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WEAKLY COERCIVE MAPPINGS SHARING A VALUE

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Abstract. Some sufficient conditions are provided that guarantee that the difference of a compact mapping and a proper mapping defined between any two Banach spaces over \mathbb{K} has at least one zero. When conditions are strengthened, this difference has at most a finite number of zeros throughout the entire space. The proof of the result is constructive and is based upon a continuation method.

Keywords: zero point, continuation method, C^1 -homotopy, surjective implicit function theorem, proper mapping, compact mapping, coercive mapping, Fredholm mapping

MSC 2010: 58C30, 65H10

1. PRELIMINAIRES

Let X and Y be two Banach spaces. If $u: D(F) \subseteq X \rightarrow Y$ is a continuous mapping, then one way of solving the equation

$$(1) \quad u(x) = 0$$

is to embed (1) in a continuum of problems

$$(2) \quad H(x, t) = 0 \quad (0 \leq t \leq 1),$$

which can easily be resolved when $t = 0$. When $t = 1$, the problem (2) becomes (1). In the case when it is possible to continue the solution for all t in $[0, 1]$ then (1) is solved. This method is called continuation with respect to a parameter [1]–[9].

In this paper some sufficient conditions are provided in order to guarantee that the difference of a compact and a proper weakly coercive C^1 -mapping has at least

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one zero. If these conditions are conveniently strengthened this difference has at most a finite number of zeros on X . Other conditions, sufficient to guarantee the existence of fixed points, have been given by the author in a finite-dimensional setting, see for example [7], and in an infinite-dimensional setting, see for example [8]. A continuation method was used in the proofs of these papers. The proofs supply the existence of implicitly defined continuous mappings whose ranges reach zero points [1]–[9]. A continuation method is also used here. The key is the use of the surjective implicit function theorem [10], and the properties of proper and Fredholm C^1 -mappings (see [9]).

We briefly recall some theorems and concepts to be used.

Definitions [26], [27]. Henceforth we will assume that X and Y are Banach spaces over \mathbb{K} , where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

A mapping $F: D(F) \subseteq X \rightarrow Y$ is called *weakly coercive* if and only if $\|F(x)\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

A mapping $F: D(F) \subseteq X \rightarrow Y$ is said to be *compact* whenever it is continuous and the image $F(B)$ is relatively compact (i.e. its closure $\overline{F(B)}$ is compact in Y) for every bounded subset $B \subset D(F)$.

A mapping F is said to be *proper* whenever the pre-image $F^{-1}(K)$ of every compact subset $K \subset Y$ is also a compact subset of $D(F)$.

The symbol *dim* means dimension, *codim* means codimension, *ker* means kernel, $R(L)$ stands for the range of the mapping L .

That $L: X \rightarrow Y$ is a *linear Fredholm* mapping means that L is linear and continuous and both the numbers $\dim(\ker(L))$ and $\text{codim}(R(L))$ are finite, and therefore $\ker(L) = X_1$ is a Banach space and has the topological complement X_2 , since $\dim(X_1)$ is finite. The integer number $\text{Ind}(L) = \dim(\ker(L)) - \text{codim}(R(L))$ is called the *index* of L .

Let $F: D(F) \subseteq X \rightarrow Y$. If $D(F)$ is open, then the mapping F is said to be a *Fredholm* mapping if and only if F is a C^1 -mapping and $F'(x): X \rightarrow Y$ is a Fredholm linear mapping for all $x \in D(F)$. If $\text{Ind}(F'(x))$ is constant with respect to $x \in D(F)$, then we call this number the index of F and write it as $\text{Ind}(F)$.

X, Y are called *isomorphic* if and only if there is a linear homeomorphism (*isomorphism*) $L: X \rightarrow Y$.

Let $\mathcal{F}(X, Y)$ denote the set of all linear Fredholm mappings $A: X \rightarrow Y$.

Let $\mathcal{L}(X, Y)$ denote the set of all linear continuous mappings $L: X \rightarrow Y$.

Let $\text{Isom}(X, Y)$ denote the set of all isomorphisms $L: X \rightarrow Y$.

Let $F: D(F) \subseteq X \rightarrow Y$ with $D(F)$ open be a C^1 -mapping. The point $u \in X$ is called a *regular point* of F if and only if $F'(u) \in \mathcal{L}(X, Y)$ maps onto Y . A point

$v \in Y$ is called a *regular value* of F if and only if the pre-image $F^{-1}(v)$ is empty or consists solely of regular points.

Theorem 1. The Surjective Implicit Function Theorem. [10, Section 4–13, Theorem 4–H]. *Let X, Y, Z be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let*

$$F: U(u_0, v_0) \subseteq X \times Y \rightarrow Z$$

be a C^1 -mapping on an open neighbourhood of the point (u_0, v_0) . Suppose that

- (i) $F(u_0, v_0) = 0$, and
- (ii) $F_v(u_0, v_0): Y \rightarrow Z$ is surjective.

Then the following assertion is true:

Let $r > 0$. There is a number $\varrho > 0$ such that, for each given $u \in X$ with $\|u - u_0\| < \varrho$, the equation

$$F(u, v) = 0$$

has a solution v such that $\|v - v_0\| < r$.

Theorem 2 [9, Section 7–9, Theorem 7–33]. *Let $g: D(g) \subseteq X \rightarrow Y$ be a compact mapping, where $a \in D(g)$ and $D(g)$ is open. If the derivative $g'(a)$ exists, then $g'(a) \in \mathcal{L}(X, Y)$ is also a compact mapping.*

Theorem 3 [9, Section 8–4, Example 8–16]. *Let $S \in \mathcal{F}(X, Y)$. The perturbed mapping $S + C$ verifies $S + C \in \mathcal{F}(X, Y)$ and $\text{Ind}(S + C) = \text{Ind}(S)$ provided $C \in \mathcal{L}(X, Y)$ and C is a compact mapping.*

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Clearly, if we define $u := f - g$, then u has a zero if and only if f and g share a value, that is, there is $x \in X$ with $f(x) = g(x)$. We thereby establish our result in terms of f, g .

Theorem 4. *Let $f, g: D \subseteq X \rightarrow Y$ be two C^1 -mappings, where X and Y are two Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and D is open.*

We assume

- (i) f is a compact mapping, g is a proper mapping and $tf(x) - g(x)$ is weakly coercive, jointly in both coordinates.
- (ii) The mapping g has a zero, x_0 .
- (iii) For any fixed t , belonging to $[0, 1]$, the zero of Y is a regular value of the mapping $tf(x) - g(x)$.

Then the following assertion holds

- (a) f and g share at least one value, i.e., there is at least one x^* such that $f(x^*) = g(x^*)$.

If in addition the condition

- (iv) g is a Fredholm mapping of index zero is satisfied, then in addition we have
 (b) f and g share at most a finite number of values on D , and at least one value.

Proof.

(a) Conclusion (a) will be proved in this section.

(a1) Let us define a mapping

$$H: D \times [0, 1] \subseteq X \times [0, 1] \rightarrow Y, \quad \text{where } H(x, t) := tf(x) - g(x).$$

We will prove here that H is a proper mapping, which will imply that $H^{-1}(0)$ is a compact set, since $\{0\} \subset Y$ is a compact set and H is proper.

Let \mathcal{C} be any fixed compact subset of Y , and let a sequence be fixed such that $(H(x_n, t_n))_{n \geq 1}$ belongs to \mathcal{C} . It suffices to show that the sequence $((x_n, t_n))_{n \geq 1}$ contains a convergent subsequence $((x_{n''}, t_{n''}))_{n'' \geq 1}$, which will imply that $H^{-1}(\mathcal{C})$ is relatively compact, and since

$$(x_{n''}, t_{n''}) \rightarrow (u, t) \quad \text{as } n'' \rightarrow \infty,$$

H is continuous and \mathcal{C} compact, therefore $H(u, t) \in \mathcal{C}$, that is, $(u, t) \in H^{-1}(\mathcal{C})$, and hence $H^{-1}(\mathcal{C})$ is compact.

Since the set \mathcal{C} is bounded and the mapping H is weakly coercive, $((x_n, t_n))_{n \geq 1}$ is a bounded sequence. Consequently $(x_n)_{n \geq 1}$ and $(t_n)_{n \geq 1}$ are bounded sequences. Since f is a compact mapping and $(x_n)_{n \geq 1}$ a bounded sequence, there exists a subsequence $(x_{n'})_{n' \geq 1}$ such that

$$f(x_{n'}) \rightarrow w' \quad \text{as } n' \rightarrow \infty$$

for some $w' \in Y$, and furthermore, since $(t_n)_{n \geq 1}$ is a bounded sequence of real numbers, there is $t \in \mathbb{R}$ with

$$t_{n'} \rightarrow t \quad \text{as } n' \rightarrow \infty.$$

Therefore

$$t_{n'} f(x_{n'}) \rightarrow tw' := w \quad \text{as } n' \rightarrow \infty$$

for some $w \in Y$.

All $H(x_{n'}, t_{n'})$, $n' \geq 1$ lie in the compact set \mathcal{C} and therefore, there is a subsequence $(H(x_{n''}, t_{n''}))_{n'' \geq 1}$ such that

$$H(x_{n''}, t_{n''}) \rightarrow v \quad \text{as} \quad n'' \rightarrow \infty$$

for some $v \in \mathcal{C}$. Therefore

$$g(x_{n''}) \rightarrow w - v \quad \text{as} \quad n'' \rightarrow \infty.$$

Since g is proper, $(x_{n''})$ has a convergent subsequence

$$x_{n'''} \rightarrow u \quad \text{as} \quad n'' \rightarrow \infty.$$

On the other hand,

$$t_{n'''} \rightarrow t \quad \text{as} \quad n'' \rightarrow \infty,$$

hence

$$(x_{n'''}, t_{n'''}) \rightarrow (u, t) \quad \text{as} \quad n'' \rightarrow \infty$$

as required.

(a2) Let us suppose that

$$(3) \quad H(x_a, t_a) = 0$$

for a fixed (x_a, t_a) . We will prove that there is $\varrho > 0$ such that for any $t \in (t_a - \varrho, t_a + \varrho)$, there exists $x = x(t)$ such that $H(x(t), t) = 0$.

Since zero is a regular value for the mappings $\{tf(x) - g(x)\}$ by hypothesis (iii),

$$H_x(x, t) = tf'(x) - g'(x) \in \mathcal{L}(X, Y)$$

is surjective for every pair (x, t) such that $H(x, t) = 0$. In particular, the mapping

$$H_x(x_a, t_a) = t_a f'(x_a) - g'(x_a) \in \mathcal{L}(X, Y)$$

is surjective, which together with identity (3) and Theorem 1 implies the existence of $\varrho > 0, r > 0$ such that for any $t \in (t_a - \varrho, t_a + \varrho)$ there is $x(t)$ with $H(x(t), t) = 0$ and $\|x(t) - x_a\| < r$.

(a3) We will prove that for every t in $[0, 1]$ there exists $x(t)$ such that $H(x(t), t) = 0$.

Let M denote the set of all t such that there is a solution $x(t)$. By assumption (ii) this set is not empty. By (a2), the set is relatively open. Finally, since the set is the projection of $H^{-1}(0)$ into the second component (i.e. the t component), and since the set is the image of a compact set by the continuous function projection,

it is therefore compact, and hence also closed. However, a relatively open, closed, non-empty subset of $[0, 1]$ is the whole interval, since $[0, 1]$ is connected.

Thus

$$H(x(1), 1) = f(x(1)) - g(x(1)) = 0 \Rightarrow f(x(1)) = g(x(1)),$$

which is conclusion (a) of the theorem, where $x^* := x(1)$.

(b) We include the hypothesis (iv) in Section (b).

(b1) We will see that the C^1 -mapping $u: D \subseteq X \rightarrow Y$, $u := f - g$ is a proper Fredholm mapping of index zero.

Since f is a compact mapping, Theorem 2 implies that for each $x \in X$, the mapping $f'(x) \in \mathcal{L}(X, Y)$ is also a compact mapping.

Since $f'(x)$ is a linear compact mapping and given that $g'(x)$ is a linear Fredholm mapping of index zero, Theorem 3 implies that

$$u'(x) = f'(x) - g'(x) \in \mathcal{L}(X, Y), \quad \forall x \in D$$

is a Fredholm linear mapping and $\text{Ind}(u'(x)) = \text{Ind}(g'(x)) = 0, \forall x \in D$. Therefore the non-linear C^1 -mapping u is a Fredholm mapping of index zero.

Furthermore, Section (a1) implies that u is a proper mapping. In fact, this is the particular case in which $t = 1$, i.e. $u(x) = H(x, 1)$.

(b2) We will see that, if any $x \in D$ exists which verifies $u(x) = 0$, then u is a local C^1 -diffeomorphism at x .

Let $x \in D$ exist such that $u(x) = 0$. Since zero is a regular value of u , the linear Fredholm mapping $u'(x)$ maps onto Y , and since $\text{Ind}(u'(x)) = 0$, therefore $\dim(\ker(u'(x))) = 0$. Thus $u'(x) \in \text{Isom}(X, Y)$. By the Local Inverse Theorem [10], u is a local diffeomorphism at x .

(b3) We will prove here that f and g share at most a finite number of values on D .

Since u is proper, $u^{-1}(0)$ is a compact set. If there were an infinite sequence

$$(x_n)_{n \geq 1} \subset D \quad \text{with } x_n \neq x_m \text{ when } n \neq m$$

verifying $u(x_n) = 0, \forall n \in \mathbb{N}$, there would be a subsequence $(x_{n'})_{n' \geq 1} \subset u^{-1}(0)$, which would converge at a point $x \in u^{-1}(0)$, and x would be a non-isolated zero of u . However, since u is a local diffeomorphism at x , given in Section (b2), x is an isolated zero of u . This is a contradiction. Hence there is not an infinite number of zeros of u on D . Thus f and g share at most a finite number of values on D . \square

Example. Let us consider an integral mapping

$$(Au)(x) = \int_a^b F(x, y, u(y)) dy, \quad \forall x \in [a, b],$$

where $-\infty < a < b < +\infty$. Define

$$Q := \{(x, y, u) \in \mathbb{R}^3 : x, y \in [a, b] \text{ and } |u| < r \text{ for fixed } r > 0\}.$$

Suppose that the function

$$F: \{(x, y, u) \in \mathbb{R}^3 : x, y \in [a, b] \text{ and } |u| \leq r\} \rightarrow \mathbb{R}$$

is twice continuously differentiable. Define $X := C[a, b]$ and $M := \{u \in X : \|u\| < r\}$, where $\|u\| = \max_{a \leq y \leq b} |u(y)|$. It can be easily proved that the mapping $A: M \rightarrow X$ is compact, and it is twice continuously differentiable.

Let B be the mapping $B: M \times [0, 1] \rightarrow X$, $B(u, t) = u$ which is $C^\infty(M \times [0, 1], X)$, is proper, has a zero, and since it is defined only on a bounded set, it is trivially weakly coercive.

The mapping $H(u, t) = t(Au) - (Bu): M \times [0, 1] \rightarrow X$ is weakly coercive, since it is defined only on a bounded set. Theorem 4 implies that if zero is a regular value, then there is $u \in M$ such that

$$(Au)(x) = (Bu)(x), \quad \forall x \in [a, b].$$

If we do not know that zero is a regular value, it is possible to prove the existence of $u \in M$ with $(Au)(x)$ as near to $u(x)$, $\forall x \in [a, b]$ as wanted in the following way:

Since $B'(u, t) \in \mathcal{L}(X \times \mathbb{R}, X)$ is surjective and $\dim(\ker(B'(u, t))) = 1$ for $\forall(u, t) \in M \times [0, 1]$, B is a Fredholm mapping of index one.

Define a mapping $A^*: M \times [0, 1] \rightarrow X$, $A^*(u, t) := t(Au)$ that is differentiable and compact. Theorem 2 implies that $(A^*)'(u, t)$ is compact. Theorem 3 implies that $H'(u, t)$ is a linear Fredholm mapping of index one for $\forall(u, t) \in M \times [0, 1]$, and hence $H(u, t)$ is a Fredholm mapping of index one. Thus we obtain from the Sard-Smale theorem [9, Theorem 4.K] that the set of regular values of the proper mapping H is open and dense in X .

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