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Gerhard Dorfer; Dietmar W. Dorninger; Helmut Länger
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ON THE STRUCTURE OF NUMERICAL EVENT SPACES

Gerhard Dorfer, Dietmar Dorninger and Helmut Länger

The probability $p(s)$ of the occurrence of an event pertaining to a physical system which is observed in different states s determines a function p from the set S of states of the system to $[0,1]$. The function p is called a numerical event or multidimensional probability. When appropriately structured, sets P of numerical events form so-called algebras of S-probabilities. Their main feature is that they are orthomodular partially ordered sets of functions p with an inherent full set of states. A classical physical system can be characterized by the fact that the corresponding algebra P of S-probabilities is a Boolean lattice. We give necessary and sufficient conditions for systems of numerical events to be a lattice and characterize those systems which are Boolean. Assuming that only a finite number of measurements is available our focus is on finite algebras of S-probabilties.

Keywords: orthomodular poset, full set of states, numerical event

Classification: 06C15, 03G12, 81P16

1. INTRODUCTION

Let S be the set of states a physical system can accept during a certain experiment and $p(s)$ the probability of an event which is observed when the system is in state $s \in S$.

Studying the physical system with regard to the occurrence of different events leads to a set P of functions from S to $[0,1]$ which can be partially ordered by the order $\leq$ of functions. We assume that

(1) $0 \in P$ (0 denotes the constant function with value 0),

(2) $p' := 1 - p \in P$ for all $p \in P$ (1 denotes the constant function with value 1),

(3) If $p, q, r \in P$ are pairwise orthogonal, i.e. $p \leq q'$, $q \leq r'$ and $r \leq p'$, then $p + q + r \in P$.

($+$ and $-$ refers to the sum and difference in $\mathbb{R}$, respectively.)

If P satisfies (1) – (3) it is called an *algebra of S-probabilities* or *algebra of numerical events* (cf. [2] and [3]).

We denote the orthogonality relation by $\perp$ ($p \perp q$ means $p \leq q'$) and call three pairwise orthogonal elements an *orthogonal triple*.

We point out that (1) and (3) imply that if $p, q \in P$ and $p \perp q$ then $p + q \in P$, and moreover, that in this case $p + q$ is the supremum $p \cup q$ of p and q (see [7]).

Axiom (3) is motivated by classical event fields, for which the pairwise orthogonality of a triple A, B, C of events implies $A \subseteq B' \cap C' = (B \cup C)'$, which in terms of functions means $p \leq 1 - (q + r)$.

We recall that an *orthomodular poset* $(P, \leq, ')$ is a poset $(P, \leq)$ with a least element 0, a largest element 1 and a unary operation $'$ which satisfies the following for all $x, y \in P$:

(i) $'$ is an *orthocomplementation*: $x \leq y$ implies $x' \geq y'$, $x'' = x$, $x \cap x' = 0$ and $x \cup x' = 1$ ($\cap$ and $\cup$ denotes here the infimum and the supremum in $(P, \leq)$, respectively),

(ii) if $x \perp y$, i.e., $x \leq y'$ then $x \cup y$ exists in $(P, \leq)$,

(iii) if $x \leq y$ then $y = x \cup (y \cap x')$ (*orthomodular law*).

Any algebra of S-probabilities is an orthomodular poset (with respect to $\leq$ and $'$) which admits a full set of states $\{\lambda_s \mid s \in S\}$ induced by S where $\lambda_s : P \to [0, 1]$ is defined by $\lambda_s(p) = p(s)$ for all $p \in P$. Vice versa, any orthomodular poset which admits a full set of states is isomorphic to an algebra of S-probabilities (cf. [7]).

Algebras of S-probabilities have been studied mainly because they allow to distinguish a classical mechanical behaviour from a quantum mechanical one, namely, a system is classical if and only if $(P, \leq)$ is a Boolean lattice (cf. [2] and [3]).

The first step in checking if one deals with a Boolean lattice often is to find out whether one deals with a lattice at all. This is also of interest when studying algebras of S-probabilities from the point of view of so-called Boolean quasirings which correspond to orthomodular lattices in a way Boolean rings and Boolean lattices do (cf. [1] and[4]).

According to this our main goal is to characterize among classes of algebras of S-probabilities the ones which are lattices and those which are Boolean algebras (in order to discern quantum phenomena and classical ones). We begin by studying properties of algebras of numerical events that show how to perform calculations within these structures and by giving some examples. Then we derive some general results revealing the structure of algebras of numerical events and eventually we focus on finite systems by studying representations by means of atoms. Most of the obtained results can be generalized to the infinite case which, however, seems to be less relevant for practical purposes.

2. ELEMENTARY PROPERTIES AND EXAMPLES

We agree to denote the infimum and supremum of two elements of P, if they exist, by $\cap$ and $\cup$, respectively, and summarize some properties concerning the internal structure of algebras of S-probabilities:

Proposition 2.1. *Let $P \subseteq [0, 1]^S$ be an algebra of S-probabilities. Then* (a) – (c) *hold:*

(a) *If* $p, q \in P$ *and* $p \perp q$ *then* $p + q = p \cup q \in P$.

(b) *If* $p, q \in P$ *and* $p \leq q$ *then* $q - p \in P$ *and* $q - p = q \cap p'$.

(c) *If* $p \in P$, $u \in [0, 1]^S$ *and* $p - u \in P$ *then* $u \in P$.

Proof. (a) was already pointed out above and proven in [7].

(b): $p, q \in P$ and $p \leq q$ implies $p \perp q'$ and hence $p \cup q' = p + q'$ according to (a). Therefore, since $(P, \leq, ')$ is an orthomodular poset, $p' \cap q$ exists and $p' \cap q = (p \cup q')' = (p + q')' = 1 - (p + 1 - q) = q - p$.

(c): Since $p \in P$ and $v := p - u \in P$ it follows that $v \leq p$ and hence $u = p - v \in P$ according to (b). □

Proposition 2.2. $P \subseteq [0, 1]^S$ *is an algebra of* S*-probabilities if and only if* (1A) – (3A) *hold:*

(1A) $1 \in P$.

(2A) *If* $p, q \in P$ *and* $p \leq q$ *then* $q - p \in P$.

(3A) *If* (p, q, r) *is an orthogonal triple then* $p + q + r \leq 1$.

Proof. First suppose P is an algebra of S-probabilities. Then (1A) follows from (1) and (2), (2A) follows from Proposition 2.1 (b), and (3A) follows from (3).

Conversely, assume that (1A) – (3A) hold. Then (1) follows from (1A) and (2A), and (2) again follows from (1A) and (2A). As to (3): If $p, q \in P$ and $p \perp q$ then $p \leq q'$ and $q' \in P$ according to (1A) and (2A) and hence $q' - p \in P$ by (2A) which shows $p + q = 1 - (q' - p) \in P$ because of (1A) and (2A). Therefore, if (p, q, r) is an orthogonal triple, then $p + q \in P$ and since $(p + q) \perp r$ we have $p + q + r = (p + q) + r \in P$. □

Remark 2.3. (3) can be substituted by requiring both

(3A) If (p, q, r) is an orthogonal triple then $p + q + r \leq 1$.

(3B) If $p, q \in P$ and $p \perp q$ then $p + q \in P$.

Proof. First assume (3). Then (3A) holds. If $p, q \in P$ and $p \perp q$ then $(p, q, 0)$ is an orthogonal triple and hence $p + q = p + q + 0 \in P$.

Conversely, assume (3A) and (3B). If (p, q, r) is an orthogonal triple then $p \perp q$ and hence $p + q \in P$ according to (3B). Moreover, according to (3A), $(p + q) \perp r$ and therefore $p + q + r = (p + q) + r \in P$ again by (3B). □

Proposition 2.4. *Let* $P \subseteq [0, 1]^S$ *be an algebra of* S*-probabilities and* $p \in P \setminus \{0, 1\}$. *Then neither* $p \leq 1/2$ *nor* $p \geq 1/2$.

Proof. Assume $p \in P \setminus \{0\}$ and $p \leq 1/2$ and put $m := \max\{k \geq 1 \,|\, kp \leq 1\}$. Then $m \geq 2$, and because of $p \perp p$ it follows that $2p \in P$. If $m \geq 3$ then $2p \perp p$ and hence $3p \in P$. The same argument leads us to $tp \in P$ for $t = 1, \ldots, m$. Since $((m-1)p, p, p)$ is an orthogonal triple, $(m+1)p \in P$ contradicting the definition of m. Therefore $p \not\leq 1/2$.

With p' in the role of p it follows that no $p \in P \setminus \{1\}$ can be $\geq 1/2$. □

Next we point out that not only sums of orthogonal triples exist in P but also sums within an arbitrary finite orthogonal set: $O \subseteq P$ is called *orthogonal* if $p \perp q$ for all distinct $p, q \in O$.

Proposition 2.5. *Let $P \subseteq [0,1]^S$ be an algebra of S-probabilities and suppose $O \subseteq P$ is a finite orthogonal set. Then $\sum_{p \in O} p \in P$.*

Proof. We show the assertion by induction on $n := |O|$. For $n \leq 3$ the proposition is clear.

$n \to n+1$: Let $O = \{p_1, \ldots, p_{n+1}\}$. By the induction hypothesis we may assume that $\sum_{i=1}^{n} p_i, \sum_{i=2}^{n+1} p_i \in P$. In particular this means that $\sum_{i=1}^{n} p_i \leq 1$ and $\sum_{i=2}^{n+1} p_i \leq 1$. The former implies $p_1 \perp \sum_{i=2}^{n} p_i$, from the latter we infer $\sum_{i=2}^{n} p_i \perp p_{n+1}$. Since O is orthogonal we also have $p_1 \perp p_{n+1}$. Hence $(p_1, \sum_{i=2}^{n} p_i, p_{n+1})$ is an orthogonal triple and by (3) we obtain $\sum_{i=1}^{n+1} p_i \in P$. □

Next we give some examples for algebras of S-probabilities and specify a procedure how to construct examples.

Example 2.6. An important example of an algebra of S-probabilities which is not a Boolean lattice but, indeed, is a lattice, is the following (cf. [3]): Let H be a Hilbert space, S the set of one-dimensional subspaces of H, and for every $s \in S$ let a_s be a fixed unit vector in s. With $P(H)$ for the set of orthogonal projections of H and $\langle .,. \rangle$ for the inner product in H, the set of functions $\{s \mapsto \langle Qa_s, a_s \rangle \,|\, Q \in P(H)\}$ is an algebra of S-probabilities which is a lattice, more precisely, an orthomodular lattice.

Example 2.7. Suppose that every $p \in P$ can only assume the values 0 and 1. In this case one can show (cf. [5]) that $P = \{I_X \mid X \in \mathcal{M}\}$, where I_X denotes the indicator function on X and $\mathcal{M} \subseteq 2^S$ satisfies (i) $\emptyset \in \mathcal{M}$, (ii) if $A \in \mathcal{M}$ then $A' := S \setminus A \in \mathcal{M}$, and (iii) if $A, B \in \mathcal{M}$ and $A \cap B = \emptyset$ then $A \cup B \in \mathcal{M}$. Moreover, $(P, \leq)$ is a lattice if and only if $(\mathcal{M}, \subseteq)$ is a lattice, and $(P, \leq)$ is a Boolean lattice if $A \cup B \in \mathcal{M}$ holds for all $A, B \in \mathcal{M}$ without the restriction $A \cap B = \emptyset$. In literature these algebras of S-probabilities are known as concrete logics, cf., e.g., [8].

Theorem 2.8. *For every $i \in I$ let S_i be mutually disjoint sets, P_i an algebra of S_i-probabilities, $S := \bigcup_{i \in I} S_i$ and $P := \{f \in [0,1]^S \,:\, f{\restriction}_{S_i} \in P_i$ for all $i \in I\}$. Then P is an algebra of S-probabilities and $(P, \leq)$ is order-isomorphic to $\prod_{i \in I}(P_i, \leq)$.*

Proof. $0 \in P_i$ for all $i \in I$ and hence $0 \in P$.

If $f \in P$ then $f\restriction_{S_i} \in P_i$ for all $i \in I$ and hence $(1-f)\restriction_{S_i} = 1 - f\restriction_{S_i} \in P_i$ for all $i \in I$, from which we obtain $1 - f \in P$.

If (f, g, h) is an orthogonal triple in $(P, \leq)$ then for all $i \in I$ $(f\restriction_{S_i}, g\restriction_{S_i}, h\restriction_{S_i})$ is an orthogonal triple in $(P_i, \leq)$ and therefore $(f+g+h)\restriction_{S_i} = f\restriction_{S_i} + g\restriction_{S_i} + h\restriction_{S_i} \in P_i$ for all $i \in I$, whence $f + g + h \in P$.

This shows that P is an algebra of S-probabilities. Obviously $f \mapsto (f\restriction_{S_i}; i \in I)$ is an order isomorphism from $(P, \leq)$ to $\prod_{i\in I}(P_i, \leq)$. □

Remark 2.9. $\prod_{i\in I}(P_i, \leq)$ is a lattice if and only if $(P_i, \leq)$ is a lattice for every $i \in I$.

3. GENERAL RESULTS

First we assume that the number of states is ≤ 2. The case $|S| = 1$ is trivial: Because of Proposition 2.4 it immediately follows that there is only one algebra of S-probabilities, namely the two-element Boolean algebra consisting of the constant functions 0 and 1.

Lemma 3.1. *$P \subseteq [0,1]^2$ is an algebra of $\{1,2\}$-probabilities if and only if there exists an antichain A in $([0,1/2) \times (1/2,1], \leq)$ such that $P = A \cup A' \cup \{0,1\}$ (where $A' = \{\alpha' \mid \alpha \in A\}$).*

Proof. Assume P to be an algebra of $\{1,2\}$-probabilities, define $A := P \cap ([0,1/2) \times (1/2,1])$ and suppose there exist $\alpha = (a,b)$ and $\beta = (c,d)$ in A with $\alpha \neq \beta$ and $\alpha \leq \beta$. Then $\alpha, \beta' \in P$, and because of $\alpha \perp \beta'$ it follows that $\alpha + \beta' \in P$. Since $b > 1/2$ and $c' > 1/2$ we have $a + c' > 1/2$ and $b + d' > 1/2$, which, according to Proposition 2.4, means that $\alpha + \beta' = 1$. Thus $\beta' = \alpha'$ and hence $\alpha = \beta$, a contradiction. Therefore $(A, \leq)$ is an antichain. According to Proposition 2.4, $P = A \cup A' \cup \{0,1\}$.

Conversely, assume $P = A \cup A' \cup \{0,1\}$ with A an arbitrary antichain in $([0,1/2) \times (1/2,1], \leq)$. Clearly, (1) and (2) are fulfilled. According to Remark 2.3 we are done if (3A) and (3B) are satisfied.

(3A): Let (p,q,r) be an orthogonal triple in P. If $\{p,q,r\} \cap \{0,1\} \neq \emptyset$ then $0 \in \{p,q,r\}$ and hence $p + q + r \leq 1$. If $\{p,q,r\} \cap \{0,1\} = \emptyset$ then $p,q,r \in A \cup A'$ contradicting the fact that these elements are pairwise orthogonal because $(A, \leq)$ is an antichain: If, without loss of generality, $p \in A$ then because of $p \leq q'$ also $q' \in A$, which means $q = p'$, and analogously that $r = p'$. Therefore $q = r$ and $q \perp r$, i.e. $q \leq 1/2$ contradicting Proposition 2.4.

(3B): Assume $p, q \in P$ and $p \perp q$. If $\{p,q\} \cap \{0,1\} \neq \emptyset$ then $0 \in \{p,q\}$ and hence $p + q \in P$. If $\{p,q\} \cap \{0,1\} = \emptyset$ and, without loss of generality, $p \in A$ then $q' \in A$ since $p \leq q'$ and hence $p = q'$. Therefore $p + q = 1 \in P$. □

Theorem 3.2. *For $|S| = 2$ all algebras of S-probabilities are lattices, and the only Boolean lattices among them are the two-element and the four-element one. In particular, any algebra of S-probabilities with $|S| = 2$ is isomorphic to the two-element lattice or a lattice MO_n, n a positive integer, or MO_∞, respectively.*

P r o o f. The proof follows from Lemma 3.1. □

In the following we always assume that P is an arbitrary algebra of S-probabilities. As we have already indicated in Section 1, if $(P, \le)$ is a lattice, it is orthomodular.

Lemma 3.3. *If* $(P, \le)$ *is a lattice then* (P1) *holds:*

(P1) *For all* $p, q \in P$ *there exists a unique* $r \in P$ *with* $r \ge p, q$ *and* $(r-p) \cap (r-q) = 0$.

P r o o f. $p \cup q \ge p, q$, and by Proposition 2.1 (b)

$$((p \cup q) - p) \cap ((p \cup q) - q) = ((p \cup q) \cap p') \cap ((p \cup q) \cap q') = (p \cup q) \cap p' \cap q' = 0.$$

So $p \cup q$ might serve for an r as required in (P1).

If s is another element of P with $s \ge p, q$ and $(s - p) \cap (s - q) = 0$ then $s \ge p \cup q$ and hence, due to orthomodularity

$$s = (p \cup q) \cup (s \cap (p \cup q)') = (p \cup q) \cup (s \cap p' \cap q') = (p \cup q) \cup 0 = p \cup q,$$

because $(s \cap p') \cap (s \cap q')$ was assumed to be 0. □

Theorem 3.4. $(P, \le)$ *is a lattice if and only if* (P1) *holds and for all* $u, v, w \in P \backslash \{0\}$ $(u \cap v) \cap w = 0$ *is equivalent to* $u \cap (v \cap w) = 0$ (*in the sense that if one of these two expressions exists and equals to* 0 *then the other also exists and equals to* 0).

P r o o f. According to Lemma 3.3 the conditions of Theorem 3.4 are necessary. As for the sufficient part, assume $p, q \in P$. If p and q are comparable then $p \cup q$ exists. Now assume that p and q are not comparable. Then because of (P1) there exists an $r \in P$ with $r \ge p, q$ and

$$(r - p) \cap (r - q) = (r \cap p') \cap (r \cap q') = 0.$$

$r \cap p' = 0$ would imply $q \le r = p$, a contradiction. If we assume $r = 0$ we also obtain a contradiction, namely $p = q = 0$. Also q' cannot be 0, because this would imply $q = 1 \ge p$. Hence $r \cap p', r, q' \in P \setminus \{0\}$ and therefore according to the second condition in Theorem 3.4

$$0 = ((r \cap p') \cap r) \cap q' = (r \cap p') \cap q'.$$

Since besides $r, q' \ne 0$ also $p' \ne 0$, because $p' = 0$ would mean $p = 1 \ge q$, we further obtain $0 = r \cap (p' \cap q')$ which shows that $p' \cap q'$ and hence $p \cup q$ exists. □

Theorem 3.5. *An algebra* P *of* S*-probabilities is a Boolean lattice if and only if for all* $p, q \in P$ *there exist* $g, h \in P$ *with* $g \perp p$, $h \perp q$ *and* $g \perp h$ *such that* $p + g = q + h$.

P r o o f. We define $f := p + g = q + h$. Then $p = f - g = f \cap g'$ and hence $p' = f' \cup g = f' + g$ since $f' \perp g$. Analogously, $q' = f' \cup h = f' + h$. This means that p' and q' can be represented by an orthogonal triple (f', g, h) such that $p' = f' \cup g$ and $q' = f' \cup h$ which by a result in [1] is equivalent to $(P, \le)$ being a Boolean lattice. □

An *orthomodular poset* is called a *Boolean poset* if $a \cap b = 0$ implies $a \le b'$. As shown in [7], for algebras of S-probabilities this is equivalent to

(P2) If for $p, q \in P$ the only common lower bound is 0, then $p + q \in P$.

Theorem 3.6. *If P is an algebra of S-probabilities in which* (P2) *holds then* $(P, \leq)$ *is a Boolean lattice if and only if* (P1) *holds in* $(P, \leq)$.

Proof. If $(P, \leq)$ is a Boolean lattice then $p \cap q = 0$ implies $p = p \cap (q \cup q') = (p \cap q) \cup (p \cap q') = p \cap q' \leq q'$, hence, as stated above, (P2) holds, and according to Lemma 3.3 also (P1) is true.

Conversely, assume that (P1) and (P2) hold. (P1) implies that for all $p, q \in P$ there exist $g, h \in P$ with $p \perp g$, $q \perp h$ and $g \cap h = 0$: Just take $r - p$ for g and $r - q$ for h. Because $g \cap h = 0$ we obtain $g \leq h'$, i.e. $g \perp h$, hence by Theorem 3.5 $(P, \leq)$ is a Boolean lattice. □

Performing actual measurements only the necessary conditions of Theorems 3.4, 3.5 and 3.6 will be of practical significance: In general, it is easier to show that a condition is violated than having to check all possibilities. As for Theorem 3.4, one will try to contradict (P1) by finding an appropriate pair of functions p and q.

4. REPRESENTATION BY MEANS OF ATOMS

As we have already mentioned, with respect to practical measurements we focus on finite algebras of S-probabilities. However, most of the results can be generalized to the infinite case.

In this section we assume that $(P, \leq, ')$ is a finite algebra of S-probabilities and A denotes the set of its atoms (an atom a is an upper neighbour of 0, i.e. if $0 < x \leq a$ then $x = a$).

Proposition 4.1. *Every $p \in P$ can be represented in the form*

$$p = \sum_{a \in O_p} a = \bigcup_{a \in O_p} a,$$

where O_p is an appropriate orthogonal set of atoms.

Proof. If $p = 0$ then choose $O_p = \emptyset$. Now suppose $p > 0$. Since P is finite there exists $a_1 \in A$ with $a_1 \leq p$. If $p = a_1$ we are done, otherwise we consider $p_1 := p - a_1 = p \cap a_1' \in P$ due to Proposition 2.1. We have $p = a_1 + p_1 = a_1 \cup p_1$ because $a_1 \perp p_1$. If p_1 is not an atom we apply the same argument once more and get $p_2 \in P$ and $a_2 \in A$ such that $p = a_1 + a_2 + p_2 = a_1 \cup a_2 \cup p_2$. Now $a_2 \neq a_1$ since $2a \not\leq 1$ for any $a \in P \setminus \{0\}$ according to Proposition 2.4. Repeating this procedure we finally end up with a representation of p as a sum (and therefore because of Proposition 2.5 also as a supremum) of orthogonal atoms. □

Remark 4.2. For $p, q \in P$ the join $p \cup q$ may exist in $(P, \leq)$ whereas $p + q \not\leq 1$. However, if $p + q \in P$ then $p \perp q$ and $p + q = p \cup q$.

For $p \in P$ let $A_p = \{a \in A \mid a \leq p\}$. Next we investigate the representation of elements as a sum of (orthogonal) atoms in more detail.

Lemma 4.3. *Suppose O is an orthogonal set of atoms and $O \subseteq A_p$ for some $p \in P$. Then $\sum_{a \in O} a \in P$ and $\sum_{a \in O} a \le p$.*

Proof. Induction on $n := |O|$: Obviously the assertion is true for $n = 0$ and $n = 1$.

$n \to n + 1$: Let $O = \{a_1, \ldots, a_{n+1}\}$. O is an orthogonal set, and by Proposition 2.5 we obtain $\sum_{i=1}^{n+1} a_i \in P$. This means that $a_{n+1} \perp \sum_{i=1}^{n} a_i$. By induction hypothesis we have $\sum_{i=1}^{n} a_i \le p$, and $a_{n+1} \le p$ since $O \subseteq A_p$. Hence we obtain that $(\sum_{i=1}^{n} a_i, a_{n+1}, p')$ is an orthogonal triple and therefore $\sum_{i=1}^{n+1} a_i + p' \le 1$ which implies $\sum_{i=1}^{n+1} a_i \le p$. □

Lemma 4.4. *For $p, q \in P$ we have $p \le q$ if and only if $A_p \subseteq A_q$.*

Proof. The given condition is necessary: If $p \le q$ and $a \in A_p$ then $a \le p \le q$, and thus $a \in A_q$.

Now we prove that the condition is also sufficient: By Proposition 4.1 there exists an orthogonal set $O_p \subseteq A_p$ such that $\sum_{a \in O_p} a = p$. Since we have $O_p \subseteq A_p \subseteq A_q$, Lemma 4.3 yields $p = \sum_{a \in O_p} a \le q$. □

Lemma 4.5. *If $p \le q$ and $p = \sum_{a \in O_p} a$ then there exists $O_q \subseteq A_q$, O_q orthogonal and $O_q \supseteq O_p$ such that $\sum_{a \in O_q} a = q$.*

Proof. If $p \le q$ then there is $r \in P$ with $q = p + r$ (Proposition 2.1). By Lemma 4.1 we can represent r in the form $r = \sum_{a \in O_r} a$ with O_r a finite orthogonal set of atoms. Assembling the representations for p and r we obtain $q = p + r = \sum_{a \in O_p \cup O_r} a$ with $O_p \cap O_r = \emptyset$ because $a \in O_p \cap O_r$ would imply $a \le p \le r' \le a'$, a contradiction. Thus $O_q := O_p \cup O_r$ satisfies all what is required in the condition. □

Now we are able to characterize the Boolean algebras among the algebras of S-probabilities by means of the sets A_p, $p \in P$.

Theorem 4.6. *A finite algebra of S-probabilities $(P, \le, ')$ is a Boolean algebra if and only if $(\{A_p \mid p \in P\}, \subseteq)$ is a join-semilattice with $\sup\{A_p, A_q\} = A_p \cup A_q$ for all $p, q \in P$ where $\cup$ here denotes the set-theoretical union.*

Proof. The necessity of the condition is the core of the Stone Representation Theorem for finite Boolean algebras.

Now we prove that the condition is sufficient. First we show that $\le$ is a lattice order on P. The given condition means that for all $p, q \in P$ there is $r \in P$ such that $A_r = A_p \cup A_q$. Hence we have $A_p, A_q \subseteq A_r$ and by Lemma 4.4 we infer $p, q \le r$, so r is an upper bound of p and q. Considering an arbitrary upper bound t of p and q we obtain $A_t \supseteq A_p, A_q$ and thus also $A_t \supseteq A_p \cup A_q = A_r$ which again by Lemma 4.4 provides $t \ge r$ showing that $r = p \cup q$.

Because $(P, \le, ')$ is an orthomodular poset $p \cap q = (p' \cup q')'$ showing that $(P, \le)$ is lattice ordered. Up to now we know that $(P, \le, ')$ is an orthomodular lattice $(P, \cap, \cup, 0, 1, ')$. Let us assume that $(P, \cap, \cup, 0, 1, ')$ is not a Boolean algebra. Then there exists a subalgebra of $(P, \cap, \cup, 0, 1, ')$ isomorphic to MO_2 or $\mathrm{MO}_2 \times \{0,1\}$ (cf. e. g. [6], ch. 1; MO_2 denotes the (only) orthomodular lattice with six elements where the four elements $\neq 0, 1$ are pairwise incomparable). We consider the case with a subalgebra $\{0, x, y, x', y', 1\} \cong \mathrm{MO}_2$ in detail, the other case runs along the same lines: Due to our condition there exists $z \in P$ such that $A_z = A_x \cup A_y$. As before this implies $z = x \cup y = 1$ and thus $A_x \cup A_y = A$ is the set of all atoms in P. The four atoms x, y, x', y' of the subalgebra are incomparable, in particular we have $x' \not\leq y$. By Lemma 4.4 there exists $a \in A_{x'} \setminus A_y$, i. e. $a \notin A_y$. Because $A_x \cup A_y = A$ this implies $a \in A_x$. But we also have $a \in A_{x'}$ thus $a \leq x, x'$, a contradiction. □

Remark 4.7. If $(P, \leq, ')$ is a lattice then $A_p \cap A_q = A_{p \cap q}$: By Lemma 4.4 we get $A_{p \cap q} \subseteq A_p \cap A_q$ and if $a \in A_p \cap A_q$ then $a \leq p \cap q$ which means $a \in A_{p \cap q}$.

Our next goal is to consider the uniqueness of representations. For this we first need

Lemma 4.8. *If O_p is an arbitrary maximal orthogonal subset of A_p then* $\sum\limits_{a \in O_p} a = p.$

Proof. Lemma 4.3 implies that $\sum_{a \in O_p} a =: p_1 \leq p$. Assuming $p_1 < p$ we have $p - p_1 = p \cap p_1' > 0$ according to Proposition 2.1 (b). Therefore there exists $a^* \in A$ with $a^* \leq p - p_1 \leq p$, i. e. $a^* \in A_p$. Moreover, $a^* + \sum_{a \in O_p} a = a^* + p_1 \leq p$ hence $a^* \perp a$ for all $a \in O_p$ and consequently $O_p \cup \{a^*\}$ is orthogonal which contradicts the maximality of O_p. □

Theorem 4.9. *A finite algebra of S-probabilities $(P, \leq, ')$ is a Boolean algebra if and only if the representation of an element $p \in P$ as a sum of atoms is unique for all $p \in P$.*

Proof. If $(P, \leq, ')$ is a finite Boolean algebra of S-probabilities then every element can be uniquely represented as a join of atoms. Distinct atoms are orthogonal in a Boolean algebra and thus the join of atoms coincides with their sum.

Now we suppose that the representation as a sum of atoms is unique. First we claim that $\sum_{a \in A} a = 1$: Due to Proposition 4.1 there exists a representation $1 = \sum_{a \in O_1} a$ with an appropriate orthogonal set of atoms O_1. If $O_1 \subset A$ there would exist an element $b \in A \setminus O_1$ and a maximal orthogonal set $O \subseteq A = A_1$ with $b \in O$. Due to Lemma 4.8 we conclude $\sum_{a \in O} a = 1$ and by the uniqueness of the representation of 1 we obtain $O = O_1$ and hence $b \in O_1$, a contradiction.

From the representation $\sum_{a \in A} a = 1$ we infer that all atoms are pairwise orthogonal. Therefore any subset of A can be summed up in P and one can see easily that the map $\varphi : P \to 2^A$, $\varphi(p) = A_p$, is a bijection which is compatible with order and complement. This implies that $(P, \leq, ')$ is isomorphic to $(2^A, \subseteq, ')$ and hence is a Boolean algebra of S-probabilities. □

Corollary 4.10. *Any finite algebra of S-probabilities with at most four atoms is lattice ordered.*

Proof. Let $n = |A|$ and $A = \{a_1, \ldots, a_n\}$, $n \leq 4$. The case $n \leq 2$ is obvious and leads to the Boolean algebras with two or four elements, respectively.

Now suppose $n \geq 3$. Due to Proposition 4.1 the element a_1' can be represented in the form $a_1' = \sum_{a \in \bar{O}} a$ with some orthogonal set of atoms $\bar{O}$. Since no element $\neq 0$ is orthogonal to itself (Proposition 2.4) we have $a_1 \notin \bar{O}$.

$n = 3$: In case $|\bar{O}| = 1$, say $a_1' = a_2$, we have $A = \{a_1, a_1', a_3\}$: If $a_3' = a_1 = a_2'$ or $a_3' = a_2 = a_1'$ then $a_3 = a_2$ or $a_3 = a_1$, respectively, a contradiction. The case $a_3' = a_1 + a_2 = a_1 + a_1' = 1$ is also impossible since then $a_3 = 0$.

If $|\bar{O}| = 2$ then $a_1' = a_2 + a_3$ and all atoms are pairwise orthogonal. As shown in the proof of Theorem 4.9, P is a Boolean algebra and thus a lattice.

$n = 4$: We first consider the case $|\bar{O}| = 1$, say $a_1' = a_2$, and check the possibilities for a_3': If $a_3' = a_4$ then all atoms are coatoms as well, and P is isomorphic to MO_2 which is a lattice. All other cases will lead to a contradiction to our assumptions: if $a_3' = a_1 + a_4$ then $a_2 = a_1' = a_3 + a_4$ is not an atom, if $a_3' = a_1 + a_2$ or $a_3' = a_1 + a_2 + a_4$ then $a_3 = 0$.

In case of $|\bar{O}| = 2$ with $a_1' = a_2 + a_3$ the set $O := \{a_1, a_2, a_3\}$ is a maximal orthogonal set of atoms. When forming sums over subsets of O we obtain a Boolean algebra with eight elements. Also a_4' and thus a_4 is among these sums which leads to a contradiction to $|A| = 4$.

Finally, if $|\bar{O}| = 3$ then $a_1 + a_2 + a_3 + a_4 = 1$, i.e. all atoms are pairwise orthogonal and P is isomorphic to the Boolean algebra with four atoms and thus a lattice. $\square$

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Gerhard Dorfer, Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8–10/104, 1040 Vienna. Austria.
e-mail: g.dorfer@tuwien.ac.at

Dietmar Dorninger, Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8–10/104, 1040 Vienna. Austria.
e-mail: d.dorninger@tuwien.ac.at

Helmut Länger, Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8–10/104, 1040 Vienna. Austria.
e-mail: h.laenger@tuwien.ac.at