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LINEAR MAPS THAT STRONGLY PRESERVE REGULAR MATRICES OVER THE BOOLEAN ALGEBRA

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Abstract. The set of all $m \times n$ Boolean matrices is denoted by $\mathbb{M}_{m,n}$. We call a matrix $A \in \mathbb{M}_{m,n}$ *regular* if there is a matrix $G \in \mathbb{M}_{n,m}$ such that $AGA = A$. In this paper, we study the problem of characterizing linear operators on $\mathbb{M}_{m,n}$ that strongly preserve regular matrices. Consequently, we obtain that if $\min\{m,n\} \leqslant 2$, then all operators on $\mathbb{M}_{m,n}$ strongly preserve regular matrices, and if $\min\{m,n\} \geqslant 3$, then an operator T on $\mathbb{M}_{m,n}$ strongly preserves regular matrices if and only if there are invertible matrices U and V such that $T(X) = UXV$ for all $X \in \mathbb{M}_{m,n}$, or $m = n$ and $T(X) = UX^TV$ for all $X \in \mathbb{M}_n$.

Keywords: Boolean algebra, regular matrix, (U,V)-operator

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1. Introduction

The *Boolean algebra* consists of the set $\mathbb{B} = \{0,1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1+1=1$. Let $\mathbb{M}_{m,n}$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B}$. The usual definitions for addition and multiplication of matrices over fields are applied to $\mathbb{M}_{m,n}$ as well. If $m = n$, we use the notation $\mathbb{M}_n$ instead of $\mathbb{M}_{n,n}$.

A matrix $X \in \mathbb{M}_n$ is said to be *invertible* if there is a matrix $Y \in \mathbb{M}_n$ such that

$$XY = YX = I_n,$$

where I_n is the $n \times n$ identity matrix.

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The notion of the generalized inverse of an arbitrary matrix apparently originated in the work of Moore [5], and the generalized inverses have applications in network and switching theory and information theory ([2]).

Let A be a matrix in $\mathbb{M}_{m,n}$. Consider a matrix $X \in \mathbb{M}_{n,m}$ in the equation

$$AXA = A. \tag{1.1}$$

Then X is called a *generalized inverse* of A if X is a solution of (1.1). Furthermore, A is called *regular* if there is a solution of (1.1).

The equation (1.1) has been studied by several authors ([3], [5], [6], [7]). Rao and Rao [7] characterized all regular matrices in $\mathbb{M}_n$. Also Plemmons [6] published algorithms for computing generalized inverses of regular matrices in $\mathbb{M}_n$ under certain conditions.

In this paper, we study some properties of regular matrices over $\mathbb{B}$. We also determine the linear operators on $\mathbb{M}_{m,n}$ that strongly preserve regular matrices over the Boolean algebra.

2. Preliminaries and some results

The matrix I_n is the $n \times n$ identity matrix, $O_{m,n}$ is the $m \times n$ zero matrix, and $J_{m,n}$ is the $m \times n$ matrix all of whose entries are 1. We will suppress the subscripts on these matrices when the orders are evident from the context and we write I, O and J, respectively. For any matrix X, X^T denotes the transpose of X. A matrix in $\mathbb{M}_{m,n}$ with only one nonzero entry equal to 1 is called a *cell*. If the nonzero entry occurs in the i^{th} row and the j^{th} column, we denote the cell by $E_{i,j}$.

Matrices J and O in $\mathbb{M}_{m,n}$ are regular because $JGJ = J$ and $OGO = O$ for all cells G in $\mathbb{M}_{n,m}$. Therefore in general, a solution of (1.1), although it exists, is not necessarily unique. Furthermore, each cell $E \in \mathbb{M}_{m,n}$ is regular because $EE^TE = E$.

Proposition 2.1. *The regularity of a matrix $A \in \mathbb{M}_{m,n}$ is preserved under pre- or post-multiplication by an invertible matrix. Furthermore, the regularity of A is preserved by its transposition.*

Proof. This is an easy exercise. □

Also we can easily show that for a matrix $A \in \mathbb{M}_{m,n}$,

$$A \text{ is regular if and only if } \begin{bmatrix} A & O \\ O & B \end{bmatrix} \text{ is regular} \tag{2.1}$$

for all regular matrices $B \in \mathbb{M}_{n,q}$. In particular, all idempotent matrices in $\mathbb{M}_n$ are regular.

For matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{M}_{m,n}$, we say that A *dominates* B (written $A \sqsupseteq B$ or $B \sqsubseteq A$) if $a_{i,j} = 0$ implies $b_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathbb{M}_{m,n}$. If $A, B \in \mathbb{M}_{m,n}$ with $A \sqsupseteq B$, we define $A \setminus B$ to be the matrix $C = [c_{i,j}] \in \mathbb{M}_{m,n}$ such that $c_{i,j} = 1$ if and only if $a_{i,j} = 1$ and $b_{i,j} = 0$.

Define an upper triangular matrix Λ_n in $\mathbb{M}_n$ by

$$\Lambda_n = [\lambda_{i,j}] \equiv \left(\sum_{i \leqslant j}^{n} E_{i,j}\right) \setminus E_{1,n} = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 \\ & 1 & \dots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

Then the following lemma shows that Λ_n is not regular for $n \geqslant 3$.

Lemma 2.2. *Λ_n is regular in $\mathbb{M}_n$ if and only if $n \leqslant 2$.*

Proof. For $n \leqslant 2$, clearly Λ_n is regular because $\Lambda_n I_n \Lambda_n = \Lambda_n$.

Conversely, assume that Λ_n is regular for some $n \geqslant 3$. Then there is a nonzero $B = [b_{i,j}] \in \mathbb{M}_n$ such that $\Lambda_n = \Lambda_n B \Lambda_n$. From

$$0 = \lambda_{1,n} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} b_{i,j}$$

we obtain that all entries of the second column of B are zero except for the entry $b_{n,2}$. From

$$0 = \lambda_{2,1} = \sum_{i=2}^{n} b_{i,1}$$

we have that all entries of the first column of B are zero except for $b_{1,1}$. Also, from

$$0 = \lambda_{3,2} = \sum_{i=3}^{n} \sum_{j=1}^{2} b_{i,j}$$

we obtain $b_{n,2} = 0$. If we combine these three results, we conclude that all entries of the first two columns are zero except for $b_{1,1}$. But we have

$$1 = \lambda_{2,2} = \sum_{i=2}^{n} \sum_{j=1}^{2} b_{i,j} = 0,$$

a contradiction. Hence Λ_n is not regular for all $n \geqslant 3$. □

In particular, $\Lambda_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not regular in $\mathbb{M}_3$. Let

$$\Phi_{m,n} = \begin{bmatrix} \Lambda_3 & O \\ O & O \end{bmatrix} \tag{2.2}$$

for all $\min\{m,n\} \geqslant 3$. Then $\Phi_{m,n}$ is not regular in $\mathbb{M}_{m,n}$ by virtue of (2.1).

The *factor rank* ([1]), $b(A)$, of a nonzero $A \in \mathbb{M}_{m,n}$ is defined as the least integer k for which there are matrices B and C of orders $m \times k$ and $k \times n$, respectively, such that $A = BC$. The rank of a zero matrix is zero. Also we can easily obtain that

$$0 \leqslant b(A) \leqslant \min\{m,n\} \quad \text{and} \quad b(AB) \leqslant \min\{b(A), b(B)\} \tag{2.3}$$

for all $A \in \mathbb{M}_{m,n}$ and for all $B \in \mathbb{M}_{n,q}$.

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n]$ be a matrix in $\mathbb{M}_{m,n}$, where $\mathbf{a}_j$ denotes the j^{th} column of A for all j. Then the *column space* of A is the set $\left\{\sum_{j=1}^{n} \alpha_j \mathbf{a}_j \mid \alpha_j \in \mathbb{B}\right\}$, denoted by $\langle A\rangle$; the *row space* of A is $\langle A^T\rangle$.

For a matrix $A \in \mathbb{M}_{m,n}$ with $b(A) = k$, A is said to be *space decomposable* if there are matrices B and C of orders $m \times k$ and $k \times n$, respectively, such that $A = BC$, $\langle A\rangle = \langle B\rangle$ and $\langle A^T\rangle = \langle C^T\rangle$.

Theorem 2.3 ([7])**.** *A is regular in $\mathbb{M}_{m,n}$ if and only if A is space decomposable.*

Lemma 2.4. *If $A \in \mathbb{M}_{m,n}$ with $b(A) \leqslant 2$, then A is regular.*

Proof. If $b(A) = 0$, then $A = O$ is clearly regular. If $b(A) = 1$, there are permutation matrices P and Q such that $PAQ = \begin{bmatrix} J & O \\ O & O \end{bmatrix}$, and hence PAQ is regular by (2.1). It follows from Proposition 2.1 that A is regular.

If $b(A) = 2$, there are matrices $B = [\mathbf{b}_1\ \mathbf{b}_2]$ and $C = [\mathbf{c}_1\ \mathbf{c}_2]^T$ of orders $m \times 2$ and $2 \times n$, respectively, such that $A = BC$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are distinct nonzero columns of B, and $\mathbf{c}_1$ and $\mathbf{c}_2$ are distinct nonzero columns of C^T. Then we can easily show that all columns of A are of the forms $\mathbf{0}, \mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_1 + \mathbf{b}_2$ so that $\langle A\rangle = \langle B\rangle$. Similarly, we have $\langle A^T\rangle = \langle C^T\rangle$. Therefore A is space decomposable and hence A is regular by Theorem 2.3. □

The number of nonzero entries of a matrix $A \in \mathbb{M}_{m,n}$ is denoted by $|A|$.

Corollary 2.5. *Let A be a nonzero matrix in $\mathbb{M}_{m,n}$, where $\min\{m,n\} \geqslant 3$.*

(i) *If $|A| \leqslant 4$, then A is regular;*

(ii) *if A is a cell, there is a regular matrix B such that $A+B$ is not regular;*

(iii) *if $|A| = 3$ and $b(A) = 2$ or 3, there is a matrix C with $|C| = 2$ such that $A+C$ is not regular;*

(iv) *if $|A| = 5$ and A has a row or a column that has at least 3 nonzero entries, then A is regular.*

Proof. (i) By Lemma 2.4, we lose no generality in assuming that $b(A) \geqslant 3$ so that $b(A) = 3$ or 4. Consider $X = \begin{bmatrix} A & O \\ O & 0 \end{bmatrix}$ in $\mathbb{M}_{m+1,n+1}$. Since $|A| \leqslant 4$ and $b(A) = 3$ or 4, we can easily show that there are permutation matrices P and Q such that $PXQ = \begin{bmatrix} Y & O \\ O & O \end{bmatrix}$ for some idempotent matrix $Y \in \mathbb{M}_4$ with $|Y| = 3$ or 4. By (2.1) and Proposition 2.1, X is regular and hence A is regular by (2.1).

(ii) Consider the matrix $\Phi_{m,n}$ in (2.2). Let P and Q be permutation matrices such that $PAQ = E_{1,1}$. Consider the matrix B satisfying $PBQ = E_{1,2}+E_{2,2}+E_{2,3}+E_{3,3}$. Then

$$(PBQ)(G_{2,1} + G_{3,3})(PBQ) = PBQ \quad \text{and} \quad P(A+B)Q = \Phi_{m,n},$$

where $G_{i,j}$ are cells in $\mathbb{M}_{n,m}$. Thus $A+B$ is not regular, while B is regular by Proposition 2.1.

(iii) Similar to (ii).

(iv) If $|A| = 5$ and A has a row or a column that has at least 3 nonzero entries, then we can easily show that $b(A) \leqslant 3$. By Lemma 2.4, it suffices to consider $b(A) = 3$. Then A has either a row or a column that has just 3 nonzero entries. Suppose that a row of A has just 3 nonzero entries. Since $b(A) = 3$, there are permutation matrices P and Q such that

$$PAQ = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,i} + E_{3,j}$$

for some i and j with $i < j$. If $j \geqslant 4$, then PAQ is regular by Lemma 2.6 and (2.1) because $b(E_{1,1} + E_{1,2} + E_{1,3} + E_{2,i}) = 2$. Hence A is regular by Proposition 2.1. If $1 \leqslant i < j \leqslant 3$, there are permutation matrices P' and Q' such that $P'PAQQ' = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$, where $D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can easily show that D is idempotent in $\mathbb{M}_3$, and hence D is regular. It follows from (2.1) and Proposition 2.1 that A is regular.

If a column of A has just 3 nonzero entries, a parallel argument shows that A is regular. □

Linearity of operators on $\mathbb{M}_{m,n}$ is defined as for vector spaces over fields. A linear operator on $\mathbb{M}_{m,n}$ is completely determined by its behavior on the set of cells in $\mathbb{M}_{m,n}$.

An operator T on $\mathbb{M}_{m,n}$ is said to

(1) be *singular* if $T(X) = O$ for some nonzero $X \in \mathbb{M}_{m,n}$; otherwise T is *nonsingular*;
(2) *preserve regularity* if $T(A)$ is regular whenever A is regular in $\mathbb{M}_{m,n}$;
(3) *strongly preserve regularity* if $T(A)$ is regular if and only if A is regular in $\mathbb{M}_{m,n}$.

Example 2.6. Let A be any regular matrix in $\mathbb{M}_{m,n}$, where at least one entry of A is 1. Define an operator T on $\mathbb{M}_{m,n}$ by

$$T(X) = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} \right) A$$

for all $X = [x_{i,j}] \in \mathbb{M}_{m,n}$. Then we can easily show that T is nonsingular and T is a linear operator that preserves regularity. But T does not preserve any matrix that is not regular in $\mathbb{M}_{m,n}$. □

Thus, we are interested in linear operators on $\mathbb{M}_{m,n}$ that strongly preserve regularity.

Lemma 2.7. *Let* $\min\{m, n\} \geqslant 3$. *If* T *is a linear operator on* $\mathbb{M}_{m,n}$ *that strongly preserves regularity, then* T *is nonsingular.*

Proof. If $T(X) = O$ for some nonzero $X \in \mathbb{M}_{m,n}$, then we have $T(E) = O$ for all cells $E \sqsubseteq X$. For such a cell E, there is a regular matrix B such that $E + B$ is not regular by Corollary 2.5(ii). Nevertheless, $T(E + B) = T(B)$, a contradiction to the fact that T strongly preserves regularity. Hence $T(X) \neq O$ for all nonzero X. Thus T is nonsingular. □

For $\min\{m, n\} \leqslant 2$, all matrices in $\mathbb{M}_{m,n}$ are regular by (2.3) and Lemma 2.4. This proves:

Theorem 2.8. *If* $\min\{m, n\} \leqslant 2$, *then all operators on* $\mathbb{M}_{m,n}$ *strongly preserve regularity.*

3. Linear operators that strongly preserve regular matrices over the Boolean algebra

In this section we have characterizations of the linear operators that strongly preserve regular matrices over the Boolean algebra $\mathbb{B}$.

As shown in Theorem 2.8, each operator T on $\mathbb{M}_{m,n}$ strongly preserves regularity if $\min\{m,n\} \leqslant 2$. Thus in the sequel, unless otherwise stated, we assume that T is a linear operator on $\mathbb{M}_{m,n}$ that strongly preserves regularity for $\min\{m,n\} \geqslant 3$. Furthermore, without loss of generality, we assume that $3 \leqslant m \leqslant n$.

Lemma 3.1. *Let $A \in \mathbb{M}_{m,n}$ with $|A| = k$ and $b(A) = k$. Then $J \setminus A$ is regular if and only if $k \leqslant 2$.*

Proof. If $k \leqslant 2$, there are permutation matrices P and Q such that $P(J\setminus A)Q = J \setminus (aE_{1,1} + bE_{2,2})$, where $a, b \in \mathbb{B}$, and hence

$$P(J \setminus A)Q = \begin{bmatrix} a' & 1 \\ 1 & b' \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \end{bmatrix}$$

so that $b(J \setminus A) = b(P(J \setminus A)Q) \leqslant 2$, where $a + a' = b + b' = 1$ with $a \neq a'$ and $b \neq b'$. Thus we have that $J \setminus A$ is regular by Lemma 2.4.

Conversely, assume that $J \setminus A$ is regular for some $k \geqslant 3$. It follows from $|A| = k$ and $b(A) = k$ that there are permutation matrices U and V such that

$$U(J \setminus A)V = J \setminus \sum_{t=1}^{k} E_{t,t}.$$

Let $J \setminus \sum\limits_{t=1}^{k} E_{t,t} = X = [x_{i,j}]$. By Proposition 2.1, X is regular, and hence there is a nonzero $B = [b_{i,j}] \in \mathbb{M}_{n,m}$ such that $X = XBX$. Then the $(t,t)^{th}$ entry of XBX becomes

$$\sum_{i \in I} \sum_{j \in J} b_{i,j} \tag{3.1}$$

for all $t = 1, \dots, k$, where $I = \{1, \dots, n\} \setminus \{t\}$ and $J = \{1, \dots, m\} \setminus \{t\}$. From $x_{1,1} = 0$ and (3.1) we have

$$b_{i,j} = 0 \quad \text{for all } i = 2, \dots, n; \; j = 2, \dots, m. \tag{3.2}$$

Consider the first row and the first column of B. It follows from $x_{2,2} = 0$ and (3.1) that

$$b_{i,1} = 0 = b_{1,j} \quad \text{for all } i = 1, 3, 4, \ldots, n; \;\; j = 1, 3, 4, \ldots, m. \tag{3.3}$$

Also, from $x_{3,3} = 0$ we obtain $b_{1,2} = b_{2,1} = 0$, and hence $B = O$ by (3.2) and (3.3). This contradiction shows that $k \leqslant 2$. □

Proposition 3.2. *Let $A \in \mathbb{M}_{m,n}$ and let E be a cell with $E \not\sqsubseteq A$. If there are distinct cells F and G that are not dominated by A such that $b(E + F + G) = 3$, then $|T(A)| < |T(A + E)|$.*

P r o o f. Suppose that $|T(A)| = |T(A+E)|$. Let $B = J \setminus (E+F+G)$. It follows from $T(A) = T(A + E)$ that $T(A + B) = T(A + E + B)$, equivalently

$$T(J \setminus (E + F + G)) = T(J \setminus (F + G)),$$

a contradiction because $J \setminus (E + F + G)$ is not regular, while $J \setminus (F + G)$ is regular by Lemma 3.1. Thus the result follows. □

Proposition 3.3. *Let E, F and G be distinct cells in $\mathbb{M}_{m,n}$ with $b(E+F+G) = 3$. Then $|T(J \setminus (E + F + G))| \leqslant mn - 3$.*

P r o o f. By Proposition 3.1, $J\setminus(E+F+G)$ is not regular. If $|T(J\setminus(E+F+G))| \geqslant mn - 2$, then $b(T(J \setminus (E + F + G)) \leqslant 2$ and so $T(J \setminus (E + F + G))$ is regular by Lemma 2.6, a contradiction. Thus the result follows. □

Let $A \in \mathbb{M}_3$. If $|A| \leqslant 4$, then A is regular by Corollary 2.5(i), and if $|A| \geqslant 7$, then $b(A) \leqslant 2$ and so A is regular by Lemma 2.4. Hence if A is not regular, then $|A| = 5$ or 6 and there are permutation matrices P and Q such that PAQ is one of the following forms:

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Furthermore, if E is a cell with $E \sqsubseteq C$, there are permutation matrices P' and Q' such that $P'(C \setminus E)Q' = B$ and hence $C \setminus E$ is not regular.

Lemma 3.4. *For all cells E in $\mathbb{M}_3$, $T(E)$ is a cell.*

Proof. It suffices to show that $|T(E)| = 1$ for all cells E. Suppose that $|T(E)| \geqslant 2$ for some cell E. Without loss of generality we assume that $E = E_{1,1}$. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Since A is not regular, neither is $T(A)$ and hence $|T(A)| \in \{5, 6\}$. Let $B \in \mathbb{M}_3$ be a matrix with $B \sqsubseteq A$ and $|B| = 4$. If $|T(B)| \geqslant 5$, then $T(B)$ is not regular, while B is regular by Corollary 2.5(i), a contradiction. Hence there is no matrix B with $B \sqsubseteq A$ and $|B| = 4$ such that $|T(B)| \geqslant 5$.

It follows from Proposition 3.2 that

$$\begin{aligned} |T(E_{1,1})| &< |T(E_{1,1} + E_{2,2})| < |T(E_{1,1} + E_{2,2} + E_{3,3})| \\ &< |T(E_{1,1} + E_{2,2} + E_{3,3} + E_{1,2})|, \end{aligned}$$

and hence $|T(E_{1,1}+E_{2,2}+E_{3,3}+E_{1,2})| \geqslant 5$ because $|T(E_{1,1})| \geqslant 2$. This is impossible. Thus $|T(E)| \leqslant 1$ and so $|T(E)| = 1$ for all cells E by Lemma 2.7, equivalently $T(E)$ is a cell for all cells E. □

The following example is good for showing that $|T(A)| \leqslant 3$ for all matrices $A \in \mathbb{M}_{m,n}$ with $|A| = 2$, where $n \geqslant 4$.

Example 3.5. Consider $\mathbb{M}_{3,4}$. By Propositions 3.2 and 3.3 we have

$$\begin{aligned} |T(E_{1,1} + E_{2,2})| &< |T(E_{1,1} + E_{2,2} + E_{1,2})| \\ &< |T(E_{1,1} + E_{2,2} + E_{1,2} + E_{3,3})| \\ &< |T(E_{1,1} + E_{2,2} + E_{1,2} + E_{3,3} + E_{3,4})| \\ &< |T(E_{1,1} + E_{2,2} + E_{1,2} + E_{3,3} + E_{3,4} + E_{1,3})| \\ &< |T(E_{1,1} + E_{2,2} + E_{1,2} + E_{3,3} + E_{3,4} + E_{1,3} + E_{2,1})| \\ &< |T(E_{1,1} + E_{2,2} + E_{1,2} + E_{3,3} + E_{3,4} + E_{1,3} + E_{2,1} + E_{3,1})| \\ &\leqslant |T(J \setminus (E_{1,4} + E_{2,3} + E_{3,2}))| \leqslant 3 \cdot 4 - 3 = 9. \end{aligned}$$

From this inequality, we have $|T(E_{1,1} + E_{2,2})| \leqslant 3$. □

A matrix $L \in \mathbb{M}_{m,n}$ is called a *line matrix* if

$$L = \sum_{s=1}^{n} E_{i,s} \quad \text{or} \quad L = \sum_{t=1}^{m} E_{t,j}$$

for some $i \in \{1, \dots, m\}$ or $j \in \{1, \dots, n\}$; $R_i = \sum\limits_{s=1}^{n} E_{i,s}$ is the i^{th} *row matrix* and $C_j = \sum\limits_{t=1}^{m} E_{t,j}$ is the j^{th} *column matrix.*

Proposition 3.6. *Let A be a matrix in $\mathbb{M}_{m,n}$ with $|A| = 2$, where $n \geqslant 4$. Then we have $|T(A)| \leqslant 3$.*

P r o o f. Without loss of generality we assume that

$$A = E_{1,1} + E_{2,2}, \quad A = E_{1,1} + E_{1,2} \quad \text{or} \quad A = E_{1,1} + E_{2,1}.$$

Let $B = E_{1,1} + E_{1,2} + E_{2,2}$. By Proposition 3.2 we have $|T(E_{1,1} + E_{2,2})| < |T(B)|$ and $|T(E_{1,1} + E_{1,2})| < |T(B)|$. Furthermore, we have

$$\begin{aligned} |T(B)| &< |T(B + E_{4,1})| < \ldots < |T(B + R_4)| \\ &< |T(B + R_4 + E_{5,1})| < \ldots < |T(B + R_4 + R_5)| \\ &< \ldots < |T(B + R_4 + \ldots + R_m)|. \end{aligned}$$

Let $X_1 = B + R_4 + \ldots + R_m$. Again by Proposition 3.2,

$$|T(X_1)| < |T(X_1 + E_{3,3})| < \ldots < |T(X_1 + E_{3,3} + \ldots + E_{3,n})|.$$

Let $X_2 = X_1 + E_{3,3} + \ldots + E_{3,n}$. By Proposition 3.2 we have

$$|T(X_2)| < |T(X_2 + E_{1,3})| < \ldots < |T(X_2 + E_{1,3} + \ldots + E_{1,n-1})|.$$

Let $X_3 = X_2 + E_{1,3} + \ldots + E_{1,n-1}$. It follows from Propositions 3.2 and 3.3 that

$$\begin{aligned} |T(X_3)| &< |T(X_3 + E_{2,1})| < |T(X_3 + E_{2,1} + E_{2,3})| \\ &< \ldots < |T(X_3 + E_{2,1} + E_{2,3} + \ldots + E_{2,n-2})| \\ &< |T(X_3 + E_{2,1} + E_{2,3} + \ldots + E_{2,n-2} + E_{3,1})| \\ &\leqslant |T(J \setminus (E_{1,n} + E_{2,n-1} + E_{3,2}))| \leqslant mn - 3. \end{aligned}$$

Thus we have

$$\begin{aligned} |T(X_3)| &\leqslant mn - 3 - (n-2) = (m-1)n - 1, \\ |T(X_2)| &\leqslant (m-1)n - 1 - (n-3) = (m-2)n + 2, \\ |T(X_1)| &\leqslant (m-2)n + 2 - (n-2) = (m-3)n + 4, \\ |T(B)| &\leqslant (m-3)n + 4 - (m-3)n = 4. \end{aligned}$$

Hence $|T(E_{1,1} + E_{2,2})| \leqslant 3$ and $|T(E_{1,1} + E_{1,2})| \leqslant 3$.

A parallel argument shows that $|T(E_{1,1} + E_{2,1})| \leqslant 3$. Thus the result follows. □

Lemma 3.7. *$T(E)$ is a cell for all cells $E \in \mathbb{M}_{m,n}$.*

Proof. For the case $n = 3$, we are done by Lemma 3.4. Assume that $n \geqslant 4$.

It suffices to show that $|T(E)| = 1$ for all cells E. First we claim that $|T(E)| \leqslant 2$ for all cells E. Let F be a cell different from E. By Propositions 3.2 and 3.6 we have $|T(E)| < |T(E+F)| \leqslant 3$. Hence $|T(E)| \leqslant 2$.

Now, suppose that $|T(E)| \geqslant 2$ for some cell E. Without loss of generality we assume that $E = E_{1,1}$. It follows from Propositions 3.2 and 3.6 that

$$|T(E_{1,1})| = 2 \quad \text{and} \quad |T(E_{1,1} + E_{i,j})| = 3$$

for all $(i,j) \neq (1,1)$. This means that for each cell $E_{i,j}$ with $(i,j) \neq (1,1)$, there is a single cell G such that $G \not\sqsubseteq T(E_{1,1})$, $G \sqsubseteq T(E_{i,j})$ and

$$T(E_{1,1} + E_{i,j}) = T(E_{1,1}) + G.$$

Let (s,t) be an arbitrary pair different from $(1,1)$ and (i,j). Similarly there is a single cell H such that $H \not\sqsubseteq T(E_{1,1})$, $H \sqsubseteq T(E_{s,t})$ and $T(E_{1,1}+E_{s,t}) = T(E_{1,1})+H$. It follows from Proposition 3.2 that $G \neq H$. Thus we have

$$|T(J \setminus (E_{1,3} + E_{2,2} + E_{3,1}))| = 2 + (mn - 4) = mn - 2,$$

a contradiction because $J \setminus (E_{1,3} + E_{2,2} + E_{3,1})$ is not regular by Lemma 3.1, while $T(J \setminus (E_{1,3} + E_{2,2} + E_{3,1}))$ is regular by Lemma 2.4. Thus the result follows. □

Corollary 3.8. *T is bijective on the set of cells in $\mathbb{M}_{m,n}$.*

Proof. By Lemma 3.7, it suffices to show that $T(E) \neq T(F)$ for all distinct cells E and F. Suppose $T(E) = T(F)$ for some distinct cells E and F. Then we have $T(E) = T(E+F)$. But this is impossible because $|T(E)| < |T(E+F)|$ by Proposition 3.2. Thus the result follows. □

Lemma 3.9. *If $A \in \mathbb{M}_{m,n}$ is a matrix with $|A| = 3$ and $b(A) = 1$, then $b(T(A)) = 1$.*

Proof. Suppose that $A \in \mathbb{M}_{m,n}$ is a matrix with $|A| = 3$ and $b(A) = 1$. By Corollary 3.8, we have $|T(A)| = 3$. If $b(T(A)) \neq 1$, then $b(T(A)) \in \{2,3\}$ and hence there is a matrix B with $|B| = 2$ such that $T(A) + B$ is not regular by Corollary 2.5(iii). Furthermore Corollary 3.8 implies that there is a matrix C with $|C| = 2$ such that $T(C) = B$. But it follows from Corollary 2.5(iv) that $A + C$ is regular, while $T(A+C) = T(A) + B$ is not regular, a contradiction. Hence we have $b(T(A)) = 1$. □

Corollary 3.10. *T preserves all line matrices.*

Proof. By Corollary 3.8, T is bijective on the set of cells. If T does not map some line matrix into a line matrix, there is a matrix $A \in \mathbb{M}_{m,n}$ with $|A| = 2$ and $b(A) = 1$ such that $|T(A)| = 2$ and $b(T(A)) = 2$. Take a cell E with $|A+E| = 3$ and $b(A+E) = 1$. Then by Lemma 3.9, we have $b(T(A+E)) = 1$. But this is impossible because $b(T(A)) = 2$. Therefore the result follows. □

A linear operator T on $\mathbb{M}_{m,n}$ is called a (U,V)-*operator* if there are invertible matrices U and V such that $T(X) = UXV$ for all $X \in \mathbb{M}_{m,n}$, or $m = n$ and $T(X) = UX^TV$ for all $X \in \mathbb{M}_n$.

Recall that the $n \times n$ permutation matrices are the only invertible matrices in $\mathbb{M}_n$. Now, we are ready to prove the main theorem.

Theorem 3.11. *Let T be a linear operator on $\mathbb{M}_{m,n}$ with $\min\{m,n\} \geqslant 3$. Then the followings are equivalent:*

(a) *T strongly preserves regularity;*

(b) *T is a (U,V)-operator.*

Proof. It follows from Proposition 2.1 that (b) implies (a). To prove that (a) implies (b), assume that T strongly preserves regularity. Then T is bijective on the set of cells by Corollary 3.8 and T preserves all line matrices by Corollary 3.10. Since no combination of s row matrices and t column matrices can dominate $J_{m,n}$ where $s + t = \min\{m,n\}$ unless $s = 0$ or $t = 0$, we have that either

(1) the image of each row matrix is a row matrix and the image of each column matrix is a column matrix, or

(2) the image of each row matrix is a column matrix and the image of each column matrix is a row matrix.

If (1) holds, then there are permutations σ and τ of $\{1,\ldots,m\}$ and $\{1,\ldots,n\}$, respectively, such that $T(R_i) = R_{\sigma(i)}$ and $T(C_j) = C_{\tau(j)}$ for all i and j. Let U and V be permutation (i.e., invertible) matrices corresponding to σ and τ, respectively. Then we have

$$T(E_{i,j}) = E_{\sigma(i),\tau(j)} = UE_{i,j}V$$

for all cells $E_{i,j}$. Let $X = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} x_{i,j}E_{i,j}$ be any matrix in $\mathbb{M}_{m,n}$. By the action of T on the cells, we have $T(X) = UXV$. If (2) holds, then $m = n$ and a parallel argument shows that there are invertible matrices U and V such that $T(X) = UX^TV$ for all $X \in \mathbb{M}_n$. □

Thus, as shown in Theorems 2.8 and 3.11, we have characterizations of the linear operators that strongly preserve regular matrices over the Boolean algebra.

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