^* \geqslant t_0$, then (using $|K_\sigma| \leqslant 1$)

$$c_0 \leqslant \int_{K_\sigma} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \leqslant |K_\sigma^*(t^*)| \cdot M + \frac{c_0}{2}|K_\sigma| \leqslant |K_\sigma^*(t^*)| \cdot M + \frac{c_0}{2}.$$

Consequently, $|K_\sigma^*(t^*)| \geqslant \xi := c_0/(2M) > 0$. Next,

$$\begin{aligned}\int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x &\geqslant \xi\frac{c_0}{2} \geqslant \xi \int_{K_\sigma \setminus K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \\ &= \xi \int_{K_\sigma} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x - \xi \int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x.\end{aligned}$$

It follows that

$$\int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \geqslant \frac{\xi}{1+\xi}\int_{K_\sigma} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x = \frac{c_0}{2M+c_0}\psi_\sigma(t^*).$$

Since K is bounded, we can choose R such that K is a subset of the ball of radius R centered at the origin. Then for sufficiently large $\sigma \geqslant 0$

$$\begin{aligned} x \cdot v = x_1 v_1 + x_2 v_2 &\geqslant -|x_1 - \sigma| v_1 + v_1\sigma - R|v_2| \\ &\geqslant R(-v_1 - |v_2|) + v_1\sigma \geqslant \frac{1}{2}v_1\sigma \qquad (x \in K_\sigma).\end{aligned}$$

Hence, for sufficiently large $\sigma \geqslant 0$, using (h2) one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi_\sigma(t^*) &= \int_{K_\sigma} \Delta u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x + \int_{K_\sigma} h(x\cdot v) f(u(x,t^*))\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant \int_{K_\sigma} u(x,t^*)\Delta\varphi_\sigma(x)\,\mathrm{d}x + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma} f(u(x,t^*))\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant \int_{K_\sigma} u(x,t^*)\Delta\varphi_\sigma(x)\,\mathrm{d}x + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma^*(t^*)} \frac{f(u(x,t^*))}{M} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant -\mu\psi_\sigma(t^*) + h\left(\frac{1}{2}v_1\sigma\right) f\left(\frac{c_0}{2}\right)\frac{1}{M}\int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant \psi_\sigma(t^*)\left[-\mu + h\left(\frac{1}{2}v_1\sigma\right) f\left(\frac{c_0}{2}\right)\frac{1}{M}\frac{c_0}{2M+c_0}\right] \\ &\geqslant \psi_\sigma(t^*).\end{aligned}$$

Thus, if $\psi_\sigma(t^*) \geqslant c_0$, then $\psi'_\sigma(t^*) \geqslant 0$, and consequently $\psi'_\sigma(t) \geqslant \psi_\sigma(t) \geqslant c_0$ for each $t \geqslant t^*$. Since $\psi_\sigma(t_0) \geqslant c_0$, one has $\psi'_\sigma(t) \geqslant c_0 > 0$ for each $t > t_0$. Therefore $\psi_\sigma(t) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of u. □

3. Proofs of main results

In this section we use the notation introduced in the previous sections. Especially, recall the definitions of $\mathbb{R}^N_\lambda$ (see (2.1)), H_λ (see (2.2)), x^λ (see (2.4)), and d_p (see (1.18)).

Our main technical tools are the following doubling lemmas.

Lemma 3.1. *Let (X, d) be a compact metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Theta := \Sigma \setminus D$. Also, let $M\colon D \to (0, \infty)$ be a bounded function on compact subsets of D, and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)d(y, \Theta) > 2k,$$

then there exists $x \in D$ such that

$$M(x)d(x, \Theta) > 2k, \quad M(x) \geqslant M(y),$$

and

$$M(z) \leqslant 2M(x) \qquad (z \in D \cap B^*(x, kM^{-1}(x))), \tag{3.1}$$

where $B^*(y, R) := \{x \in X\colon d^*(x, y) \leqslant R\}$ and $d^*(x, y) = |d(x, \Theta) - d(y, \Theta)|$.

Lemma 3.2. *The statement of Lemma 3.1 holds true if (X, d) is a complete metric space and $B^*(x, kM^{-1}(x))$ in (3.1) is replaced by $B(x, kM^{-1}(x))$, where $B(x, R) := \{x \in X\colon d(x, y) \leqslant R\}$.*

Lemma 3.2 was proved in [25, Lemma 5.1]. The proof of Lemma 3.1 is analogous to the proof of [25, Lemma 5.1]. One only replaces every d by d^* and uses compactness of X when passing to the limit.

P r o o f of Theorem 1.1. This proof is partly inspired by the proofs of the corresponding results in [7], [26], [36]. We use the equivalent formulation introduced in Remark 1.3. If (1.17) fails, then there exist $(T_k)_{k\in\mathbb{N}} \subset (0, \infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1) with T replaced by T_k, and $(y_k, s_k)_{k\in\mathbb{N}} \subset \Omega \times (0, T_k)$ such that

$$M_k(y_k, s_k) := u_k^{(p-1)/3}(y_k, s_k) > 2k(1 + d_k^{-1}(s_k)) \qquad (k \in \mathbb{N}),$$

where $d_k(s) := \min\{s, T_k - s\}^{1/2}$. Now, for each $k \in \mathbb{N}$, Lemma 3.2 with $X_k = \Sigma_k = \bar{\Omega} \times [0, T_k]$, $d = d_P$, $D_k = \bar{\Omega} \times (0, T_k)$ and $\Theta_k = \Omega \times \{0, T_k\}$ implies the existence of $(x_k, t_k) \in \bar{\Omega} \times (0, T_k)$ with

$$
\begin{aligned}
M_k(x_k, t_k) &\geqslant M_k(y_k, s_k) > 2k d_k^{-1}(t_k), \\
M_k(x_k, t_k) &\geqslant M_k(y_k, s_k) > 2k, \\
2M_k(x_k, t_k) &\geqslant M_k(x, t) \qquad ((x, t) \in G_k),
\end{aligned}
\tag{3.2}
$$

where

$$G_k := \{(x, t) \in \Omega \times (0, T_k)\colon\ d_P((x, t), (x_k, t_k)) < k\lambda_k\},$$

and

$$\lambda_k := M_k^{-1}(x_k, t_k) \to 0 \quad \text{as } k \to \infty.$$

Here we have used that $d_P((x, t), \Theta_k) = d_k(t)$ for each $(x, t) \in \Sigma_k$. By (3.2)

$$|t - t_k| < k^2 \lambda_k^2 < \frac{d_k^2(t_k)}{4} = \frac{1}{4} \min\{t_k, T_k - t_k\} \qquad ((x, t) \in G_k),$$

and therefore

$$\left\{x \in \Omega\colon\ |x - x_k| < \frac{k\lambda_k}{2}\right\} \times \left(t_k - \frac{k^2\lambda_k^2}{4}, t_k + \frac{k^2\lambda_k^2}{4}\right) \subset G_k.$$

Since the function a is bounded, we can, after passing to a subsequence, assume that $\mathcal{A} := \lim\limits_{k\to\infty} a(x_k)$ exists.

Case (1). First assume $\mathcal{A} \neq 0$. We define a sequence $(v_k)_{k\in\mathbb{N}}$ of rescaled copies of u as

$$v_k(x, t) := \lambda_k^{3/(p-1)} u\big(x_k + \lambda_k^{3/2} x, t_k + \lambda_k^3 t\big) \qquad ((x, t) \in D_k),$$

where

$$D_k := \left\{x \in \lambda_k^{-3/2}(\Omega - x_k)\colon\ |x| < \frac{k}{2\lambda_k^{1/2}}\right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k}\right). \tag{3.3}$$

Then $v_k(0, 0) = 1$ and, by (3.2), $0 \leqslant v_k(x, t) \leqslant 2$ for each $(x, t) \in D_k$. Moreover, v_k satisfies

$$(v_k)_t = \Delta v_k + a\big(x_k + \lambda_k^{\frac{3}{2}} x\big) v_k^p, \quad (x, t) \in D_k, \tag{3.4}$$

$$v_k = 0, \quad (x, t) \in \left\{y \in \lambda_k^{-3/2}(\partial\Omega - x_k)\colon\ |y| < \frac{k}{2\lambda_k^{\frac{1}{2}}}\right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k}\right). \tag{3.5}$$

By passing to a suitable subsequence we may assume either

$$\text{(i)}\quad \frac{\operatorname{dist}(x_k,\partial\Omega)}{\lambda_k^{\frac{3}{2}}}\to\infty \qquad \text{or} \qquad \text{(ii)}\quad \frac{\operatorname{dist}(x_k,\partial\Omega)}{\lambda_k^{\frac{3}{2}}}\to c^*\geqslant 0.$$

If (i) holds, then (3.4), the L^p estimates, and Schauder's estimates yield a subsequence of $(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\mathrm{loc}}(\mathbb{R}^N\times\mathbb{R})$, $\sigma\in(0,1)$ to a function v_∞ satisfying

$$(v_\infty)_t=\Delta v_\infty+\mathcal{A}v_\infty^p,\qquad (x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

Moreover, $v_\infty(0,0)=1$ and $v_\infty\leqslant 2$. However, if $\mathcal{A}>0$ and $p<p_B(N)$ (for the definition of $p_B(N)$ see (1.10)) this contradicts [5, Remark 2.6]. If $\mathcal{A}<0$ and $p>1$ we have a contradiction to Lemma 2.1.

If (ii) holds, then after an application of a suitable orthogonal change of coordinates, the L^p estimates and Schauder's estimates yield a subsequence of $(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\mathrm{loc}}(\mathbb{R}^N_{c^*}\times\mathbb{R})$ to a function v_∞ satisfying

$$\begin{aligned}(v_\infty)_t&=\Delta v_\infty+\mathcal{A}v_\infty^p, &&(x,t)\in\mathbb{R}^N_{c^*}\times\mathbb{R},\\ v_\infty&=0, &&(x,t)\in\partial\mathbb{R}^N_{c^*}\times\mathbb{R},\end{aligned}$$

with $v_\infty(0,0)=1$ and $v_\infty\leqslant 2$. However, if $\mathcal{A}>0$ and $p<p_S(N)\leqslant p_B(N-1)$, then this contradicts [26, Theorem 2.1]. If $\mathcal{A}<0$ and $p>1$, we have a contradiction to Lemma 2.2.

Case (2). Assume $\mathcal{A}=0$. Since a is bounded in $C^2(\bar{\Omega})$, we can assume, after passing to a subsequence, that there exists a vector $\mathcal{B}:=\lim\limits_{k\to\infty}\nabla a(x_k)\in\mathbb{R}^N$. Then (1.3) implies $\mathcal{B}\neq 0$.

If $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence, we can, after appropriate restriction, assume the existence of $x_\infty:=\lim\limits_{k\to\infty}x_k$. Then $\mathcal{A}=a(x_\infty)=0$. Set $\tilde{z}_k:=x_\infty$ and $V_k:=\mathcal{V}:=\Omega$ for each $k\in\mathbb{N}$

If $(x_k)_{k\in\mathbb{N}}$ has no convergent subsequence, we can assume $|x_k-x_l|\geqslant 3$ for each $k\neq l$. Let V_k be the connected component of $B_1(x_k)\cap\Omega$ containing x_k, where $B_1(y)$ is the unit ball centered at y. By [16, Lemma 6.37], there exists an extension of $a\in C^2(\overline{V}_k)$ to $C^2(\overline{B}_1(x_k))$, which we denote again by a. Since $V_k\cap V_l=\emptyset$ for $k\neq l$, the function a is well defined on $\mathcal{V}:=\bigcup\limits_{k\in\mathbb{N}}\overline{B}_1(x_k)$.

Denote $\tilde{\Gamma}:=\{x\in\overline{\mathcal{V}}\colon\ a(x)=0\}$. Since $a\in C^2(\mathcal{V})$, $\mathcal{A}=0$, and $\mathcal{B}\neq 0$, there is $(\tilde{z}_k)_{k\in\mathbb{N}}\subset\tilde{\Gamma}$ with $|x_k-\tilde{z}_k|\to 0$ as $k\to\infty$. Define δ_k and $(z_k)_{k\in\mathbb{N}}\subset\tilde{\Gamma}$ such that

$$\delta_k:=|z_k-x_k|=\operatorname{dist}(x_k,\tilde{\Gamma})\leqslant|x_k-\tilde{z}_k|\to 0.$$

Then $a \in C^2(\mathcal{V})$ yields $\lim_{k\to\infty} \nabla a(z_k) = \lim_{k\to\infty} \nabla a(x_k) \neq 0$. Thus we may assume $|\nabla a(z_k)| \neq 0$, and therefore

$$\delta_k = \frac{|\nabla a(z_k)(x_k - z_k)|}{|\nabla a(z_k)|} \qquad (k \in \mathbb{N}).$$

Using that $z_k \in \tilde{\Gamma}$, that is, $a(z_k) = 0$, we obtain

$$a(x_k + \lambda_k x) = \nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2 |x|^2). \tag{3.6}$$

We define a sequence $(w_k)_{k\in\mathbb{N}}$ of rescaled copies of u as

$$w_k(x,t) := \lambda_k^{3/(p-1)} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x,t) \in \tilde{D}_k),$$

where

$$\tilde{D}_k := \left\{ x \in \lambda_k^{-1}(V_k - x_k)\colon\ |x| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right).$$

Then $w_k(0,0) = 1$ and $0 \leqslant w_k(x,t) \leqslant 2$ for each $(x,t) \in \tilde{D}_k$, and w_k satisfies

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k} a(x_k + \lambda_k x) w_k^p, \quad (x,t) \in \tilde{D}_k, \tag{3.7}$$

$$w_k = 0, \quad (x,t) \in \left\{ y \in \lambda_k^{-1}(\partial\Omega - x_k)\colon\ |y| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right). \tag{3.8}$$

Hence, by (3.6),

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k}\big[\nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2|x|^2)\big] w_k^p, \qquad (x,t) \in \tilde{D}_k. \tag{3.9}$$

Case (2a). Assume that there is a suitable subsequence of $(x_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{\nabla a(z_k)(x_k - z_k)}{\lambda_k} = \pm|\mathcal{B}| \lim_{k\to\infty} \frac{\delta_k}{\lambda_k} =: d^* \in \mathbb{R}.$$

By passing to a yet another subsequence we may assume that either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to \infty \qquad \textit{or} \qquad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to c^* \geqslant 0.$$

If (i) holds, then (3.9), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N \times \mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t = \Delta w_\infty + (d^* + \mathcal{B} \cdot x) w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

satisfying $w_\infty(0,0) = 1$, $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory implies that w_∞ is in fact a classical solution. After a suitable orthogonal transformation and translation, we obtain a nontrivial nonnegative bounded solution of the problem

$$(w_\infty)_t = \Delta w_\infty \pm |\mathcal{B}| x_n w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

a contradiction to [23, Theorem 1.1] for any $p > 1$.

If (ii) holds, then $\mathrm{dist}(x_k, \partial\Omega) \to 0$ as $k \to \infty$. After a suitable rotation we have $\nu_\Omega(x_k) \to -e_1$ as $k \to \infty$. Then (3.9), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N_{c^*} \times \mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned} (w_\infty)_t &= \Delta w_\infty + (d^* + \mathcal{B} \cdot x) w_\infty^p, \qquad && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}, \\ w_\infty &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}, \end{aligned}$$

with $w_\infty(0,0) = 1$ and $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory yields that w_∞ is in fact a classical solution. Also, $a \in C^2(\bar{\Omega})$, $\mathrm{dist}(x_k, \partial\Omega) \to 0$ and (1.13) imply

$$0 < \frac{\tilde{c}}{2} \leqslant \liminf_{k\to\infty} \left| \frac{\nabla a(x_k)}{|\nabla a(x_k)|} + e_1 \right| = \left| \frac{\mathcal{B}}{|\mathcal{B}|} + e_1 \right|.$$

Thus, $\mathcal{B}$ is not a multiple of $-e_1$. Now, after a suitable translation, we obtain a contradiction to Corollary 2.4 for any $p > 1$.

Case (2b). After passing to a subsequence, we may assume that

$$\lim_{k\to\infty} \frac{\nabla a(z_k)(x_k - z_k)}{\lambda_k} = \pm|\mathcal{B}| \lim_{k\to\infty} \frac{\delta_k}{\lambda_k} = \pm\infty.$$

Setting

$$y = \frac{x}{\alpha_k}, \quad s = \frac{t}{\alpha_k^2},$$

where

$$\alpha_k := \left(\frac{\lambda_k}{\delta_k |\nabla a(z_k)|} \right)^{\frac{1}{2}} = \left(\frac{\lambda_k}{|\nabla a(z_k)(x_k - z_k)|} \right)^{\frac{1}{2}} \to 0$$

we transform (3.9) to

$$\begin{aligned} (w_k)_s &= \Delta_y w_k + \frac{\alpha_k^2}{\lambda_k} a(x_k + \lambda_k \alpha_k y) w_k^p \\ &= \Delta_y w_k + \frac{\nabla a(z_k)(x_k - z_k + \lambda_k x) + O(\delta_k^2 + \lambda_k^2 |x|^2)}{|\nabla a(z_k)(x_k - z_k)|} w_k^p \\ &= \Delta_y w_k + [\pm 1 + \alpha_k^3 \nabla a(z_k) y + O(\delta_k + \alpha_k^4 \lambda_k |y|^2)] w_k^p, \qquad (y,s) \in \hat{D}_k, \end{aligned}$$

where

$$\hat{D}_k := \left\{ y \in (\lambda_k\alpha_k)^{-1}(\Omega - x_k)\colon\ |y| < \frac{k}{2\alpha_k} \right\} \times \left(-\frac{k^2}{4\alpha_k^2}, \frac{k^2}{4\alpha_k^2} \right).$$

Moreover, by (3.8)

$$w_k = 0, \quad (y, s) \in \left\{ y \in (\lambda_k\alpha_k)^{-1}(\partial\Omega - x_k)\colon\ |y| < \frac{k}{2\alpha_k} \right\} \times \left(-\frac{k^2}{4\alpha_k^2}, \frac{k^2}{4\alpha_k^2} \right).$$

By passing to a yet another subsequence, we may assume either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k\alpha_k} \to \infty \qquad \text{or} \qquad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k\alpha_k} \to c^* \geqslant 0.$$

If (i) holds, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N \times \mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t = \Delta w_\infty \pm w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

and $w_\infty(0,0) = 1$, $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory implies that w_∞ is a classical solution. However, this contradicts [5] (with "+" sign) for any $1 < p < p_B(N)$ and Lemma 2.1 (with "−" sign) for any $p > 1$.

If (ii) holds, then after a suitable orthogonal change of coordinates and a translation, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N_{c^*} \times \mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned} (w_\infty)_t &= \Delta w_\infty \pm w_\infty^p, \qquad && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}, \\ w_\infty &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}, \end{aligned}$$

and $w_\infty(0,0) = 1$, $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory implies that w_∞ is a classical solution. However, this contradicts [26, Theorem 2.1] (with "+" sign) for any $1 < p < p_S(N) \leqslant p_B(N-1)$ and Lemma 2.2 (with "−" sign) for any $p > 1$. □

Let us formulate a sufficient condition that guarantees (1.20).

Lemma 3.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $1 < p < p_B(N)$, and assume that $a \in C^2(\bar{\Omega})$. For a nonnegative classical solution u of* (1.1), (1.2) *define $x^*\colon (0,T) \to \Omega$ such that*

$$u(x^*(t), t) = \sup_{x\in\Omega} u(x,t) \qquad (t \in (0,T)).$$

If there exist $\varepsilon^* > 0$ and $t_0 \in [0,T]$ such that $\operatorname{dist}(x^*(t), \Gamma) \geqslant \varepsilon^*$ for each $t \in [t_0, T]$, then (1.20) *holds with C depending on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$, ε^* and t_0.*

P r o o f. As in the proof of Theorem 1.1, we use the equivalent formulation introduced in Remark 1.3. Assume that (1.20) fails. Then there exist $(T_k)_{k\in\mathbb{N}} \subset (0,\infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1), and a sequence $(y_k, s_k)_{k\in\mathbb{N}} \subset \Omega\times(0,T_k)$ such that

$$\tilde{M}_k(y_k, s_k) > 2k(1 + d_k^{-1}(s_k)),$$

where

$$\tilde{M}_k := u_k^{(p-1)/2}, \qquad d_k(t) = \min\{t, T_k - t\}^{1/2}.$$

Now, Lemma 3.1 with compact $X_k = \Sigma_k = \bar{\Omega}\times[0,T_k]$, $D_k = \bar{\Omega}\times(0,T_k)$ and $\Theta_k = \bar{\Omega}\times\{0,T_k\}$ implies the existence of a sequence $(x_k', t_k) \in \Omega\times(0,T_k)$ with

$$\begin{aligned} \tilde{M}_k(x_k', t_k) &\geqslant \tilde{M}_k(y_k, s_k) > 2kd_k^{-1}(t_k), \\ \tilde{M}_k(x_k', t_k) &\geqslant \tilde{M}_k(y_k, s_k) > 2k, \\ 2\tilde{M}_k(x_k', t_k) &\geqslant \tilde{M}_k(x,t) \qquad ((x,t)\in G_k'), \end{aligned} \tag{3.10}$$

where

$$G_k' := \{(x,t)\in\Omega\times(0,T)\colon\ d_k^*((x,t),(x_k',t_k)) < k\lambda_k'\},$$
$$d_k^*((x,t),(y,s)) := |d_k(t) - d_k(s)| \qquad ((x,t),(y,s)\in X_k),$$

and

$$\lambda_k' := \tilde{M}^{-1}(x_k', t_k) \to 0 \quad \text{as } k\to\infty.$$

Observe that d_k^* does not depend on x, and therefore (3.10) remains true if we replace x_k' by $x_k := x^*(t_k)$ and G_k' by

$$G_k := \{(x,t)\in\Omega\times(0,T)\colon\ d_k^*((x,t),(x_k,t_k)) < k\lambda_k\} \subset G_k',$$

where

$$\lambda_k := \tilde{M}^{-1}(x_k, t_k) \to 0.$$

By our assumptions $\lim\limits_{k\to\infty} a(x_k) \neq 0$. The rest of the proof is now the same as Case (1) in the proof of Theorem 1.1 (see also [26, Theorem 4.1]) with v_k replaced by

$$v_k(x,t) := \lambda^{2/(p-1)} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x,t)\in D_k),$$

and D_k by

$$D_k := \left\{(x,t)\in\lambda_k^{-1}(\Omega - x_k)\colon\ |x| < \frac{k}{2}\right\}\times\left(-\frac{k^2}{2}, \frac{k^2}{2}\right).$$

□

P r o o f of Proposition 1.5. In the proof we implicitly assume that all constants depend on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$ and T. Fix any $\xi \in \partial\Omega$ with $a(\xi) = 0$. Since Ω is convex, we can, after a suitable rotation, assume

$$\xi_1 = \sup_{x\in\Omega} x_1, \quad \text{and therefore} \quad \nu_\Omega(\xi) = e_1.$$

Since ξ is a local minimizer of a in $\bar{\Omega}$, all tangential derivatives of a vanish at ξ. Then (1.7) implies $\partial_{x_1} a(\xi) < 0$. Denote

$$\Omega_\lambda := \{x \in \Omega\colon\ x_1 > \lambda\}.$$

Assume $u \not\equiv 0$, otherwise the statement is trivial. Observe that u satisfies

$$u_t = \Delta u + \alpha(x,t)u, \qquad (x,t) \in \Omega \times (0,T),$$

where $\alpha(x,t) = a(x)u^{p-1}$. By Theorem 1.1, α is bounded on $\Omega \times (0, T/2)$ and the bound depends only on the constants implicitly assumed. Next, the Hopf boundary lemma (see [19, Lemma 2.6]) implies $\partial_{e_1} u(\xi, T/2) < 0$. By the convexity of Ω, we can choose $\lambda < \xi_1$, sufficiently close to ξ_1 such that

$$w_\lambda(x,t) := u(x^\lambda,t) - u(x,t) \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is well defined (for the definition of x^λ and Ω_λ see (2.4)). Since $\partial_{x_1} u(\xi, T/2) < 0$ and $\partial_{x_1} a(\xi) < 0$, we can increase $\lambda < \xi_1$ such that

$$w_\lambda(x, T/2) > 0, \quad \text{and} \quad a(x^\lambda) > a(x) \qquad (x \in \Omega_\lambda).$$

Observe that $\xi_1 - \lambda \geqslant c_1 > 0$, where c_1 is independent of ξ. Since $a(x^\lambda) > a(x)$ for $x \in \Omega_\lambda$, w_λ satisfies

$$(w_\lambda)_t \geqslant \Delta w_\lambda + \alpha^*(x,t) w_\lambda \qquad (x,t) \in \Omega_\lambda \times (0,T),$$

where

$$\alpha^*(x,t) := a(x)\frac{u^p(x^\lambda,t) - u^p(x,t)}{u(x^\lambda,t) - u(x,t)} \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is bounded on compact subintervals of $(0,T)$. Similarly to (2.5)

$$w_\lambda(x,t) \geqslant 0 \qquad ((x,t) \in \partial\Omega_\lambda \times (0,T)).$$

Now, the maximum principle implies $w_\lambda > 0$ in $\Omega_\lambda \times (T/2, T)$. Therefore $|x^*(t) - \xi| \geqslant c_0$ for each $t \in (T/2, T)$. Since c_0 is independent of ξ and $\Gamma \subset \partial\Omega$, one has

$$\operatorname{dist}(x^*(t), \Gamma) \geqslant \operatorname{dist}(x^*(t), \partial\Omega) \geqslant c_0 > 0 \qquad (t \in (T/2, T)),$$

and the statement of the proposition follows from Lemma 3.3. □

Lemma 3.4. *Let $N = 1$, $\Omega = (0,1)$ and fix $\mu \in [0, \frac{1}{2})$. Assume $a \in C^2([0,1])$ has exactly one nondegenerate zero $\mu \in [0, 2\mu]$. Also assume $a(x) < 0$ for $x \in [0, \mu)$ and*

$$u_0(x) \leqslant u_0(x^\mu) \qquad (x \in (0, \mu)). \tag{3.11}$$

If $u \not\equiv 0$ is a nonnegative solution of the problem (1.1), (1.2), *then $|x^*(t) - \mu| \geqslant c_0 > 0$ and c_0 depends on N, p, a, $\|u_0\|_{L^\infty((0,1))}$, T.*

P r o o f. For each $\lambda \in (0, \frac{1}{2})$, define $w_\lambda \colon (0, \lambda) \times (0, \infty) \to \mathbb{R}$ as $w_\lambda(x,t) := u(x^\lambda, t) - u(x,t)$. Since $a(x^\mu) \geqslant 0 \geqslant a(x)$ for each $x \in [0, \mu]$,

$$a(x^\mu) u^p(x^\mu, t) - a(x) u^p(x,t) \geqslant 0 \qquad ((x,t) \in [0,\mu] \times (0,T)).$$

Thus,

$$(w_\mu)_t - (w_\mu)_{xx} \geqslant 0 \qquad ((x,t) \in (0,\mu) \times (0,T)).$$

By (3.11)

$$w_\mu(x, 0) = u_0(x^\mu) - u_0(x) \geqslant 0 \qquad (x \in (0,\mu)).$$

Since $u \not\equiv 0$, the maximum principle implies $u > 0$ in $(0,1) \times (0,T)$. Then similarly to (2.5)

$$w_\mu(0,t) > 0 \quad \text{and} \quad w_\mu(\mu, t) = 0 \qquad (t \in (0,T)).$$

Then by the maximum principle $w_\mu > 0$ in $(0, \mu) \times (0,T)$ and $\partial_x w_\mu(\mu, t) < 0$ for $t \in (0,T)$. Hence, for sufficiently small $\varepsilon_0 > 0$ we obtain

$$w_\lambda(x, T/2) \geqslant 0 \qquad (x \in (0,\lambda), \lambda \in [\mu, \mu + \varepsilon_0)).$$

As above one can show

$$w_\lambda(0,t) > 0 \quad \text{and} \quad w_\lambda(\lambda, t) = 0 \qquad (t \in (T/2, T)).$$

Since $a'(\mu) > 0$, we can decrease $\varepsilon_0 > 0$ to obtain $a(x^\lambda) \geqslant a(x)$ for each $x \in (0, \lambda)$ and each $\lambda \in [\mu, \mu + \varepsilon_0)$. Then

$$(w_\lambda)_t - \Delta w_\lambda \geqslant a(x)[u^p(x^\lambda, t) - u^p(x,t)] = c(x,t) w_\lambda \quad ((x,t) \in (0,\lambda) \times (t_0, T)),$$

where $c(x,t)$ is a continuous function on $[0,\lambda] \times [t_0, T)$ (possibly unbounded as $t \to T$). The maximum principle implies $w_\lambda(x,t) > 0$ for each $(x,t) \in (0,\lambda) \times (t_0,T)$. In particular, $x^*(t) \geqslant \lambda > \mu$ and therefore $|x^*(t) - \mu| \geqslant c_0 > 0$ for each $t \in (t_0, T)$. □

P r o o f of Proposition 1.7. Lemma 3.4 with $\mu = \mu_1$ implies $|x^*(t) - \mu_1| > \varepsilon^* > 0$. If we replace x by $1 - x$ and use Lemma 3.4 with $\mu = 1 - \mu_2$ again, we obtain $|x^*(t) - \mu_2| > \varepsilon^* > 0$. Now, the proposition follows from Lemma 3.3. □

P r o o f of Proposition 1.6. Without loss of generality assume $a(0) \leqslant 0$, otherwise replace x by $1-x$. If $\mu < \frac{1}{2}$, then the proposition follows from Lemma 3.4 and Lemma 3.3. Assume $\mu \in [\frac{1}{2}, 1]$. Similarly to the proof of Lemma 3.4, we can show that $w_\mu(x,t) := u(x^\mu, t) - u(x,t)$ is well defined on $[\mu, 1]$ and satisfies

$$w_\mu(x,t) < 0 \qquad ((x,t) \in (\mu,1)\times(0,T)) \quad \text{and} \quad w'_\mu(\mu,t) < 0 \qquad (t \in (0,T)).$$

Hence, for $\lambda > \mu$ sufficiently close to μ we have $w_\lambda(x, T/2) < 0$ for any $x \in (\lambda, 1)$. Similarly to Lemma 3.4 (using the maximum principle), we prove $w_\lambda(x,t) < 0$ for any $(x,t) \in (\lambda,1)\times(T/2,T)$. Consequently, $|x^*(t) - \mu| > \lambda - \mu > 0$ for all $t \in (T/2, T)$ and the proposition follows from Lemma 3.3. □

References

[1] *N. Ackermann, T. Bartsch, P. Kaplický and P. Quittner*: A priori bounds, nodal equilibria and connecting orbits in indefinite superlinear parabolic problems. Trans. Am. Math. Soc. *360* (2008), 3493–3539.

[2] *H. Amann*: Existence and regularity for semilinear parabolic evolution equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. *11* (1984), 593–676.

[3] *D. Andreucci and E. DiBenedetto*: On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. *18* (1991), 363–441.

[4] *P. Baras and L. Cohen*: Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation. J. Funct. Anal. *71* (1987), 142–174.

[5] *M. F. Bidaut-Véron*: Initial blow-up for the solutions of a semilinear parabolic equation with source term. In: Équations aux dérivées partielles et applications. Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998, pp. 189–198.

[6] *X. Cabré*: On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations. Commun. Pure Appl. Math. *48* (1995), 539–570.

[7] *Y. Du and S. Li*: Nonlinear Liouville theorems and a priori estimates for indefinite superlinear elliptic equations. Adv. Diff. Equ. *10* (2005), 841–860.

[8] *A. Farina*: Liouville-type theorems for elliptic problems. Handbook of differential equations: Stationary partial differential equations, Vol. IV (M. Chipot, ed.). Elsevier/North-Holland, Amsterdam, 2007, pp. 60–116.

[9] *M. Fila and P. Souplet*: The blow-up rate for semilinear parabolic problems on general domains. NoDEA Nonlinear Differ. Equ. Appl. *8* (2001), 473–480.

[10] *M. Fila, P. Souplet, and F. B. Weissler*: Linear and nonlinear heat equations in L^q_δ spaces and universal bounds for global solutions. Math. Ann. *320* (2001), 87–113.

[11] *A. Friedman and B. McLeod*: Blow-up of positive solutions of semilinear heat equations. Indiana Univ. Math. J. *34* (1985), 425–447.

[12] *B. Gidas and J. Spruck*: A priori bounds for positive solutions of nonlinear elliptic equations. Commun. Partial Differ. Equations *6* (1981), 883–901.

[13] *Y. Giga and R. V. Kohn*: Characterizing blowup using similarity variables. Indiana Univ. Math. J. *36* (1987), 1–40.

[14] *Y. Giga, S. Matsui, and S. Sasayama*: Blow up rate for semilinear heat equations with subcritical nonlinearity. Indiana Univ. Math. J. *53* (2004), 483–514.

[15] *Y. Giga, S. Matsui, and S. Sasayama*: On blow-up rate for sign-changing solutions in a convex domain. Math. Methods Appl. Sci. *27* (2004), 1771–1782.
[16] *D. Gilbarg and N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics. Springer, Berlin, 2001. Reprint of the 1998 edition.
[17] *M. A. Herrero and J. J. L. Velázquez*: Blow-up behaviour of one-dimensional semilinear parabolic equations. Ann. Inst. Henri Poincaré Anal. Non Linéaire *10* (1993), 131–189.
[18] *N. V. Krylov*: Nonlinear Elliptic and Parabolic Equations of the Second Order. Mathematics and its Applications (Soviet Series). Vol. 7. D. Reidel Publishing Co., Dordrecht, 1987.
[19] *G. M. Lieberman*: Second Order Parabolic Differential Equations. World Scientific Publishing Co., River Edge, NJ, 1996.
[20] *J. López-Gómez and P. Quittner*: Complete and energy blow-up in indefinite superlinear parabolic problems. Discrete Contin. Dyn. Syst. *14* (2006), 169–186.
[21] *A. Lunardi*: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Progress in Nonlinear Differential Equations and their Applications. Vol. 16. Birkhäuser, Basel, 1995.
[22] *F. Merle and H. Zaag*: Optimal estimates for blowup rate and behavior for nonlinear heat equations. Commun. Pure Appl. Math. *51* (1998), 139–196.
[23] *P. Poláčik and P. Quittner*: Liouville type theorems and complete blow-up for indefinite superlinear parabolic equations. In: Nonlinear elliptic and parabolic problems. Progr. Nonlinear Differential Equations Appl., Vol. 64. Birkhäuser, Basel, 2005, pp. 391–402.
[24] P. Poláčik and P. Quittner A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation. Nonlinear Anal. *64* (2006), 1679–1689.
[25] *P. Poláčik, P. Quittner, and P. Souplet*: Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. Duke Math. J. *139* (2007), 555–579.
[26] *P. Poláčik, P. Quittner, and P. Souplet*: Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations. Indiana Univ. Math. J. *56* (2007), 879–908.
[27] *P. Quittner and F. Simondon*: A priori bounds and complete blow-up of positive solutions of indefinite superlinear parabolic problems. J. Math. Anal. Appl. *304* (2005), 614–631.
[28] *P. Quittner and P. Souplet*: Superlinear parabolic problems. Blow-up, global existence and steady states. Birkhäuser Advanced Texts: Basel Textbooks. Birkhäuser, Basel, 2007.
[29] *P. Quittner, P. Souplet, and M. Winkler*: Initial blow-up rates and universal bounds for nonlinear heat equations. J. Differ. Equations *196* (2004), 316–339.
[30] *J. Serrin*: Entire solutions of nonlinear Poisson equations. Proc. London. Math. Soc. (3) *24* (1972), 348–366.
[31] *J. Serrin*: Entire solutions of quasilinear elliptic equations. J. Math. Anal. Appl. *352* (2009), 3–14.
[32] *S. D. Taliaferro*: Isolated singularities of nonlinear parabolic inequalities. Math. Ann. *338* (2007), 555–586.
[33] *S. D. Taliaferro*: Blow-up of solutions of nonlinear parabolic inequalities. Trans. Amer. Math. Soc. *361* (2009), 3289–3302.
[34] *F. B. Weissler*: Single point blow-up for a semilinear initial value problem. J. Differ. Equations *55* (1984), 204–224.
[35] *F. B. Weissler*: An L^∞ blow-up estimate for a nonlinear heat equation. Commun. Pure Appl. Math. *38* (1985), 291–295.

[36] *R. Xing*: The blow-up rate for positive solutions of indefinite parabolic problems and related Liouville type theorems. Acta Math. Sin. (Engl. Ser.) *25* (2009), 503–518.

Author's address: Juraj Földes, Vanderbilt University, Nashville, TN 37240, USA, e-mail: juraj.foldes@vanderbilt.edu.