PROPERTIES OF RELATIVELY PSEUDOCOMPLEMENTED DIRECTOIDS

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Abstract. The concept of a relatively pseudocomplemented directoid was introduced recently by the first author. It was shown that the class of relatively pseudocomplemented directoids forms a variety whose axiom system contains seven identities. The aim of this paper is three-fold. First we show that these identities are not independent and their independent subset is presented. Second, we modify the adjointness property known for relatively pseudocomplemented semilattices in the way which is suitable for relatively pseudocomplemented directoids. Hence, they can also be considered as residuated structures in a rather modified version. We also get two important congruence properties, namely congruence distributivity and 3-permutability valid in the variety $\mathcal{V}$ of relatively pseudocomplemented directoids. Then we show some basic results connected with subdirect irreducibility in $\mathcal{V}$. Finally, we show another way how to introduce pseudocomplementation on directoids via relative pseudocomplementation.

Keywords: directoid, relatively pseudocomplemented directoid, congruence distributivity, 3-permutability, residuated structure, adjointness property, variety

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1. Introduction

Relatively pseudocomplemented lattices and semilattices play an important role in the investigation of intuitionistic logics and their reducts. They were intensively studied by G. T. Jones [8]. The operation of relative pseudocomplementation serves as an algebraic counterpart of the intuitionistic connective implication. For the readers' convenience, we can refer to the compendium [5] where essential results on relatively pseudocomplemented semilattices and lattices are gathered.

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To investigate some more general algebraic systems connected with non-classical logic (as e.g. BCK-algebras, BCI-algebras, etc.), we often study ordered sets which are not necessarily semilattices. However, a bit weaker structure was introduced by J. Ježek and R. Quackenbush [7] as follows.

By a *directoid* (a commutative directoid in [7]) we mean a groupoid $\mathcal{D} = (D; \sqcap)$ satisfying the identities

(D1) $x \sqcap x = x$ (idempotency),

(D2) $x \sqcap y = y \sqcap x$ (commutativity),

(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$ (weak associativity).

Of course, every $\wedge$-semilattice is a directoid but not vice versa. It can be shown that every directoid $\mathcal{D} = (D; \sqcap)$ can be converted into an ordered set $(D; \leqslant)$ via

$$x \leqslant y \quad \text{if and only if} \quad x \sqcap y = x$$

and every downward directed ordered set $(D; \leqslant)$ can be organized into a directoid taking

$$x \sqcap y = y \sqcap x \in L(x, y) = \{z \in D;\ z \leqslant x \ \text{ and } \ z \leqslant y\}$$

arbitrarily for non-comparable elements x, y and

$$x \sqcap y = y \sqcap x = x \quad \text{when} \quad x \leqslant y,$$

see [7] or [5] for details. It is worth noticing that the operation $\sqcap$ is not isotone in general, in fact we have

$$x \leqslant y \quad \Rightarrow \quad x \sqcap z \leqslant y \sqcap z \quad \text{for all} \quad x, y, z \in D$$

if and only if $(D; \sqcap)$ is an $\wedge$-semilattice where $\sqcap$ coincides with the infimum $\wedge$ (with respect to $\leqslant$).

A natural question arises if a directoid with a least element 0 can be equipped with pseudocomplementation. This task was investigated by the first author in [2] where an axiom system for pseudocomplementation on directoids was presented. Another problem is how to define and characterize relatively pseudocomplemented directoids. As mentioned in [3], x is a greatest element satisfying $a \wedge x \leqslant b$ if and only if x is a greatest element satisfying $a \wedge x = a \wedge b$ in any relatively pseudocomplemented semilattice. However, if for some a, b of an $\wedge$-semilattice $\mathcal{S}$ the relative pseudocomplement does not exist then the conditions need not coincide, see the following

Example 1. Let $\mathcal{S} = (S, \wedge)$ be an $\wedge$-semilattice where $S = \{0, a, b, c, 1\}$ whose diagram is depicted in Fig. 1.

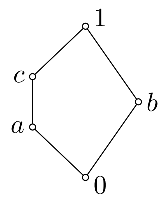


Fig. 1

Then there exists no pseudocomplement of c with respect to a since $c \wedge b = 0 \leqslant a$, $c \wedge a = a$ but there is no greatest $x \in S$ with $c \wedge x \leqslant a$. On the other hand, there exists a greatest $x \in S$ with $c \wedge x = c \wedge a$, namely $x = a$.

Of course, the aforementioned conditions are not equivalent in directoids either, see [3] for details. Hence we use the following definition of relative pseudocomplementation in directoids which was introduced in [3].

Definition. Let $\mathcal{D} = (D; \sqcap)$ be a directoid and $a, b \in D$. An element x is called a *relative pseudocomplement of a with respect to b* if it is a greatest element of D such that $a \sqcap x = a \sqcap b$. It is denoted by $a * b$. A directoid $\mathcal{D}$ is *relatively pseudocomplemented* if there exists $a * b$ for every $a, b \in D$.

The fact that $\mathcal{D}$ is a relatively pseudocomplemented directoid will be expressed by the notation $\mathcal{D} = (D; \sqcap, *)$. As shown in [3], every relatively pseudocomplemented directoid has a greatest element (which is denoted by 1).

Let us mention that if the above definition of relative pseudocomplementation is used for $\wedge$-semilattices, what we get is nothing else than the definition of the so-called sectional pseudocomplementation as defined in [1].

As already noticed, in a relatively pseudocomplemented semilattice our new definition of $a * b$ coincides with the usual relative pseudocomplement of a with respect to b. Hence, every relatively pseudocomplemented semilattice belongs to the class of relatively pseudocomplemented directoids.

Example 2. For the semilattice $\mathcal{S}$ from Example 1, the operation table for $*$ (defined by the above Definition) is

$*$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	c	c	1	c	1
c	b	a	b	1	1
1	0	a	b	c	1

Hence, if $\mathcal{S}$ is considered as a directoid then it is relatively pseudocomplemented.

2. Axiom system

It was shown in [3] that the class of relatively pseudocomplemented directoids forms a variety which is determined by additional four identities as follows.

Proposition (See Theorem 2 in [3]). *Let $(D; \sqcap)$ be a directoid and let $*$ be a binary operation on D. Then $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid if and only if it satisfies the following identities:*

(S1) $x \sqcap (x * y) = x \sqcap y$,
(S2) $(x * y) \sqcap y = y$,
(S3) $x * y = x * (x \sqcap y)$,
(S4) $x * x = y * y$.

A natural question is if the axioms (S1)–(S4) or, more generally, the axioms (D1)–(D3), (S1)–(S4) are independent. An immediate reflexion shows that it is not the case. In fact, we can prove the following result.

Theorem 1. *Let $\mathcal{D} = (D; \sqcap, *)$ be an algebra with two binary operations. Then $\mathcal{D}$ is a relatively pseudocomplemented directoid if and only if it satisfies the identities* (D2), (D3), (S1), (S2) *and* (S3).

P r o o f. All we need to prove is that the above listed identities imply (D1) and (S4). For (D1) this is immediate because $x \sqcap x = x \sqcap (x * x)$ by (S1) and $x \sqcap (x * x) = x$ by (S2), thus $x \sqcap x = x$. It remains to prove (S4).

By using (S3) and (S2), we infer

$$x * (y * x) = x * (x \sqcap (y * x)) = x * x \tag{1}$$

and

$$(x \sqcap y) * x = (x \sqcap y) * ((x \sqcap y) \sqcap x) = (x \sqcap y) * (x \sqcap y).$$

Due to symmetry, also

$$(x \sqcap y) * y = (x \sqcap y) * (x \sqcap y)$$

and hence

$$(x \sqcap y) * x = (x \sqcap y) * y. \tag{2}$$

Since $y \leqslant x * y$ by (S2), and $y \sqcap z \leqslant y$, we have $y \sqcap z \leqslant x * y$ and hence

$$(x * y) \sqcap (y \sqcap z) = (y \sqcap z). \tag{3}$$

Replacing y by $z * y$ and z by y in (3), we get

$$(x * (z * y)) \sqcap ((z * y) \sqcap y) = (z * y) \sqcap y,$$

i.e.

$$y \sqcap (x * (z * y)) = y. \tag{4}$$

This yields by (2)

$$y = y \sqcap (x * ((x \sqcap y) * y)) = y \sqcap (x * ((x \sqcap y) * x)).$$

We can apply (1) to conclude

$$y = y \sqcap (x * x)$$

for each $x, y \in D$. This yields that $x * x$ is the greatest element of D which will be denoted by 1. Hence $x * x = 1 = y * y$ proving (S4). □

Now, we are going to prove that the remaining identities of Theorem 1 are really independent.

Theorem 2. *The identities* (D2), (D3), (S1), (S2) *and* (S3) *are independent.*

Proof. As usual, we prove the independence of axioms by creating models satisfying all the axioms except one.

(a) Let $D = \{0, 1\}$ and let $\sqcap$, $*$ be defined as follows:

$\sqcap$	0	1
0	0	1
1	0	1

$*$	0	1
0	0	1
1	0	1

Then $(D; \sqcap, *)$ satisfies all the identities except (D2): $1 \sqcap 0 = 0 \neq 1 = 0 \sqcap 1$.

(b) Let $D = \{0, a, 1\}$ and let $\sqcap$, $*$ be defined by the tables

$\sqcap$	0	a	1
0	0	a	0
a	a	a	1
1	0	1	1

$*$	0	a	1
0	1	a	1
a	0	0	1
1	0	a	a

Then $(D; \sqcap, *)$ satisfies all the identities except (D3) since

$$(a \sqcap 1) \sqcap 0 = 0 \neq a = a \sqcap ((a \sqcap 1) \sqcap 0).$$

In all the remaining cases we have $D = \{0, 1\}$.

(c) If $\sqcap, *$ are defined by

$$\begin{array}{c|cc} \sqcap & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

then $(D; \sqcap, *)$ satisfies all the identities except (S1) since

$$0 \sqcap 1 = 1 \neq 0 = 0 \sqcap (0 * 1).$$

(d) If $\sqcap, *$ are defined by

$$\begin{array}{c|cc} \sqcap & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

then $(D; \sqcap, *)$ satisfies all the identities except (S2) since

$$(1 * 0) \sqcap 0 = 1 \neq 0.$$

(e) Finally, if $\sqcap, *$ are defined by

$$\begin{array}{c|cc} \sqcap & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \qquad \begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}$$

then $(D; \sqcap, *)$ satisfies all the identities except (S3) since

$$0 * 1 = 1 \neq 0 = 0 * (0 \sqcap 1).$$

$\square$

3. Relative pseudocomplement as a residuum

It is well-known that relatively pseudocomplemented $\wedge$-semilattices can be considered alternatively as residuated structures where the relationship between the operations $\wedge$ and $*$ is established by the so-called *adjointness property*:

(AP) $$a \wedge x \leqslant b \quad \text{if and only if} \quad x \leqslant a * b.$$

As mentioned above, this cannot be translated to directoids since the operation $\sqcap$ is not isotone. A natural question is if also (AP) can be modified for directoids to characterize relative pseudocomplementation as a residual operation. By replacing $a \wedge x \leqslant b$ by $a \sqcap x = a \sqcap b$, we can easily infer

(I) $$a \sqcap x = a \sqcap b \quad \Rightarrow \quad x \leqslant a * b.$$

Unfortunately, the converse implication fails for relatively pseudocomplemented directoids, see the following

Example 3. Let $\mathcal{D} = (D, \sqcap, *)$ be a relatively pseudocomplemented directoid whose Hasse diagram is depicted in Fig. 2 (one can easily enumerate the operation $*$).

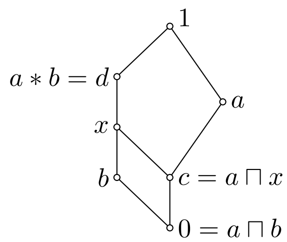


Fig. 2

Although $x \leqslant d = a * b$, we have $a \sqcap x = c \neq 0 = a \sqcap b$.

So, the right hand side of implication (I) must be completed to reach the condition in the form of equivalence. Our solution follows.

Theorem 3. *Let $(D; \sqcap)$ be a directoid and let $*$ be a binary operation on D. Then $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid if and only if the following adjointness property holds:*

(AD) $$a \sqcap x = a \sqcap b \quad \text{if and only if} \quad x \leqslant a * b \;\text{ and }\; a \sqcap (a * b) = a \sqcap x.$$

P r o o f. Assume first that $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid. We prove that $\mathcal{D}$ satisfies (AD). If $a \sqcap x = a \sqcap b$ then $x \leqslant a * b$ directly by (I). Using (S1), we infer

$$a \sqcap x = a \sqcap b = a \sqcap (a * b).$$

Conversely, if $x \leqslant a * b$ and $a \sqcap (a * b) = a \sqcap x$ then using (S1) we have

$$a \sqcap x = a \sqcap b$$

and hence (AD) is satisfied in $\mathcal{D}$.

Now, we suppose that $(D; \sqcap)$ is a directoid, $*$ is a binary operation on D and the condition (AD) is satisfied. We need only to verify the identities (S1), (S2) and (S3).

Put $x = a * b$. Then the right hand side of (AD) is satisfied, thus also the left hand side which is $a \sqcap (a * b) = a \sqcap b$ proving (S1).

If we consider $x = b$, then the left hand side of (AD) holds and hence $b \leqslant a * b$ which yields $(a * b) \sqcap b = b$ proving (S2).

Finally, for $x = a * (a \sqcap b)$ we have by (S1)

$$a \sqcap (a * (a \sqcap b)) = a \sqcap (a \sqcap b) = a \sqcap b,$$

thus, applying (AD), we infer $a * (a \sqcap b) \leqslant a * b$. Taking $x = a * b$ on the left hand side of (AD), we infer by (S1) $a \sqcap (a * b) = a \sqcap b = a \sqcap (a \sqcap b)$ and hence $a * b \leqslant a * (a \sqcap b)$. Altogether, also (S3) holds. By Theorem 1, $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid. □

Although the condition (AD) is more complex than (AP), relatively pseudocomplemented directoids satisfy also a condition which is more similar to the adjointness property.

Theorem 4. *Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid. The following condition is satisfied in $\mathcal{D}$ for all $a, b, x \in D$:*

(A) $$a \sqcap x \leqslant b \quad \text{if and only if} \quad x \leqslant (a \sqcap x) * b.$$

P r o o f. Assume $x \leqslant (a \sqcap x) * b$. Since $a \sqcap x \leqslant x$, we have $a \sqcap x \leqslant (a \sqcap x) * b$ and hence

$$a \sqcap x = (a \sqcap x) \sqcap ((a \sqcap x) * b) = (a \sqcap x) \sqcap b$$

due to (S1). This yields $a \sqcap b \leqslant b$. The converse is trivial since $a \sqcap x \leqslant b$ implies $x \leqslant 1 = (a \sqcap x) * b$. □

By putting $x = a$ in (A), we conclude

$$a \leqslant a * b \quad \Rightarrow \quad a \leqslant b.$$

By putting $a = x$ and $b = a * b$, the equivalence (A) yields

$$x \leqslant a * b \quad \text{if and only if} \quad x \leqslant x * (a * b).$$

4. Congruence properties

It was already proved in [3] that the variety of relatively pseudocomplemented directoids is congruence distributive. Unfortunately the explicit Jónsson terms ensuring the Maltsev condition for congruence distributivity (see e.g. [4]) which are listed there are not optimal (since the terms t_0 and t_1 coincide). In what follows, we can improve it by introducing new terms $t_0, \ldots, t_4$ as follows.

Theorem 5. *The terms*
$t_0(x,y,z) = x$, $t_4(x,y,z) = z$,
$t_1(x,y,z) = x \sqcap [((z * y) \sqcap (x * z)) * (x * y)]$,
$t_2(x,y,z) = x \sqcap (y * z)$,
$t_3(x,y,z) = z \sqcap [((z * x) * (x * y)) * (z * y)]$
are Jónsson terms proving congruence distributivity of the variety of relatively pseudocomplemented directoids.

Proof. Of course, we have $x = t_0(x,y,x) = t_4(x,y,x)$. Further, using (S1)–(S4), we compute

$$\begin{aligned} t_1(x,y,x) &= x \sqcap [((x * y) \sqcap (x * x)) * (x * y)] \\ &= x \sqcap [(x * y) * (x * y)] = x \sqcap 1 = x, \\ t_2(x,y,x) &= x \sqcap (y * x) = x, \\ t_3(x,y,x) &= x \sqcap [((x * x) \sqcap (x * y)) * (x * y)] \\ &= x \sqcap [(x * y) * (x * y)] = x \sqcap 1 = x. \end{aligned}$$

For i even we compute

$$\begin{aligned} t_0(x,x,y) = x = x \sqcap 1 &= x \sqcap [((y * x) \sqcap (x * y)) * 1] \\ &= x \sqcap [((y * x) \sqcap (x * y)) * (x * x)] = t_1(x,x,y), \end{aligned}$$

$$
\begin{aligned}
t_2(x,x,y) &= x \sqcap (x*y) = x \sqcap y = y \sqcap x = y \sqcap (y*x) \\
&= y \sqcap [1*(y*x)] = y \sqcap [((y*x)*1)*(y*x)] \\
&= y \sqcap [((y*x)*(x*x))*(y*x)] = t_3(x,x,y).
\end{aligned}
$$

For i odd we have

$$
\begin{aligned}
t_1(x,y,y) &= x \sqcap [((y*y) \sqcap (x*y))*(x*y)] \\
&= x \sqcap [(x*y)*(x*y)] = x \sqcap 1 = x \sqcap (y*y) = t_2(x,y,y), \\
t_3(x,y,y) &= y \sqcap [((y*x)*(x*y))*(y*y)] \\
&= y \sqcap [((y*x)*(x*y))*1] = y \sqcap 1 = y = t_4(x,y,y).
\end{aligned}
$$

□

We can prove one more congruence condition which is satisfied by the variety of relatively pseudocomplemented directoids, namely the 3-permutability.

By the result of Hagemann and Mitschke (see [6]), a variety $\mathcal{V}$ is 3-permutable if and only if there exist ternary terms p_0, p_1, p_2 and p_3 in $\mathcal{V}$ such that the following identities hold in $\mathcal{V}$:

(P)
$$
\begin{aligned}
&p_0(x,y,z) = x,\ \ p_3(x,y,z) = z, \\
&p_i(x,x,z) = p_{i+1}(x,z,z) \ \text{ for } \ i \in \{0,1,2\}.
\end{aligned}
$$

Theorem 6. *The variety $\mathcal{V}$ of relatively pseudocomplemented directoids is congruence 3-permutable.*

P r o o f. We need to find ternary terms p_0, p_1, p_2, p_3 satisfying (P). We can take $p_0(x,y,z) = x$, $p_3(x,y,z) = z$ and $p_1(x,y,z) = x \sqcap (y*z)$, $p_2(x,y,z) = z \sqcap (y*x)$. Then clearly

$$
\begin{aligned}
&p_0(x,x,z) = x = x \sqcap 1 = x \sqcap (z*z) = p_1(x,z,z), \\
&p_1(x,x,z) = x \sqcap (x*z) = x \sqcap z = z \sqcap x = z \sqcap (z*x) = p_2(x,z,z), \\
&p_2(x,x,z) = z \sqcap (x*x) = z \sqcap 1 = z = p_3(x,z,z).
\end{aligned}
$$

□

There is an open problem if the variety $\mathcal{V}$ of relatively pseudocomplemented directoids is congruence permutable. Unfortunately, at the moment we are not able to decide. That means we have neither an example of relatively pseudocomplemented directoid $\mathcal{D} = (D; \sqcap, *)$ for which there are $\Theta, \Phi \in \mathrm{Con}\mathcal{D}$ such that $\Theta \circ \Phi \neq \Phi \circ \Theta$, nor a Maltsev term for $\mathcal{V}$ proving congruence permutability.

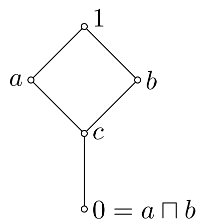
On the other hand, we are able to find proper subvarieties of $\mathcal{V}$ which are congruence permutable. Let us show some of them.

As mentioned in Introduction, every relatively pseudocomplemented semilattice is a relatively pseudocomplemented directoid. Hence, the variety $\mathcal{R}$ of relatively pseudocomplemented semilattices is a subvariety of $\mathcal{V}$ and thus also a variety $\mathcal{B}$ of Boolean algebras. However, $\mathcal{V}$ has more proper subvarieties. Let $\mathcal{W}$ be the variety of relatively pseudocomplemented directoids satisfying the identity

$$\text{(T)} \qquad ((x * y) * y) \sqcap x = x,$$

which is equivalent to $x \leqslant (x * y) * y$. It is well-known that this identity is valid in $\mathcal{R}$ but $\mathcal{W}$ does not coincide with $\mathcal{R}$, see the following

E x a m p l e 4. Let $(A; \sqcap, *)$ be a relatively pseudocomplemented directoid, where $A = \{0, c, a, b, 1\}$ and $\sqcap$, $*$ are defined by the following Hasse diagram and the table:



$*$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	b	c	1	b	1
b	a	c	a	1	1
1	0	c	a	b	1

Then $(a * b) * b = b * b = 1 \geqslant a$, $(b * a) * a = a * a = 1 \geqslant b$, $(a * c) * c = c * c = 1 \geqslant a$, analogously for $(b * c) * c$ and trivially for other combinations. Thus $(A; \sqcap, *)$ is a relatively pseudocomplemented directoid satisfying the identity (T) and hence $(A; \sqcap, *) \in \mathcal{W}$. On the other hand, $(A; \sqcap, *) \notin \mathcal{R}$ since it is not a semilattice (of course, $a \wedge b = c \neq a \sqcap b$).

It is worth noticing that (T) does not hold in $\mathcal{V}$. For example, if S is the semilattice of Example 2 considered as a directoid then $\mathcal{S}$ does not satisfy (T) since $(b * a) * a = c * a = a \ngeqslant b$.

Theorem 7. *Let $\mathcal{W}$ be a subvariety of the variety of relatively pseudocomplemented directoids satisfying the identity* (T). *Then $\mathcal{W}$ is congruence permutable and $p(x, y, z) = ((x * y) * z) \sqcap ((z * y) * x)$ is its Maltsev term.*

P r o o f. It is an easy exercise to verify that

$$p(x, z, z) = ((x * z) * z) \sqcap ((z * z) * x) = ((x * z) * z) \sqcap x = x$$

by (T) and, analogously, $p(x, x, z) = z$. □

It is a hard question to describe all subdirectly irreducible members of the variety $\mathcal{V}$ of relatively pseudocomplemented directoids. However, it is easy to discern an infinite family (in fact a proper class) of such algebras. To this end, we introduce the following notation. Let $\mathcal{S}$ be a directoid with the greatest element q. Denote by $\mathcal{S}\oplus\mathbf{1}$ the directoid which is constructed from $\mathcal{S}$ by adding a new greatest element 1. In other words, the base set of $\mathcal{S}\oplus\mathbf{1}$ is $S\cup\{1\}$ and the order in S remains and, moreover, $a < 1$ for each $a \in S$. It is evident that if $\mathcal{S}$ is a relatively pseudocomplemented directoid then $\mathcal{S} \oplus \mathbf{1}$ is so as well.

Theorem 8. *For any relatively pseudocomplemented directoid $\mathcal{S}$, the directoid $\mathcal{S} \oplus \mathbf{1}$ is a subdirectly irreducible member of $\mathcal{V}$.*

P r o o f. By Theorem 3 in [3], the variety $\mathcal{V}$ is weakly regular, i.e., for each $\mathcal{S} \in \mathcal{V}$ and every congruence $\theta \in \operatorname{Con}\mathcal{S}$, θ is fully determined by its kernel $[1]_\theta$. Hence, $\mathcal{S}$ is subdirectly irreducible if there exists the least non-trivial kernel $[1]_\mu$ for some $\mu \in \operatorname{Con}\mathcal{S}$.

Assume $\mathcal{S} \in \mathcal{V}$ and create $\overline{\mathcal{S}} = \mathcal{S}\oplus\mathbf{1}$. Let q be the greatest element of $\mathcal{S}$. Consider the congruence $\mu = \theta(q, 1)$, i.e. the one generated by the pair $\langle q, 1\rangle$. It is evident that we do not obtain another non-trivial pair induced by $\langle q, 1\rangle$ using the operation $\sqcap$ and the trivial pairs $\langle a, a\rangle$ for $a \in S$. Since $a \leqslant q$ for each $a \in S$, we have also $\langle a * 1, a * q\rangle = \langle 1, 1\rangle$. Further, $1 * a = a$ and, since $a \leqslant q$ for all $a \in S$, also $q * a = a$, thus $\langle 1 * a, q * a\rangle = \langle a, a\rangle$. Thus the operation $*$ does not create new pairs either and hence μ has only one non-singleton class which is $[1]_\mu = \{1, q\}$. Hence, $[1]_\mu$ is non-trivial and minimal, thus μ is the unique atom in $\operatorname{Con}\overline{\mathcal{S}}$. We conclude that $\overline{\mathcal{S}} = \mathcal{S} \oplus \mathbf{1}$ is a subdirectly irreducible member of $\mathcal{V}$. □

R e m a r k. Unfortunately, the directoids of type $\mathcal{S} \oplus \mathbf{1}$ do not exhaust all subdirectly irreducible members of $\mathcal{V}$. When we consider the relatively pseudocomplemented directoid N_5 from Example 2 (see Fig. 1) then it is also a subdirectly irreducible member of $\mathcal{V}$ since its congruence lattice is a 4-element chain where non-trivial members μ and ϱ have partitions $\{0, b\}$, $\{a\}$, $\{c, 1\}$ and $\{0, b\}$, $\{a, c, 1\}$, respectively.

Corollary. *Every finite chain considered as a relatively pseudocomplemented directoid is subdirectly irreducible.*

5. Derived pseudocomplementation

The concept of pseudocomplement in directoids with the least element was already introduced by the first author in [2] by using the concept of supremum. Having a directoid with the least element 0, in symbols $\mathcal{D} = (D; \sqcap, 0)$, we can consider another alternative of introducing pseudocomplementation as follows: an element y is a pseudocomplement of x if it is the greatest element satisfying

$$x \sqcap y = 0. \tag{Q}$$

At the first glance, this concept differs from that in [2] and, moreover, does not satisfy several important identities from [2]. On the other hand, if $\mathcal{D} = (D; \sqcap, *, 0)$ is a relatively pseudocomplemented directoid with 0 then for each $x \in D$, $x * 0$ is just the greatest element y satisfying (Q) and hence a pseudocomplement in the afore mentioned sense. To distinguish these cases, the latest one, i.e. $x * 0$, will be called a *d-pseudocomplement of* x (an abbreviation of derived pseudocomplement from the relative pseudocomplementation). Hence, a d-pseudocomplement is fully determined by

$$x \sqcap x * 0 = 0$$

and the implication

$$x \sqcap y = 0 \quad \Rightarrow \quad y \leqslant x * 0$$

which, unfortunately, cannot be converted. This leads to the following

Definition. Let $\mathcal{D} = (D; \sqcap, 0)$ be a directoid with 0. For each $x \in D$, denote by x^0 the greatest element of D satisfying $x \sqcap x^0 = 0$ and

$$x \sqcap y = 0 \quad \Rightarrow \quad y \leqslant x^0.$$

Hence, if $\mathcal{D} = (D; \sqcap, *, 0)$ is a relative pseudocomplemented directoid with 0 then $x^0 = x * 0$. On the other hand, the binary operation $*$ need not be the relative pseudocomplementation on $(D; \sqcap, 0)$ to get the d-pseudocomplementation, see the following

Example 5. Let $\mathcal{D}$ be a directoid visualized in Fig. 4 and let $*$ be a binary

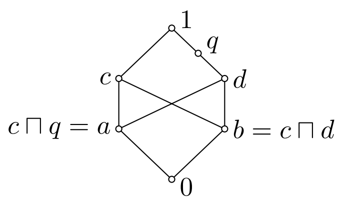


Fig. 4

operation on D whose table is

$*$	0	a	b	c	d	q	1
0	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1
b	a	a	1	1	1	1	1
c	0	d	q	1	d	q	1
d	0	c	b	c	1	1	1
q	0	a	c	c	d	1	1
1	1	a	b	c	d	q	1

Then $*$ is not a relative pseudocomplementation on D since it violates (S1):

$$c \sqcap c * b = c \sqcap q = a \neq b = c \sqcap b.$$

On the other hand, $x^0 = x * 0$ is a d-pseudocomplementation on $(D; \sqcap, 0)$.

We can state also

Lemma 1. *Let $\mathcal{D} = (D; \sqcap, 0)$ be a directoid with 0 and let $*$ be a binary operation on D satisfying the conditions* (S2), (S3) *and*

(P1) $x \sqcap x * 0 = 0$.

Then $x^0 = x * 0$ is a d-pseudocomplement of $x \in D$.

Proof. By (P1) we have $x \sqcap x^0 = 0$. Assume $x \sqcap y = 0$. Then by (S2) and (S3) we compute

$$y \leqslant x * y = x * (x \sqcap y) = x * 0 = x^0,$$

thus x^0 is a d-pseudocomplement of x. □

Lemma 2. *Let $\mathcal{D} = (D; \sqcap, ^0, 0)$ be a directoid with d-pseudocomplementation. Then the following assertions hold in $\mathcal{D}$:*

(a) $x \leqslant x^{00}$;

(b) *$\mathcal{D}$ has the greatest element $1 = 0^0$;*

(c) *$y \leqslant (x \sqcap y)^0$ if and only if $x \sqcap y = 0$.*

Proof. (a) follows directly by the definition since $x^0 \sqcap x = x \sqcap x^0 = 0$ yields $x \leqslant x^{00}$.

(b) Since $0 \sqcap x = 0$, we have $x \leqslant 0^0$ for each $x \in D$. Hence, 0^0 is the greatest element of $\mathcal{D}$.

(c) If $y \leqslant (x \sqcap y)^0$ then $x \sqcap y \leqslant y$ implies $x \sqcap y \leqslant (x \sqcap y)^0$ and hence $0 = (x \sqcap y) \sqcap (x \sqcap y)^0 = x \sqcap y$. The converse is trivial since $x \sqcap y = 0$ yields $y \leqslant 1 = 0^0 = (x \sqcap y)^0$. □

Remark. Unfortunately, the d-pseudocomplement in a directoid $\mathcal{D} = (D; \sqcap, 0)$ does not satisfy some desired identities and implications which hold for that defined in [2], e.g. $x^{000} = x^0$, $x \leqslant y \Rightarrow y^0 \leqslant x^0$, $x^0 \sqcap y^0 = (x^0 \sqcap y^0)^{00}$.

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