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PERIODIC SOLUTIONS TO A p-LAPLACIAN NEUTRAL RAYLEIGH EQUATION WITH DEVIATING ARGUMENT*

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Abstract. By using the coincidence degree theory, we study a type of p-Laplacian neutral Rayleigh functional differential equation with deviating argument to establish new results on the existence of T-periodic solutions.

Keywords: deviating argument, neutral, coincidence degree theory

MSC 2010: 34B15, 34B24, 34B20

1. Introduction

In this paper we consider the p-Laplacian neutral Rayleigh functional differential equation with deviating argument of the form

$$(\varphi_p((x(t)-cx(t-\sigma))'))' + f(x'(t)) + \alpha(t)g(x(t-\tau(t))) = e(t), \tag{1.1}$$

where $\varphi_p\colon \mathbb{R}\to\mathbb{R}$, $\varphi_p(u)=|u|^{p-2}u$, $p>1$; $f,g\in C(\mathbb{R},\mathbb{R})$; α, τ, e are continuous T-periodic functions defined on $\mathbb{R}$ with $\alpha(t)>0$; $\sigma,c\in\mathbb{R}$ are constants such that $|c|\neq 1$.

Neutral functional differential equations (in short NFDEs) have been used for the study of distributed networks containing lossless transmission lines and other aspects [3], [4]. In recent papers, many researchers have obtained a lot of results for the existence of periodic solutions to NFDEs. In [8], Enrico Serra studied a kind of NFDE in the form

$$x'(t) + ax'(t-\tau) = f(t,x(t)). \tag{1.2}$$

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He proved the existence of at least one periodic solution for equation (1.2) (Theorem 3.1). In [6], Lu and Ge studied the existence of periodic solutions for NFDE

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x(t)-kx(t-\tau)) = f(x'(t)) + \alpha(t)g(u(t)) + \sum_{i=1}^{n}\beta_i(t)g(x(t-\tau_i(t))) + e(t). \tag{1.3}$$

They used Mawhin's continuation theorem to obtain the existence of periodic solutions for equation (1.3). Liu [5] considered the first order neutral equation

$$(u(t)+Bu(t-\tau))' = g_1(t,u(t)) - g_2(t,u(t-\tau_1)) + p(t)$$

and Si [9] examined the kth-order neutral equation

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(x(t)+b_0x(t-h_0)) + \sum_{j=1}^{k}a_jx^{(k-j)}(t) + \sum_{j=1}^{k}a_jx^{(k-j)}(t-h_j) = f(t).$$

However, there have been few results for the existence of periodic solutions to p-Laplacian neutral equations. The reason for it lies in the following two facts. The first is that the differential operator $\varphi_p(u) = |u|^{p-2}u$, $p \neq 2$, is no longer linear, so the theory of coincidence degree cannot be used directly; the second is that an a priori bound of solutions is not easy to achieve. In this paper we will overcome these difficulties and obtain the existence of periodic solutions to equation (1.1).

2. Main Lemmas

Let

$$A\colon\ C_T \to C_T, \quad (Ax)(t) = x(t) - cx(t-\sigma),$$

where

$$C_T = \{\varphi\colon\ \varphi \in C(\mathbb{R},\mathbb{R}),\ \varphi(t+T) = \varphi(t)\}$$

with the norm $|\varphi|_0 = \max\limits_{t\in[0,T]}|\varphi(t)|$. In order to use Mawhin's continuation theorem to obtain the existence of T-periodic solutions of equation (1.1), we rewrite equation (1.1) in the form of the two-dimensional differential system

$$\begin{cases} (Ax_1)'(t) = \varphi_q(x_2(t)), \\ x_2'(t) = -f([A^{-1}\varphi_q(x_2)](t)) - \alpha(t)g(x_1(t-\tau(t))) + e(t), \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $1/p + 1/q = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to system (2.1), then $x_1(t)$ must be a T-periodic solution to equation (1.1). Thus, in order to prove that equation (1.1) has a T-periodic solution, it suffices to show that system (2.1) has a T-periodic solution. Now we set

$$X = Y = \{x = (x_1, x_2)^\top \in C(\mathbb{R}, \mathbb{R}^2),\ x(t+T) = x(t)\}$$

with the norm $\|x\| = \max\{|x_1|_0, |x_2|_0\}$. Clearly X and Y are two Banach spaces. Further, let

$$L\colon\ D(L) \subset X \to X, \quad Lx = \begin{pmatrix} (Ax_1)' \\ x_2' \end{pmatrix}, \tag{2.2}$$

$$N\colon\ X \longrightarrow X,\ (Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f([A^{-1}\varphi_q(x_2)](t)) - \alpha(t)g(x_1(t-\tau(t))) + e(t) \end{pmatrix}, \tag{2.3}$$

where $D(L) = \{x\colon\ x \in C^2(\mathbb{R}, \mathbb{R}^2),\ x(t+T) = x(t)\}$.

Lemma 2.1 ([7]). *If $|c| < 1$ then A has continuous inverse on C_T, and*

(1) $\|A^{-1}x\| \leqslant \dfrac{\|x\|}{\bigl|1-|c|\bigr|}\ \ \forall\, x \in C_T$;

(2) $\displaystyle\int_0^T |(A^{-1}f)(t)|\,\mathrm{d}t \leqslant \frac{1}{\bigl|1-|c|\bigr|}\int_0^T |f(t)|\,\mathrm{d}t\ \ \forall\, f \in C_T$;

(3) $\displaystyle\int_0^T |(A^{-1}f)|^2(t)\,\mathrm{d}t \leqslant \frac{1}{(1-|c|)^2}\int_0^T f^2(t)\,\mathrm{d}t\ \ \forall\, f \in C_T$.

By Hale's terminology [2], a solution of the system (2.1) is $x = (x_1, x_2)^\top \in C(\mathbb{R}, \mathbb{R}^2)$ such that $(Ax_1, x_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$ and the equalities in (2.1) are satisfied on $\mathbb{R}$. In general, x is not from $C^1(\mathbb{R}, \mathbb{R}^2)$. Nevertheless, it is easy to see that $(Ax_1)' = Ax_1'$. So a T-periodic solution x of the system (2.1) must be from $C^1(\mathbb{R}, \mathbb{R}^2)$. According to Lemma 2.1, we can easily obtain that $\operatorname{Ker} L = \mathbb{R}^2$, $\operatorname{Im} L = \bigl\{x\colon\ x \in X,\ \int_0^T x(s)\,\mathrm{d}s = 0\bigr\}$. So L is a Fredholm operator with index zero. Let the projections P and Q be

$$P\colon\ X \longrightarrow \operatorname{Ker} L, \quad Px = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s, \qquad Q\colon\ X \longrightarrow X, \quad Qy = \frac{1}{T}\int_0^T y(s)\,\mathrm{d}s.$$

Then $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Let $L_P = L|_{D(L)\cap \operatorname{Ker} P}$. We can easily prove that L_P is invertible, $L_P^{-1}\colon\ \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$, and

$$(L_P^{-1}z)(t) = \begin{pmatrix} (A^{-1}Fz_1)(t) \\ (Fz_2)(t) \end{pmatrix}, \quad (Fz)(t) = \int_0^T G(t,s)z(s)\,\mathrm{d}s,$$

where $G(t,s) = \begin{cases} \dfrac{s}{T}, & 0 \leqslant s < t \leqslant T, \\[2mm] \dfrac{s-T}{T}, & 0 \leqslant t \leqslant s \leqslant T, \end{cases} \qquad z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}.$

Lemma 2.2 ([1]). *Suppose that X and Y are two Banach spaces, and $L\colon D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N\colon \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. Let the following conditions hold:*
(1) $Lx \neq \lambda Nx \ \forall\, x \in \partial\Omega \cap D(L) \ \forall\, \lambda \in (0,1)$,
(2) $Nx \notin \operatorname{Im} L \ \forall\, x \in \partial\Omega \cap \operatorname{Ker} L$,
(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,
where $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism. Then equation $Lx = Nx$ has a solution in $\bar{\Omega} \cap D(L)$.

3. Main results

For the sake of convenience, we list the following conditions which will be needed in our study of equation (1.1).

(H_1) There is a constant $K > 0$ such that $|f(x)| \leqslant K \ \forall\, x \in \mathbb{R}$.

(H_2) There is a constant $D > 0$ such that

$$\begin{cases} g(x) < -\dfrac{|e|_0}{\alpha_m} - \dfrac{K}{\alpha_m} & \text{for } x > D, \\ g(x) > \dfrac{K}{\alpha_m} & \text{for } x < -D, \end{cases}$$

where $\alpha_m = \min\limits_{t\in[0,T]} \alpha(t)$ and K is defined by (H_1).

(H_3) There is a constant r such that

$$\limsup_{x\to-\infty} \frac{|g(x)|}{|x|^{p-1}} \leqslant r \in [0,\infty).$$

(H_3') There is a constant r such that

$$\limsup_{x\to+\infty} \frac{|g(x)|}{|x|^{p-1}} \leqslant r \in [0,\infty).$$

Theorem 3.1. *Suppose that $|c| < 1$, $\int_0^T e(s)\,\mathrm{d}s = 0$, and ($H_1$)–($H_3$) are all satisfied. Then equation (1.1) has at least one T-periodic solution if*

$$\frac{2(1+|c|)r\alpha_M T^p}{\big|1-|c|\big|^p} < 1,$$

where $\alpha_M = \max\limits_{t\in[0,T]} \alpha(t)$.

P r o o f. Consider the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1),$$

where L and N are defined by (2.2) and (2.3), respectively. Let $\Omega_1 = \{x\colon x \in D(L),\ Lx = \lambda Nx,\ \lambda \in (0,1)\}$. If $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Omega_1$ then x must satisfy

$$\begin{cases} (Ax_1)'(t) = \lambda\varphi_q(x_2(t)), \\ x_2'(t) = -\lambda f([A^{-1}\varphi_q(x_2)](t)) - \lambda\alpha(t)g(x_1(t-\tau(t))) + \lambda e(t). \end{cases} \tag{3.1}$$

From the first equation of (3.1) we get $x_2(t) = \varphi_p(\lambda^{-1}(Ax_1)'(t))$, and combining it with the second equation of (3.1) yields

$$(\varphi_p((Ax_1)'(t)))' + \lambda^p f\Big(\frac{1}{\lambda}x_1'(t)\Big) + \lambda^p\alpha(t)g(x_1(t-\tau(t))) = \lambda^p e(t). \tag{3.2}$$

Let t_0 be the point, where Ax_1 achieves its maximum on $[0,T]$, i.e., $(Ax_1)(t_0) = \max\limits_{t\in[0,T]}(Ax_1)(t)$. Then $(Ax_1)'(t_0) = 0$ and $x_2(t_0) = \varphi_p(\lambda^{-1}(Ax_1)'(t_0)) = 0$ for all $\lambda \in (0,1)$. Furthermore, we assume that $t_0 < T$. Then we have

$$x_2'(t_0) \leqslant 0. \tag{3.3}$$

In fact, if $x_2'(t_0) > 0$ then there exists a constant $\delta > 0$ such that $x_2'(t) > 0$ for $t \in [t_0, t_0+\delta]$, and then $x_2(t) > x_2(t_0) = 0$ for $t \in [t_0, t_0+\delta]$. So $(Ax_1)'(t) = \lambda\varphi_q(x_2(t)) > 0$ for $t \in [t_0, t_0+\delta]$ and then $(Ax_1)(t) > (Ax_1)(t_0)$, which contradicts the assumption on t_0. This proves (3.3). From the second equation of (3.1) we have

$$-\lambda f([A^{-1}\varphi_q(x_2)](t_0)) - \lambda\alpha(t_0)g(x_1(t_0-\tau(t_0))) + \lambda e(t_0) \leqslant 0.$$

Hence,

$$g(x_1(t_0-\tau(t_0))) \geqslant -\frac{|e|_0}{\alpha_m} - \frac{K}{\alpha_m}.$$

Assumption (H_2) implies

$$x_1(t_0-\tau(t_0)) \leqslant D. \tag{3.4}$$

Integrating both sides of (3.2) over $[0,T]$, we get

$$\int_0^T f\Big(\frac{1}{\lambda}x_1'(t)\Big)\,\mathrm{d}t + \int_0^T \alpha(t)g(x_1(t-\tau(t)))\,\mathrm{d}t = 0. \tag{3.5}$$

From the integral mean value theorem and (3.5) we know that there exists a constant $t_1 \in [0,T]$ such that

$$\alpha(t_1)g(x_1(t_1-\tau(t_1))) + f\Big(\frac{1}{\lambda}x_1'(t_1)\Big) = 0.$$

Then we have

$$g(x_1(t_1-\tau(t_1))) \leqslant \frac{K}{\alpha_m}.$$

Assumption (H_2) implies

$$x_1(t_1-\tau(t_1)) \geqslant -D. \tag{3.6}$$

From (3.4) and (3.6) it is easy to prove that there exists a constant $\xi \in [0,T]$ such that

$$|x_1(\xi)| \leqslant D. \tag{3.7}$$

In fact, by (3.4) we know $x_1(t_0-\tau(t_0)) \in [-D,D]$, or $x_1(t_0-\tau(t_0)) < -D$.

(1) If $x_1(t_0-\tau(t_0)) \in [-D,D]$ then $t_0-\tau(t_0) = k\pi + \xi$, $k \in \mathbb{Z}$, $\xi \in [0,T]$. This proves (3.7).

(2) If $x_1(t_0-\tau(t_0)) < -D$ then by (3.6) and the fact that $x_1(t)$ is continuous on $\mathbb{R}$, there is a point t_2 between $t_0-\tau(t_0)$ and $t_1-\tau(t_1)$ such that $|x_1(t_2)| \leqslant D$. Let $t_2 = k\pi+\xi$, $k \in \mathbb{Z}$, and $\xi \in [0,T]$. This also proves (3.7). Hence, we get

$$\begin{aligned} |x_1|_0 &= \max_{t\in[0,T]}\Big|x_1(\xi) + \int_\xi^t x_1'(s)\,\mathrm{d}s\Big| \\ &\leqslant |x_1(\xi)| + \int_0^T |x_1'(s)|\,\mathrm{d}s \leqslant D + \int_0^T |x_1'(s)|\,\mathrm{d}s. \end{aligned} \tag{3.8}$$

Let

$$\begin{aligned} E_1 &= \{t \in [0,T]\colon\ x_1(t-\tau(t)) < -\varrho\}, \\ E_2 &= \{t \in [0,T]\colon\ |x_1(t-\tau(t))| \leqslant \varrho\}, \\ E_3 &= \{t \in [0,T]\colon\ x_1(t-\tau(t)) > \varrho\}, \end{aligned}$$

where $\varrho > D > 0$ is a given constant. Integrating the two sides of (3.2) on $[0,T]$, we get

$$\int_0^T \alpha(t)g(x_1(t-\tau(t)))\,\mathrm{d}t = -\int_0^T f\Big(\frac{1}{\lambda}x_1'(t)\Big)\,\mathrm{d}t.$$

Therefore, using (H_1) and (H_2), we obtain

$$\begin{aligned}
(3.9)\qquad \int_{E_3} \alpha(t)|g(x_1(t-\tau(t)))|\,\mathrm{d}t &= -\int_{E_3} \alpha(t)g(x_1(t-\tau(t)))\,\mathrm{d}t \\
&= \int_{E_1\cup E_2} \alpha(t)g(x_1(t-\tau(t)))\,\mathrm{d}t + \int_0^T f\Big(\frac{1}{\lambda}x_1'(t)\Big)\,\mathrm{d}t \\
&\leqslant \int_{E_1\cup E_2} \alpha(t)|g(x_1(t-\tau(t)))|\,\mathrm{d}t + KT.
\end{aligned}$$

Since $\big(2(1+|c|)r\alpha_M T^p\big)/\big(\big|1-|c|\big|^p\big) < 1$, there exists a constant $\varepsilon > 0$ such that

$$(3.10)\qquad \frac{2(1+|c|)(r+\varepsilon)\alpha_M T^p}{\big|1-|c|\big|^p} < 1.$$

For such ε, by assumption (H_3), there exists a constant $\varrho > 0$ such that $\varrho > D$ and

$$(3.11)\qquad |g(u)| \leqslant (r+\varepsilon)|u|^{p-1} \quad \text{for } \ u < -\varrho.$$

From (3.9) and (3.11) we get

$$\begin{aligned}
(3.12)\qquad \int_0^T \alpha(t)|g(x_1(t-\tau(t)))|\,\mathrm{d}t &= \int_{E_1\cup E_2\cup E_3} \alpha(t)|g(x_1(t-\tau(t)))|\,\mathrm{d}t \\
&\leqslant 2\int_{E_1\cup E_2} \alpha(t)|g(x_1(t-\tau(t)))|\,\mathrm{d}t + KT \\
&\leqslant 2(r+\varepsilon)\alpha_M T|x_1|_0^{p-1} + 2g_\varrho\alpha_M T + KT,
\end{aligned}$$

where $g_\varrho = \max\limits_{t\in E_2} |g(x_1(t-\tau(t)))|$. On the other hand, multiplying the two sides of equation (3.2) by $(Ax_1)(t)$, integrating them over $[0,T]$ and combining it with (3.12), we arrive at

$$\begin{aligned}
(3.13)\qquad &\int_0^T |(Ax_1)'(t)|^p\,\mathrm{d}t \\
&\leqslant (1+|c|)|x_1|_0\Big(\int_0^T \Big|f\Big(\frac{1}{\lambda}x_1'(t)\Big)\Big|\,\mathrm{d}t + \int_0^T \alpha(t)|g(x_1(t-\tau(t)))|\,\mathrm{d}t + T|e|_0\Big) \\
&\leqslant (1+|c|)|x_1|_0\int_0^T \alpha(t)|g(x_1(t-\tau(t)))|\,\mathrm{d}t + (1+|c|)|x_1|_0(T|e|_0 + KT) \\
&\leqslant 2(1+|c|)(r+\varepsilon)\alpha_M T|x_1|_0^p + (1+|c|)(2g_\varrho\alpha_M T + 2KT + T|e|_0)|x_1|_0.
\end{aligned}$$

For simplicity, let $k_1 = 2(1+|c|)(r+\varepsilon)\alpha_M T$, $k_2 = (1+|c|)(2g_\varrho\alpha_M T + 2KT + T|e|_0)$. From (3.8) and (3.13) we have

$$\begin{aligned}(3.14)\quad \int_0^T |(Ax_1)'(t)|^p \,\mathrm{d}t &\leqslant k_1|x_1|_0^p + k_2|x_1|_0 \\ &\leqslant k_1\Bigl(D + \int_0^T |x_1'(t)|\,\mathrm{d}t\Bigr)^p + k_2\int_0^T |x_1'(t)|\,\mathrm{d}t + Dk_2.\end{aligned}$$

By applying the second part of Lemma 2.1 and the Hölder inequality, we get

$$\begin{aligned}(3.15)\qquad \int_0^T |x_1'(t)|\,\mathrm{d}t &= \int_0^T |(A^{-1}Ax_1')(t)|\,\mathrm{d}t \\ &\leqslant \frac{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}{\bigl|1-|c|\bigr|} \leqslant \frac{T^{1/q}\bigl(\int_0^T |(Ax_1')(t)|^p\,\mathrm{d}t\bigr)^{1/p}}{\bigl|1-|c|\bigr|}.\end{aligned}$$

Case 1. If $\int_0^T |(Ax_1')(t)|\,\mathrm{d}t = 0$ then $\int_0^T |x_1'(t)|\,\mathrm{d}t = 0$, by (3.8), $|x_1|_0 \leqslant D$.

Case 2. If $\int_0^T |(Ax_1')(t)|\,\mathrm{d}t > 0$ then by (3.14) and (3.15) we have

$$(3.16)\quad \int_0^T |(Ax_1')(t)|^p\,\mathrm{d}t \leqslant k_1\Bigl(D + \frac{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}{\bigl|1-|c|\bigr|}\Bigr)^p + k_2\frac{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}{\bigl|1-|c|\bigr|} + Dk_2.$$

Clearly,

$$\begin{aligned}(3.17)\qquad &\Bigl(D + \frac{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}{\bigl|1-|c|\bigr|}\Bigr)^p \\ &\qquad = \frac{1}{\bigl|1-|c|\bigr|^p}\Bigl(\int_0^T |(Ax_1')(t)|\,\mathrm{d}t\Bigr)^p\Bigl(1 + \frac{D\bigl|1-|c|\bigr|}{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}\Bigr)^p.\end{aligned}$$

By classical elementary inequalities, we see that there is a constant $h(p) > 0$ which is dependent on p only, such that

$$(3.18)\qquad (1+u)^p < 1 + (1+p)u \quad \forall\, u \in (0, h(p)].$$

If $(D\bigl|1-|c|\bigr|)/\int_0^T |(Ax_1')(t)|\,\mathrm{d}t > h$ then $\int_0^T |(Ax_1')(t)|\,\mathrm{d}t < (D\bigl|1-|c|\bigr|)/h$. By (3.8) and (3.15), $|x_1|_0 < D + D/h$. If $(D\bigl|1-|c|\bigr|)/\int_0^T |(Ax_1')(t)|\,\mathrm{d}t \leqslant h$ then by (3.17) and (3.18) we have

$$\begin{aligned}(3.19)\quad &\Bigl(D + \frac{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}{\bigl|1-|c|\bigr|}\Bigr)^p \\ &\leqslant \frac{1}{\bigl|1-|c|\bigr|^p}\Bigl(\int_0^T |(Ax_1')(t)|\,\mathrm{d}t\Bigr)^p\Bigl(1 + \frac{(p+1)D\bigl|1-|c|\bigr|}{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}\Bigr) \\ &\leqslant \frac{\bigl(\int_0^T |(Ax_1')(t)|\,\mathrm{d}t\bigr)^p}{\bigl|1-|c|\bigr|^p} + (p+1)D\bigl|1-|c|\bigr|^{1-p}\Bigl(\int_0^T |(Ax_1')(t)|\,\mathrm{d}t\Bigr)^{p-1}.\end{aligned}$$

By (3.16) and (3.19),

$$
\begin{aligned}
(3.20)\qquad & \int_0^T |(Ax_1')(t)|^p \,\mathrm{d}t \\
&\leqslant \frac{k_1}{\bigl|1-|c|\bigr|^p}\Bigl(\int_0^T |(Ax_1')(t)|\,\mathrm{d}t\Bigr)^p \\
&\quad + k_1(p+1)D\bigl|1-|c|\bigr|^{1-p}\Bigl(\int_0^T |(Ax_1')(t)|\,\mathrm{d}t\Bigr)^{p-1} \\
&\quad + k_2\frac{\int_0^T |(Ax_1')(t)|\,\mathrm{d}t}{\bigl|1-|c|\bigr|} + Dk_2 \\
&\leqslant \frac{k_1}{\bigl|1-|c|\bigr|^p}T^{p/q}\int_0^T |(Ax_1')(t)|^p\,\mathrm{d}t \\
&\quad + k_1(p+1)D\bigl|1-|c|\bigr|^{1-p}T^{(p-1)/q}\Bigl(\int_0^T |(Ax_1')(t)|^p\,\mathrm{d}t\Bigr)^{(p-1)/p} \\
&\quad + \frac{k_2}{\bigl|1-|c|\bigr|}T^{1/q}\Bigl(\int_0^T |(Ax_1')(t)|^p\,\mathrm{d}t\Bigr)^{1/p} + Dk_2.
\end{aligned}
$$

In view of the definition of the number k_1, by virtue of (3.10), (3.20), $(p-1)/p<1$ and $1/p<1$, there is a constant $M_1>0$ such that $\int_0^T |(Ax_1')(t)|^p\,\mathrm{d}t \leqslant M_1$. It follows from (3.15) that $\int_0^T |x_1'(t)|\,\mathrm{d}t \leqslant (T^{1/q}(M_1)^{1/p})/(\bigl|1-|c|\bigr|) := M_2$. By (3.8) we get

$$|x_1|_0 \leqslant D + M_2 := M_3.$$

Consequently, in both cases 1 and 2, we have $|x_1|_0 \leqslant M_3$. In view of the first equation of (3.1) we have $\int_0^T |x_2(t)|^{q-2}x_2(t)\,\mathrm{d}t = 0$. By the integral mean value theorem there exists a constant $\eta\in[0,T]$ such that $x_2(\eta)=0$. Hence, $|x_2|_0 \leqslant \int_0^T |x_2'(t)|\,\mathrm{d}t$. By the second equation of (3.1) we get

$$
\begin{aligned}
\int_0^T |x_2'(t)|\,\mathrm{d}t &\leqslant \int_0^T \Bigl|f\Bigl(\frac{1}{\lambda}x_1'(t)\Bigr)\Bigr|\,\mathrm{d}t + \int_0^T \alpha_M|g(x_1(t-\tau(t)))|\,\mathrm{d}t + \int_0^T |e(t)|\,\mathrm{d}t \\
&\leqslant KT + T\alpha_M g_{M_3} + T|e|_0,
\end{aligned}
$$

where $g_{M_3} = \max\limits_{|u|<M_3} |g(u)|$. So we obtain

$$|x_2|_0 \leqslant KT + T\alpha_M g_{M_3} + T|e|_0 =: M_4.$$

We have proved that if $x=(x_1,x_2)^\top \in D(L)$, $Lx=\lambda Nx$, $\lambda\in(0,1)$, then $|x_1|_0\leqslant M_3$ and $|x_2|_0 \leqslant M_4$. Let $M=\max\{M_3,M_4\}$ and

$$\Omega = \{x=(x_1,x_2)^\top \in X\colon\ |x_1|_0\leqslant M,\ |x_2|_0\leqslant M\}.$$

Then $M > D$ and it is clear that the assumption (1) of Lemma 2.2 is satisfied. Moreover, for any $x = (x_1, x_2)^\top \in X$ we have

$$QNx = \begin{pmatrix} \dfrac{1}{T}\displaystyle\int_0^T \varphi_q(x_2(t))\,\mathrm{d}t \\ \dfrac{1}{T}\displaystyle\int_0^T \big(-f([A^{-1}\varphi_q(x_2)](t)) - \alpha(t)g(x_1(t-\tau(t)))\big)\,\mathrm{d}t \end{pmatrix}.$$

Since $\operatorname{Ker} L = \mathbb{R}^2$ and $\operatorname{Im} L = \operatorname{Ker} Q$, if $QNx = 0$ for some $x = (x_1, x_2)^\top \in \partial\Omega \cap \operatorname{Ker} L$, then $x_2 \equiv 0$, $|x_1| \equiv M$, and

$$g(x_1) = -\frac{f(0)}{\dfrac{1}{T}\displaystyle\int_0^T \alpha(t)\,\mathrm{d}t}.$$

By assumptions (H_1) and (H_2), one has $M = |x_1| \leqslant D$, which is a contradiction. So $QNx \neq 0$ for all $x \in \partial\Omega \cap \operatorname{Ker} L$ and thus the assumption (2) of Lemma 2.2 is satisfied. It remains to verify condition (3) of Lemma 2.2. In order to prove it, let

$$J\colon \operatorname{Im} Q \to \operatorname{Ker} L, \quad J(x_1, x_2)^\top = (x_2, x_1)^\top,$$

and $H(x, \mu) = \mu x + (1-\mu)JQNx$ for $(x, \mu) \in X \times [0, 1]$. Then we have

$$H(x, \mu) = \begin{pmatrix} \mu x_1 + \dfrac{(1-\mu)}{T}\displaystyle\int_0^T \big(-f([A^{-1}\varphi_q(x_2)](t)) - \alpha(t)g(x_1(t-\tau(t)))\big)\,\mathrm{d}t \\ \mu x_2 + \dfrac{(1-\mu)}{T}\displaystyle\int_0^T \varphi_q(x_2(t))\,\mathrm{d}t \end{pmatrix}.$$

It is not difficult to verify that, using (H_2), for any $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\mu \in [0, 1]$, we have $H(x, \mu) \neq 0$. Therefore,

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\{H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\} \\ &= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \end{aligned}$$

Therefore, by using Lemma 2.2, we see that equation $Lx = Nx$ has a solution $x = (x_1, x_2)^\top$ in $\bar{\Omega}$, i.e., equation (1.1) has a T-periodic solution x_1. □

Corollary 3.2. *Suppose that* $|c| < 1$, $\int_0^T e(s)\,\mathrm{d}s = 0$ *and that* (H_1), (H_2), *and* (H_3') *are satisfied. Then equation* (1.1) *has at least one* T*-periodic solution provided* $2(1+|c|)r\alpha_M T^p/(\bigl|1-|c|\bigr|^p) < 1$.

As an application, we consider the following NFDE:

$$(\varphi_3((x(t)-0.1x(t-\pi))'))' \\ + 5\sin x'(t) + (1+\tfrac{1}{2}\sin t)g(x(t-\tfrac{1}{2}\cos t)) = 20\cos t, \tag{3.21}$$

where

$$g(u) = \begin{cases} -\frac{1}{1000}u^2, & u > 10, \\ 5 - \frac{51}{100}u, & u \in [-10, 10], \\ 10 + \frac{1}{1000}u^2, & u < -10. \end{cases}$$

Clearly, equation (3.21) is a particular case of (1.1) in which

$$p = 3,\ \ c = 0.1,\ \ \sigma = \pi,\ \ \alpha(t) = 1 + 1/2\sin t,\ \ \tau(t) = 1/2\cos t,$$
$$e(t) = 20\cos t,\ \ f(u) = 5\sin u.$$

Then we have $T = 2\pi$, $\alpha_M = 3/2$ and $r = 1/1000$, and thus

$$\frac{2(1+|c|)r\alpha_M T^p}{(1-|c|)^p} = \frac{1.1\times 3\times (2\pi)^3}{0.9^3\times 1000} < 1.$$

Here assumptions (H_1)–(H_3) are satisfied. By using Theorem 3.1, we conclude that equation (3.21) has at least one 2π-periodic solution.

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