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IMPULSIVE SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH MULTIVALUED JUMPS

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Abstract. In this paper we establish sufficient conditions for the existence of mild solutions and extremal mild solutions for some densely defined impulsive semilinear neutral functional differential inclusions in separable Banach spaces. We rely on a fixed point theorem for the sum of completely continuous and contraction operators.

Keywords: impulsive semilinear neutral functional differential equation, densely defined operator, infinite delay, phase space, fixed point, mild solutions, extremal mild solution

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1. Introduction

In this paper we are concerned with the existence of mild solutions and extremal mild solutions defined on a compact real interval for first order impulsive semilinear neutral functional inclusions in a separable Banach space. More precisely, we will consider the following first order impulsive semilinear neutral functional differential inclusions:

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)-g(t,y_t)]-A[y(t)-g(t,y_t)]\in F(t,y_t), \quad \text{a.e. } t\in J=[0,b],\ t\neq t_k,\ k=1,\dots,m, \tag{1}$$

$$\Delta y|_{t=t_k}\in I_k(y(t_k^-)),\qquad k=1,\dots,m, \tag{2}$$

$$y(t)=\varphi(t),\quad t\in(-\infty,0], \tag{3}$$

where $F\colon J\times D\to\mathcal{P}(E)$ is a compact and convex valued multivalued map, $g\colon J\times D\to E$ is a given function, $0=t_0<t_1<\dots<t_m<t_{m+1}=b$, $\varphi\in D$, where

D is the phase space that will be specified later, $I_k\colon E \to \mathcal{P}(E)$, $k = 1, 2, \dots, m$, are bounded valued multivalued maps, $\mathcal{P}(E)$ is the collection of all subsets of E, $A\colon D(A) \subset E \to E$ is a densely defined closed linear operator on E, and E is a real separable Banach space with a norm $|\cdot|$. For any function y defined on $(-\infty, b] \setminus \{t_1, t_2, \dots, t_m\}$ and any $t \in J$, we denote by y_t the element of D defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

Functional and neutral functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Hale [21], Hale and Verduyn Lunel [23], Kolmanovskii and Myshkis [32], Kuang [33] and Wu [45], and the references therein. Impulsive differential and partial differential equations are used to describe various models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. That is why in recent years they have been the object of investigations. We refer to the monographs of Bainov and Simeonov [7], Benchohra *et al* [10], Lakshmikantham *et al* [34], and Samoilenko and Perestyuk [42], where numerous properties of their solutions are studied, and a detailed bibliography is given. Semilinear functional differential equations and inclusions with or without impulses have been extensively studied where the operator A generates a C_0-semigroup. The existence and uniqueness, among other things, have been derived; see the books of Ahmed [3], [4], Benchohra *et al* [9], Heikkila and Lakshmikantam [24], Kamenskii *et al* [29] and the papers by Ahmed [5], [6], Liu [37], and Rogovchenko [40], [41]. In [2] Abada *et al* have studied the controllability of a class of impulsive semilinear functional differential inclusions in Fréchet spaces by means of the extrapolation method ([13], [18]), and in [1] the existence of mild and extremal mild solutions for first-order semilinear densely defined impulsive functional differential inclusions in separable Banach spaces with local and nonlocal conditions has been considered. To the best of our knowledge, there are very few results for impulsive evolution inclusions with multivalued jump operators; see [1], [11], [38]. The notion of the phase space D plays an important role in the study of both the qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [22] (see also Kappel and Schappacher [30] and Schumacher [43]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [27]. For the case where the impulses are absent (i.e. $I_k = 0$, $k = 1, \dots, m$), an extensive theory has been developed for the problem (1)–(3). We refer to Belmekki *et al* [8], Corduneanu and Lakshmikantham [12], Hale and Kato [22], Hino *et al* [27], Lakshmikantham *et al* [35] and Shin [44]. The study of first order abstract neutral functional differential equations with unbounded delay was initiated by Hernandez and Henriquez [25], [26]. The goal of the present paper is to give existence results for first order impulsive neutral functional differential inclusions with multivalued jump functions and infinite delay. The paper is organized as follows. In Section 2 we recall briefly some basic definitions and preliminaries facts which will be used throughout the subsequent sections. In Section 3 we establish sufficient conditions for the existence of mild solutions for the problem (1)–(3) by relying on a fixed point theorem due to Dhage. In Section 4 sufficient conditions for the existence of extremal mild solutions for the problem (1)–(3) are established. The last section is devoted to an example illustrating the abstract theory.

2. Preliminaries

In this section we state some facts about semigroups, notation and definitions that are used throughout this paper. $C(J,E)$ is the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| \colon t \in J\},$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup\{|N(y)| \colon |y| = 1\}.$$

Let $L^1(J,E)$ denote the Banach space of measurable functions $y\colon J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\,\mathrm{d}t.$$

In order to define the phase space and the solution of (1)–(3) we shall consider the space

$$PC = \{y\colon J \to E,\ y(t_k^-),\, y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-),\ y_k \in C(J_k, E)\},$$

where y_k is the restriction of y to $J_k = (t_k,\, t_{k+1}]$, $k = 0,\dots,m$. Let $\|\cdot\|_{PC}$ be the norm in PC defined by

$$\|y\|_{PC} = \sup\{|y(s)| \colon 0 \leqslant s \leqslant b\}, \quad y \in PC.$$

We will assume that D satisfies the following axioms:

(A) If $y\colon (-\infty, b] \to E$, $b > 0$ and $y(t_k^-)$, $y(t_k^+)$ exist with $y(t_k) = y(t_k^-)$, $k = 1,\dots,m$ and $y_0 \in D$, then for every t in $[0,b]$ the following conditions hold:

(i) $\|y_t\|_D \leqslant K(t)\sup\{|y(s)|\colon\ 0 \leqslant s \leqslant t\} + M(t)\|y_0\|_D$,

(ii) $|y(t)| \leqslant H\|y_t\|_D$,

where $H \geqslant 0$ is a constant, $K\colon\ [0,\infty) \to [0,\infty)$ is continuous, $M\colon\ [0,\infty) \to [0,\infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(B) The space D is complete.

Set

$$D_b = \{y\colon\ (-\infty, b] \to E,\ y \in PC \cap D\},$$

and let $\|\cdot\|_b$ be the seminorm in D_b defined by

$$\|y\|_b := \|y_0\|_D + \sup\{|y(t)|\colon\ 0 \leqslant t \leqslant b\}, \quad y \in D_b.$$

Denote

$$K_b = \sup\{K(t)\colon\ t \in J\} \quad \text{and} \quad M_b = \sup\{M(t)\colon\ t \in J\}.$$

Let (X, d) be a metric space. We use the notation:

$$P_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ closed}\}, \quad P_{\mathrm{bd}}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ bounded}\},$$
$$P_{\mathrm{cv}}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ convex}\}, \quad P_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ compact}\}.$$

Consider $H_d\colon\ \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\Big\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{\mathrm{bd,cl}}(X), H_d)$ is a metric space and $(P_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [31]).

A multivalued map $N\colon\ J \to P_{\mathrm{cl}}(X)$ is said to be measurable if, for each $x \in X$, the function $Y\colon\ J \to \mathbb{R}$ defined by

$$Y(t) = d(x, N(t)) = \inf\{d(x, z)\colon\ z \in N(t)\}$$

is measurable.

Definition 2.1. A measurable multivalued function $F\colon\ J \to P_{\mathrm{bd,cl}}(X)$ is said to be integrably bounded if there exists a function $w \in L^1(J, \mathbb{R}^+)$ such that $\|v\| \leqslant w(t)$ a.e. $t \in J$ for all $v \in F(t)$.

A multivalued map $F\colon\ X \to \mathcal{P}(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in P_b(X)$, i.e. $\sup_{x\in B}\{\sup\{|y|\colon\ y \in F(x)\}\} < \infty$. The function F is called upper semi-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subseteq \mathcal{U}$. The function G is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_{\mathrm{bd}}(X)$. If the multivalued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has closed graph, i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_*)$ implies $y_* \in F(x_*)$.

Definition 2.2. A multivalued map $F\colon J \times D \to \mathcal{P}(E)$ is said to be L^1-Carathéodory if

(i) $t \mapsto F(t,u)$ is measurable for each $u \in D$;
(ii) $u \mapsto F(t,u)$ is u.s.c. for almost all $t \in J$;
(iii) for each $q > 0$ there exists $\varphi_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,u)\| = \sup\{|v|\colon\ v \in F(t,u)\} \leqslant \varphi_q(t) \quad \text{for all } \|u\|_D \leqslant q \text{ and for a.e. } t \in J.$$

Definition 2.3. A multivalued operator $N\colon\ J \to P_{\mathrm{cl}}(X)$

(a) is called contraction if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leqslant \gamma d(x,y) \quad \text{for each } x, y \in X,$$

with $\gamma < 1$,

(b) has a fixed point if there exists $x \in X$ such that $x \in N(x)$.

For more details on multivalued maps and proofs of the known results cited in this section we refer the interested reader to the books of Deimling [17], Gorniewicz [20], and Hu and Papageorgiou [28]. Details on semigroup theory can be found in the books by Ahmed [3] and Pazy [39]. The key tool in our approach is the following form of the fixed point theorem of Dhage [14], [16].

Theorem 2.1. *Let X be a Banach space, $\mathcal{A}\colon X \to P_{\mathrm{cl,cv,bd}}(X)$ and $\mathcal{B}\colon X \to P_{\mathrm{cp,cv}}(X)$ two multivalued operators satisfying*

(a) *$\mathcal{A}$ is a contraction, and*
(b) *$\mathcal{B}$ is completely continuous.*

Then either

(i) *the operator inclusion $\lambda x \in \mathcal{A}x + \mathcal{B}x$ has a solution for $\lambda = 1$, or*
(ii) *the set $\mathcal{E} = \{u \in X\colon\ u \in \lambda\mathcal{A}u + \lambda\mathcal{B}u,\ 0 \leqslant \lambda \leqslant 1\}$ is unbounded.*

3. Existence of mild solutions

In this section we present an existence result of mild solutions for the problem (1)–(3). First, we define what we mean by a mild solution.

Definition 3.1. A function $y \in D_b$ is said to be a mild solution of system (1)–(3) if $y(t) = \varphi(t)$ for all $t \in (-\infty, 0]$, and there exist $v(\cdot) \in L^1(J_k, E)$ and $\mathcal{I}_k \in I_k(y(t_k^-))$ such that $v(t) \in F(t, y_t)$ a.e. $t \in J$, and y satisfies the integral equation

$$\begin{aligned} y(t) &= T(t)(\varphi(0) - g(0, \varphi(0))) + g(t, y_t) \\ &\quad + \int_0^t T(t-s)v(s)\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k, \quad t \in J. \end{aligned}$$

Here $T(t)$, $t \geqslant 0$, denotes the semigroup generated by the operator A. For each $y \in D_b$ define the set of selections of the multivalued F by

$$S_{F,y} = \{v \in L^1(J, E)\colon\ v(t) \in F(t, y_t) \text{ a.e. } t \in J\}.$$

In our proof we use the following result due to Lasota and Opial [36].

Lemma 3.1. *Let E be a Banach space and F an L^1-Carathéodory multivalued map with compact convex values, and let $\Gamma\colon\ L^1(J, E) \to C(J, E)$ be a linear continuous mapping. Then the operator*

$$\Gamma \circ S_F\colon\ C(J, E) \to P_{\mathrm{cp,cv}}(C(J, E))$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Let us introduce the following hypotheses:

(H1) $A\colon D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}$, $t \in J$ which is compact for $t > 0$ in the Banach space E, and there exists a constant $M > 0$ such that

$$\|T(t)\|_{B(E)} \leqslant M; \quad t \in J.$$

(H2) There exist constants $c_k > 0$, $k = 1, \ldots, m$ such that

$$H_d(I_k(y), I_k(x)) \leqslant c_k|y - x| \quad \text{for each } x, y \in E.$$

(H3) The function $g(t, \cdot)$ is continuous on J and there exists a constant $l_g > 0$ such that

$$|g(t, u) - g(t, v)| \leqslant l_g\|u - v\|_D \quad \text{for each } u, v \in D.$$

(H4) There exist constants $\alpha_1, \alpha_2 > 0$ such that

$$|g(t,u)| \leqslant \alpha_1 \|u\|_D + \alpha_2 \quad \text{for each } (t,u) \in [0,b] \times D,$$

and

$$M \sum_{k=1}^{m} c_k + K_b \alpha_1 < 1.$$

(H5) F is L^1-Carathéodory with compact convex values.

(H6) There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi\colon\ [0,\infty) \to (0,\infty)$ such that

$$\|F(t,x)\| \leqslant p(t)\psi(\|x\|_D) \quad \text{for a.e. } t \in J \text{ and each } x \in D,$$

with

$$\limsup_{u \to +\infty} \frac{\Big(1 - \alpha_1 K_b - M \sum\limits_{k=1}^{m} c_k\Big)u}{C_0 + M\|p\|_{L^1}\psi(K_b u + (MK_b + M_b)\|\varphi\|_D)} > 1, \tag{4}$$

where

$$\begin{aligned} C_0 = {} & \alpha_1(MK_b + M_b)\|\varphi\|_D) + \alpha_2 + M(\alpha_1\|\varphi\|_D + \alpha_2) \\ & + M^2 \sum_{k=1}^{m} c_k \|\varphi\|_D + M \sum_{k=1}^{m} \|I_k(0)\|. \end{aligned} \tag{5}$$

Theorem 3.1. *Assume that* (H1)–(H6) *hold. If*

$$l_g + M \sum_{k=1}^{m} c_k < 1, \tag{6}$$

then IVP (1)–(3) *has at least one mild solution on* $(-\infty, b]$.

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the multivalued operator $N\colon\ D_b \to \mathcal{P}(D_b)$ defined by

$$N(y) = \left\{ h \in D_b\colon\ h(t) = \left\{ \begin{array}{ll} \varphi(t) & \text{if } t \leqslant 0; \\ T(t)(\varphi(0) - g(0,\varphi(0))) + g(t, y_t) & \\ \quad + \displaystyle\int_0^t T(t-s)v(s)\,\mathrm{d}s & \\ \quad + \displaystyle\sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k, & \\ \qquad v \in S_{F,y},\ \mathcal{I}_k \in I_k(y(t_k^-)) & \text{if } t \in J. \end{array} \right. \right\}$$

It is clear that the fixed points of N are mild solutions of IVP (1)–(3). For $\varphi \in D_b$ define a function $x(\cdot)\colon (-\infty, b] \to E$ such that

$$x(t) = \begin{cases} \varphi(t) & \text{if } t \leqslant 0, \\ T(t)\varphi(0) & \text{if } t \in J. \end{cases}$$

Then $x(\cdot)$ is an element of D_b, and $x_0 = \varphi(s)$, $s \leqslant 0$. Set

$$y(t) = z(t) + x(t).$$

Obviously, if y satisfies the integral equation

$$\begin{aligned} y(t) &= T(t)(\varphi(0) - g(0, \varphi(0))) + g(t, y_t) \\ &\quad + \int_0^t T(t-s)v(s)\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k, \quad t \in J, \end{aligned}$$

then z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= g(t, z_t + x_t) - T(t)g(0, \varphi(0)) \\ &\quad + \int_0^t T(t-s)v(s)\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k, \quad t \in J, \end{aligned}$$

where $v(t) \in F(t, z_t + x_t)$ for a.e. $t \in J$ and $\mathcal{I}_k \in I_k(z(t_k^-) + x(t_k^-))$.

Let $D_b^0 = \{z \in D_b\colon\ z_0 = 0\}$. For any $z \in D_b^0$ we have

$$\|z\|_b = \|z_0\|_D + \sup\{|z(s)|\colon\ 0 \leqslant s \leqslant b\} = \sup\{|z(s)|\colon\ 0 \leqslant s \leqslant b\}.$$

Thus $(D_b^0, \|\cdot\|_b)$ is a Banach space. Let the operator $P\colon D_b^0 \to \mathcal{P}(D_b^0)$ defined by

$$P(z) = \left\{ h \in D_b^0\colon\ h(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0]; \\ g(t, z_t + x_t) - T(t)g(0, \varphi(0)) & \\ \quad + \displaystyle\int_0^t T(t-s)v(s)\,\mathrm{d}s & \\ \quad + \displaystyle\sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k & \text{if } t \in J. \end{cases} \right\}$$

The operator N having a fixed point is equivalent to P having one, so it suffices to prove that P has a fixed point. Consider the operators $\mathcal{A}, \mathcal{B}\colon D_b^0 \to \mathcal{P}(D_b^0)$ defined by

$$\mathcal{A}(z) := \left\{ h \in D_b^0\colon\ h(t) = \begin{cases} 0 & \text{if } t \leqslant 0; \\ g(t, z_t + x_t) - T(t)g(0, \varphi(0)) & \\ + \displaystyle\sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k,\ \mathcal{I}_k \in I_k(z(t_k^-) + x(t_k^-)) & \text{if } t \in J, \end{cases} \right\}$$

and

$$\mathcal{B}(z) := \left\{ h \in D_b^0 \colon\ h(t) = \begin{cases} 0 & \text{if } t \leqslant 0; \\ \displaystyle\int_0^t T(t-s)v(s)\,\mathrm{d}s & \text{if } t \in J, \end{cases} \right\}$$

where

$$v \in S_{F,z+x} = \{v \in L^1(J,E)\colon\ v(t) \in F(t, z_t + x_t) \text{ for a.e. } t \in J\}.$$

It is clear that $P = \mathcal{A} + \mathcal{B}$. Hence, the problem of finding mild solutions of (1)–(3) is then reduced to finding solutions of the operator inclusion $z \in \mathcal{A}(z) + \mathcal{B}(z)$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Theorem 2.1. The proof will be given in several steps.

Step 1: $\mathcal{A}$ is a contraction.

Let $z_1, z_2 \in D_b^0$. Then (H2) yields

$$\begin{aligned} &H_d(\mathcal{A}(z_1), \mathcal{A}(z_2)) \\ &\quad \leqslant |g(t, z_{1_t} + x_t) - g(t, z_{2_t} + x_t)| \\ &\qquad + H_d\bigg(\sum_{0<t_k<t} T(t-t_k) I_k(z_1(t_k^-) + x(t_k^-)), \sum_{0<t_k<t} T(t-t_k) I_k(z_2(t_k^-) + x(t_k^-))\bigg) \\ &\quad \leqslant l_g \|z_1 - z_2\|_{D_b^0} + M \sum_{k=1}^m c_k |z_1(t_k^-) - z_2(t_k^-)| \\ &\quad \leqslant \bigg(l_g + M \sum_{k=1}^m c_k\bigg) \|z_1 - z_2\|_{D_b^0}. \end{aligned}$$

Hence, by (6), $\mathcal{A}$ is a contraction.

Step 2: $\mathcal{B}$ has compact, convex values, and it is completely continuous.

This will be proved in several claims.

Claim 1. *$\mathcal{B}$ has compact values.*

The operator $\mathcal{B}$ is equivalent to the composition $\mathcal{L} \circ S_F$ on $L^1(J,E)$, where $\mathcal{L}\colon L^1(J,E) \to D_b^0$ is the continuous operator defined by

$$\mathcal{L}(v)(t) = \int_0^t T(t-s)v(s)\,\mathrm{d}s, \quad t \in J.$$

Hence, it suffices to show that $\mathcal{L} \circ S_F$ has compact values on D_b^0.

Let $z \in D_b^0$ be arbitrary and let v_n be a sequence in $S_{F,z+x}$, then $v_n(t) \in F(t, z_t + x_t)$ for a.e. $t \in J$. Since $F(t, z_t + x_t)$ is compact, we may pass to a subsequence. Suppose that $v_n \to v$ in $L_w^1(J,E)$ (the space endowed with the weak topology), where $v(t) \in F(t, z_t + x_t)$ a.e. $t \in J$. An application of Mazur's theorem ([46]) implies that v_n converges strongly to v and hence $v \in S_{F,z+x}$. From the continuity of $\mathcal{L}$ it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equicontinuous sequence. Let $\tau_1, \tau_2 \in J$. Then we have

$$\begin{aligned}|\mathcal{L}(v_n(\tau_1)) - \mathcal{L}(v_n(\tau_2))| &= \left|\int_0^{\tau_1} T(\tau_1 - s)v_n(s)\,\mathrm{d}s - \int_0^{\tau_2} T(\tau_2 - s)v_n(s)\,\mathrm{d}s\right| \\ &\leqslant \int_0^{\tau_1} \|T(\tau_1 - s) - T(\tau_2 - s)\|_{B(E)}|v_n(s)|\,\mathrm{d}s \\ &\quad + \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}|v_n(s)|\,\mathrm{d}s.\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. Since $T(t)$ is a strongly continuous operator and due to the compactness of $T(t)$, $t > 0$, the uniform continuity follows (see [3], [39]). Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Arzéla-Ascoli theorem implies that there exists a subsequence which is uniformly convergent. Then we have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S_F)(z)$ as $j \to \infty$, and so $(\mathcal{L} \circ S_F)(z)$ is compact. Therefore, $\mathcal{B}$ is a compact valued multivalued operator on D_b^0.

Claim 2. *$\mathcal{B}(z)$ is convex for each $z \in D_b^0$.*

Let $h_1, h_2 \in \mathcal{B}(z)$, then there exist $v_1, v_2 \in S_{F,z+x}$ such that for each $t \in J$ we have $(i = 1, 2)$

$$h_i(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ \displaystyle\int_0^t T(t-s)v_i(s)\,\mathrm{d}s & \text{if } t \in J. \end{cases}$$

Let $0 \leqslant \delta \leqslant 1$. Then for each $t \in J$ we have

$$(\delta h_1 + (1-\delta)h_2)(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ \displaystyle\int_0^t T(t-s)[\delta v_1(s) + (1-\delta)v_2(s)]\,\mathrm{d}s & \text{if } t \in J. \end{cases}$$

Since $F(t, z_t + x_t)$ has convex values, one has

$$\delta h_1 + (1-\delta)h_2 \in \mathcal{B}(z).$$

Claim 3. *$\mathcal{B}$ maps bounded sets into bounded sets in D_b^0.*

Let $B = \{z \in D_b^0;\ \|z\|_{D_b^0} \leqslant q\}$, let $q \in \mathbb{R}^+$ be a bounded set in D_b^0. For each $h \in \mathcal{B}(z)$ and each $z \in B$ there exists $v \in S_{F,z+x}$ such that

$$h(t) = \int_0^t T(t-s)v(s)\,\mathrm{d}s.$$

From (A) we have

$$\|z_s + x_s\|_D \leqslant \|z_s\|_D + \|x_s\|_D \leqslant K_b q + K_b M|\varphi(0)| + M_b\|\varphi\|_D = q_*.$$

Then by (H6) we have

$$|h(t)| \leqslant M\psi(q_*)\int_0^t p(s)\,\mathrm{d}s \leqslant M\psi(q_*)\|p\|_{L^1} := l.$$

This further implies that

$$\|h\|_{D_b^0} \leqslant l.$$

Hence, $\mathcal{B}(B)$ is bounded.

Claim 4. *$\mathcal{B}$ maps bounded sets into equicontinuous sets.*

Let B be, as above, a bounded set and let $h \in \mathcal{B}(z)$ for some $z \in B$. Then there exists $v \in S_{F,z+x}$ such that

$$h(t) = \int_0^t T(t-s)v(s)\,\mathrm{d}s, \quad t \in J.$$

Let $\tau_1, \tau_2 \in J \setminus \{t_1, t_2, \ldots, t_m\}$, $\tau_1 < \tau_2$. Thus if $\varepsilon > 0$, we have

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| \leqslant{} & \int_0^{\tau_1-\varepsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}|v(s)|\,\mathrm{d}s \\
& + \int_{\tau_1-\varepsilon}^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}|v(s)|\,\mathrm{d}s \\
& + \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}|v(s)|\,\mathrm{d}s \\
\leqslant{} & \psi(q_*)\int_0^{\tau_1-\varepsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}p(s)\,\mathrm{d}s \\
& + \psi(q_*)\int_{\tau_1-\varepsilon}^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}p(s)\,\mathrm{d}s \\
& + \psi(q_*)\int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}p(s)\,\mathrm{d}s.
\end{aligned}$$

As $\tau_1 \to \tau_2$ and ε becomes sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the uniform continuity. This proves the equicontinuity for the case where $t \neq t_i$, $i = 1, \ldots, m+1$. It remains to examine the equicontinuity at $t = t_i$. First we prove the equicontinuity at $t = t_i^-$. We have that for some $z \in B$ there exists $v \in S_{F,z+x}$ such that

$$h(t) = \int_0^t T(t-s)v(s)\,\mathrm{d}s, \quad t \in J.$$

Fix $\delta_1 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < \varrho < \delta_1$ we have

$$\begin{aligned}|h(t_i - \varrho) - h(t_i)| \leqslant \int_0^{t_i - \varrho} &\|T(t_i - \varrho - s) - T(t_i - s)\|_{B(E)} |v(s)|\,\mathrm{d}s \\ &+ \psi(q_*) M \int_{t_i - \varrho}^{t_i} p(s)\,\mathrm{d}s,\end{aligned}$$

which tends to zero as $\varrho \to 0$. Define

$$\hat{h}_0(t) = h(t), \quad t \in [0, t_1]$$

and

$$\hat{h}_i(t) = \begin{cases} h(t) & \text{if } t \in (t_i, t_{i+1}], \\ h(t_i^+) & \text{if } t = t_i. \end{cases}$$

Next, we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k\colon\ k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. Then

$$\hat{h}(t_i) = \int_0^{t_i} T(t_i - s)v(s)\,\mathrm{d}s.$$

For $0 < \varrho < \delta_2$ we have

$$\begin{aligned}|\hat{h}(t_i + \varrho) - \hat{h}(t_i)| \leqslant \int_0^{t_i} &\|T(t_i + \varrho - s) - T(t_i - s)\|_{B(E)} |v(s)|\,\mathrm{d}s \\ &+ \psi(q_*) M \int_{t_i}^{t_i + \varrho} p(s)\,\mathrm{d}s.\end{aligned}$$

The right-hand side tends to zero as $\varrho \to 0$. The equicontinuity for the cases $\tau_1 < \tau_2 \leqslant 0$ and $\tau_1 \leqslant 0 \leqslant \tau_2$ follows from the uniform continuity of φ on the interval $(-\infty, 0]$. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that $\mathcal{B}$ maps B into a precompact set in E.

Let $0 < t < b$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in B$ we define

$$h_\varepsilon(t) = T(\varepsilon) \int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\,\mathrm{d}s,$$

where $v \in S_{F,z+x}$. Since $T(t)$ is a compact operator for $t > 0$, the set

$$H_\varepsilon(t) = \{h_\varepsilon(t)\colon\ h_\varepsilon \in \mathcal{B}(z)\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $h \in \mathcal{B}(z)$ we have

$$\begin{aligned}|h(t) - h_\varepsilon(t)| &= \left|\int_0^t T(t-s)v(s)\,\mathrm{d}s - T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon)v(s)\,\mathrm{d}s\right| \\ &= \left|\int_{t-\varepsilon}^t T(t-s)v(s)\,\mathrm{d}s\right| \leqslant M\psi(q_*)\int_{t-\varepsilon}^t p(s)\,\mathrm{d}s.\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h(t)\colon\ h \in \mathcal{B}(z)\}$. Hence, the set $H(t) = \{h(t)\colon\ h \in \mathcal{B}(B)\}$ is precompact in E. Consequently, the operator $\mathcal{B}$ is totally bounded.

Claim 5. *$\mathcal{B}$ has a closed graph.*

Let $z_n \to z_*$, $h_n \in \mathcal{B}(z_n)$, and $h_n \to h_*$. We shall show that $h_* \in \mathcal{B}(z_*)$. The relation $h_n \in \mathcal{B}(z_n)$ means that there exists $v_n \in S_{F,z_n+x}$ such that

$$h_n(t) = \int_0^t T(t-s)v_n(s)\,\mathrm{d}s, \quad t \in J.$$

We must prove that there exists $v_* \in S_{F,z_*+x}$ such that

$$h_*(t) = \int_0^t T(t-s)v_*(s)\,\mathrm{d}s.$$

Consider the linear and continuous operator $\mathcal{K}\colon\ L^1(J,E) \to D_b^0$ defined by

$$(\mathcal{K}v)(t) = \int_0^t T(t-s)v(s)\,\mathrm{d}s.$$

We have

$$|h_n(t) - h_*(t)| \leqslant \|h_n - h_*\|_{D_b^0} \to 0 \quad \text{as } n \to \infty.$$

From Lemma 3.1 it follows that $\mathcal{K} \circ S_F$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$h_n(t) \in \mathcal{K} \circ S_{F,z_n+x}.$$

As $z_n \to z_*$ and $h_n \to h_*$, there is a $v_* \in S_{F,z_*+x}$ such that

$$h_*(t) = \int_0^t T(t-s)v_*(s)\,\mathrm{d}s.$$

Hence, the multivalued operator $\mathcal{B}$ is upper semi-continuous.

Step 3. A priori bounds on solutions.

Now, it remains to show that the set

$$\mathcal{E} = \{z \in D_b^0 \colon\ z \in \lambda\mathcal{A}z + \lambda\mathcal{B}z,\ 0 \leqslant \lambda \leqslant 1\}$$

is bounded.

Let $z \in \mathcal{E}$ be any element. Then there exist $v \in S_{F,z+x}$ and $\mathcal{I}_k \in I_k(z(t_k^-)+x(t_k^-))$ such that

$$\begin{aligned} z(t) = {}& \lambda g(t, z_t + x_t) - \lambda T(t)g(0,\varphi(0)) \\ & + \lambda\int_0^t T(t-s)v(s)\,\mathrm{d}s + \lambda\sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k. \end{aligned}$$

Then (H1), (H2), (H4), and (H6) yield

$$\begin{aligned} |z(t)| \leqslant{}& \alpha_1\|z_t + x_t\|_D + \alpha_2 + M(\alpha_1\|\varphi\|_D + \alpha_2) \\ & + M\int_0^t p(s)\psi(\|z_s + x_s\|_D)\,\mathrm{d}s + M\sum_{k=1}^m c_k|z(t_k^-) + x(t_k^-)| + M\sum_{k=1}^m \|I_k(0)\| \\ \leqslant{}& \alpha_1(K_b\|z\|_{D_b^0} + (MK_b + M_b)\|\varphi\|_D) + \alpha_2 + M(\alpha_1\|\varphi\|_D + \alpha_2) \\ & + M\int_0^t p(s)\psi(K_b\|z\|_{D_b^0} + (MK_b + M_b)\|\varphi\|_D)\,\mathrm{d}s \\ & + M\sum_{k=1}^m c_k|z(t_k^-)| + M\sum_{k=1}^m c_k|x(t_k^-)| + M\sum_{k=1}^m \|I_k(0)\| \\ \leqslant{}& \alpha_1(K_b\|z\|_{D_b^0} + (MK_b + M_b)\|\varphi\|_D) + \alpha_2 + M(\alpha_1\|\varphi\|_D + \alpha_2) \\ & + M\int_0^t p(s)\psi(K_b\|z\|_{D_b^0} + (MK_b + M_b)\|\varphi\|_D)\,\mathrm{d}s \\ & + M\sum_{k=1}^m c_k|z(t_k^-)| + M^2\sum_{k=1}^m c_k\|\varphi\|_D + M\sum_{k=1}^m \|I_k(0)\| \\ \leqslant{}& \alpha_1(K_b\|z\|_{D_b^0} + (MK_b + M_b)\|\varphi\|_D) + \alpha_2 + M(\alpha_1\|\varphi\|_D + \alpha_2) \\ & + M\|p\|_{L^1}\psi(K_b\|z\|_{D_b^0} + (MK_b + M_b)\|\varphi\|_D) \\ & + M\sum_{k=1}^m c_k\|z\|_{D_b^0} + M^2\sum_{k=1}^m c_k\|\varphi\|_D + M\sum_{k=1}^m \|I_k(0)\|. \end{aligned}$$

Hence, by (5) we have

$$\begin{aligned}
\Big(1-\alpha_1 K_b - M\sum_{k=1}^{m} c_k\Big)&\|z\|_{D_b^0} \\
&\leqslant \alpha_1(MK_b+M_b)\|\varphi\|_D)+\alpha_2+M(\alpha_1\|\varphi\|_D+\alpha_2)\\
&\quad + M\|p\|_{L^1}\psi(K_b\|z\|_{D_b^0}+(MK_b+M_b)\|\varphi\|_D)\\
&\quad + M^2\sum_{k=1}^{m} c_k\|\varphi\|_D + M\sum_{k=1}^{m}\|I_k(0)\|\\
&= C_0 + M\|p\|_{L^1}\psi(K_b\|z\|_{D_b^0}+(MK_b+M_b)\|\varphi\|_D).
\end{aligned}$$

Thus

$$\frac{\Big(1-\alpha_1 K_b - M\sum\limits_{k=1}^{m} c_k\Big)\|z\|_{D_b^0}}{C_0+M\|p\|_{L^1}\psi(K_b\|z\|_{D_b^0}+(MK_b+M_b)\|\varphi\|_D)}\leqslant 1. \tag{7}$$

From (4) it follows that there exists a constant $R>0$ such that for each $z\in\mathcal{E}$ with $\|z\|_{D_b^0}>R$ the condition (7) is violated. Hence, $\|z\|_{D_b^0}\leqslant R$ for each $z\in\mathcal{E}$, which means that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 2.1, $\mathcal{A}+\mathcal{B}$ has a fixed point z^* in the interval $(-\infty,b]$, so $y^*=z^*+x$ is a fixed point of the operator N which is the mild solution of problem (1)–(3). □

4. Existence of extremal mild solutions

In this section we will prove the existence of maximal and minimal solutions of problem (1)–(3) under suitable monotonicity conditions on the multivalued functions involved in it.

Definition 4.1. A nonempty closed subset C of a Banach space $(X,\|\cdot\|)$ is called a cone if

(i) $C+C\subset C$,
(ii) $\lambda C\subset C$ for $\lambda>0$,
(iii) $\{-C\}\cap\{C\}=\{0\}$.

A cone C is called normal if the norm $\|\cdot\|$ is semi-monotone on C, i.e., there exists a constant $N>0$ such that $\|x\|\leqslant N\|y\|$ whenever $x\leqslant y$. We equip the space $X=C(J,E)$ with the order relation $\leqslant$ induced by a regular cone $\mathcal{C}$ in E, that is, for all $y,\overline{y}\in X$ we put $y\leqslant\overline{y}$ if and only if $\overline{y}(t)-y(t)\in\mathcal{C}$, for all $t\in J$. In what follows we will assume that the cone C is normal. Cones and their properties are detailed in [19], [24]. Let $a, b \in X$ be such that $a \leqslant b$. Then by an order interval $[a, b]$ we mean the set of points in X given by

$$[a, b] = \{x \in X \colon\ a \leqslant x \leqslant b\}.$$

Let $Q_1, Q_2 \in P_{\rm cl}(X)$. Then by $Q_1 \leqslant Q_2$ we mean $a \leqslant b$ for all $a \in Q_1$ and $b \in Q_2$. Thus $a \leqslant Q_2$ implies that $a \leqslant b$ for all $b \in Q_2$; in particular, if $Q_1 \leqslant Q_1$, then it follows that Q_1 is a singleton set.

Definition 4.2. Let X be an ordered Banach space. A multivalued mapping $T\colon X \to P(X)$ is called isotone increasing if $T(x) \leqslant T(y)$ for any $x, y \in X$ with $x < y$. Similarly, T is called isotone decreasing if $T(x) \geqslant T(y)$ whenever $x < y$.

Definition 4.3. We say that $x \in X$ is the least fixed point of G in X if $x \in Gx$ and $x \leqslant y$ whenever $y \in X$ and $y \in Gy$. The greatest fixed point of G in X is defined similarly by reversing the inequality. If both the least and greatest fixed points of G in X exist, we call them extremal fixed points of G in X.

Very recently Dhage has proved the following assertion.

Theorem 4.1 ([15]). *Let $[a, b]$ be an order interval in a Banach space and let $B_1, B_2\colon [a, b] \to P(X)$ be two functions satisfying*

(a) *B_1 is a contraction,*
(b) *B_2 is completely continuous,*
(c) *B_1 and B_2 are strictly monotone increasing, and*
(d) *$B_1(x) + B_2(x) \in [a, b]$ for all $x \in [a, b]$.*

If the cone C in X is normal, then the inclusion $x \in B_1(x) + B_2(x)$ has at least one fixed point x_* and a greatest fixed point $x^* \in [a, b]$. Moreover, $x_* = \lim\limits_{n\to\infty} x_n$ and $x^* = \lim\limits_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are sequences in $[a, b]$ defined by

$$x_{n+1} \in B_1(x_n) + B_2(x_n),\ \ x_0 = a \quad \text{and} \quad y_{n+1} \in B_1(y_n) + B_2(y_n),\ \ y_0 = b.$$

We adopt the following definitions in the sequel.

Definition 4.4. We say that a continuous function $\tilde{v} \in D_b$ is a lower mild solution of problem (1)–(3) if $\tilde{v}(t) = \varphi(t)$, $t \in (-\infty, 0]$, and there exist $v(\cdot) \in L^1(J_k, E)$ and $\mathcal{I}_k \in I_k(\tilde{v}(t_k^-))$ such that $v(t) \in F(t, \tilde{v}_t)$ for a.e. $t \in J$, and $\tilde{v}$ satisfies

$$\begin{aligned}\tilde{v}(t) \leqslant{}& T(t)(\varphi(0) - g(0, \varphi(0))) + g(t, \tilde{v}_t)\\ &+ \int_0^t T(t-s)v(s)\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k, \quad t \in J,\ \ t \neq t_k,\end{aligned}$$

and $\tilde{v}(t_k^+) - \tilde{v}(t_k^-) \leqslant \mathcal{I}_k$, $k = 1, \ldots, m$.

Similarly an upper mild solution $\tilde{w}$ of problem (1)–(3) is defined by reversing the order.

Definition 4.5. A solution x_M of (1)–(3) is said to be maximal if for any other solution x of (1)–(3) on J we have that $x(t) \leqslant x_M(t)$ for each $t \in J$. Similarly a minimal solution of (1)–(3) is defined by reversing the order of the inequalities.

Definition 4.6. A multivalued function $F(t,x)$ is called strictly monotone increasing in x almost everywhere for $t \in J$, if $F(t,x) \leqslant F(t,y)$ for a.e. $t \in J$ for all $x, y \in X$ with $x < y$. Similarly $F(t,x)$ is called strictly monotone decreasing in x almost everywhere for $t \in J$, if $F(t,x) \geqslant F(t,y)$ for a.e. $t \in J$ for all $x, y \in X$ with $x < y$.

We consider the following assumptions in the sequel.

- (H7) The multivalued function $F(t,y)$ is strictly monotone increasing in y for almost each $t \in J$.
- (H8) The problem (1)–(3) has a lower mild solution $\tilde{v}$ and an upper mild solution $\tilde{w}$ with $\tilde{v} \leqslant \tilde{w}$.
- (H9) $T(t)$ is preserving the order, that is, $T(t)v \geqslant 0$ whenever $v \geqslant 0$.
- (H10) The functions I_k, $k = 1, \ldots, m$, are nondecreasing.

Theorem 4.2. *Assume that assumptions* (H1)–(H10) *hold. Then the problem* (1)–(3) *has minimal and maximal solutions on* D_b.

P r o o f. We can write $\tilde{v}$ and $\tilde{w}$ as

$$\tilde{v}(t) = v^*(t) + x(t), \quad \tilde{w}(t) = w^*(t) + x(t),$$

where $v^* \in D_b^0$, $w^* \in D_b^0$ and x is defined as in Section 3. Then $\tilde{v}$ is a lower solution to (1)–(3) if v^* satisfies

$$\begin{aligned} v^*(t) \leqslant{}& T(t)g(0, \varphi(0)) + g(t, v_t^* + x_t) \\ &+ \int_0^t T(t-s)v(s)\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k, \quad t \in J, \ t \neq t_k, \end{aligned}$$

and $v^*(t_k^+) - v^*(t_k^-) \leqslant \mathcal{I}_k$ with $\mathcal{I}_k \in I_k(v^*(t_k^-) + x(t_k^-))$, $k = 1, \ldots, m$.

The function $\tilde{w}$ is an upper solution to (1)–(3) if w^* satisfies the reversed inequality. It can be shown as in the proof of Theorem 3.1 that $\mathcal{A}$ is completely continuous and $\mathcal{B}$ is a contraction on $[v^*, w^*]$. We shall show that $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v^*, w^*]$. Let $z, \overline{z} \in [v^*, w^*]$ be such that $z \leqslant \overline{z}$, $z \neq \overline{z}$. Then by (H10) we have for each $t \in J$

$$\begin{aligned}\mathcal{A}(z) &= \Big\{h \in D_b^0 \colon\ h(t) = T(t)g(0,\varphi(0)) + g(t, z_t + x_t) \\ &\quad + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k,\ \mathcal{I}_k \in I_k(z(t_k^-) + x(t_k^-))\Big\} \\ &\leqslant \Big\{h \in D_b^0 \colon\ h(t) = T(t)g(0,\varphi(0)) + g(t, \overline{z}_t + x_t) \\ &\quad + \sum_{0<t_k<t} T(t-t_k)\mathcal{I}_k,\ \mathcal{I}_k \in I_k(\overline{z}(t_k^-) + x(t_k^-))\Big\} = \mathcal{A}(\overline{z}).\end{aligned}$$

Similarly, by (H7), (H9),

$$\begin{aligned}\mathcal{B}(z) &= \Big\{h \in D_b^0 \colon\ h(t) = \int_0^t T(t-s)v(s)\,\mathrm{d}s,\ v \in S_{F,z+x}\Big\} \\ &\leqslant \Big\{h \in D_b^0 \colon\ h(t) = \int_0^t T(t-s)v(s)\,\mathrm{d}s,\ f \in S_{F,\overline{z}+x}\Big\} = \mathcal{B}(\overline{z}).\end{aligned}$$

Therefore, $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v^*, w^*]$. Finally, let $y \in [v^*, w^*]$ be any element. By (H8) and (H9) we deduce that

$$v^* \leqslant \mathcal{A}(v^*) + \mathcal{B}(v^*) \leqslant \mathcal{A}(y) + \mathcal{B}(y) \leqslant \mathcal{A}(w^*) + \mathcal{B}(w^*) \leqslant w^*,$$

which shows that $\mathcal{A}(y) + \mathcal{B}(y) \in [v^*, w^*]$ for all $y \in [v^*, w^*]$. Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Theorem 4.1. Hence, problem (1)–(3) has maximal and minimal solutions on $(-\infty, b]$. This completes the proof. $\square$

5. Application

In this section we apply some of the results established in this paper. We begin by mentioning some examples of the phase space.

5.1. Phase space

Let $g\colon (-\infty, 0] \to [1, \infty)$ be a continuous, nonincreasing function with $g(0) = 1$ which satisfies the conditions (g-1), (g-2) of [27]. This means that the function

$$G(t) = \sup_{-\infty<\theta\leqslant -t} \frac{g(t+\theta)}{g(\theta)}$$

is locally bounded for $t \geqslant 0$ and that $\lim\limits_{\theta\to-\infty} g(\theta) = \infty$.

We say that $\varphi\colon (-\infty,0] \to E$ is normalized piecewise continuous, if φ is left continuous and the restriction of φ to any interval $[-r,0]$ is piecewise continuous.

Next, we modify slightly the definition of the spaces C_g, C_g^0 of [27]. We denote by $\mathcal{PC}_g((-\infty,0],E)$ the space formed by the normalized piecewise continuous functions φ such that φ/g is bounded on $(-\infty,0]$, and by $\mathcal{PC}_g^0$ the subspace of $PC_g((-\infty,0],E)$ formed by the functions φ such that

$$\lim_{\theta\to-\infty}\frac{\varphi(\theta)}{g(\theta)}=0.$$

It is easy to see that $\mathcal{D}=\mathcal{PC}_g((-\infty,0],E)$ and $\mathcal{D}=\mathcal{PC}_g^0((-\infty,0],E)$ endowed with the norm

$$\|\varphi\|_{\mathcal{D}}=\sup_{\theta\in(-\infty,0]}\frac{|\varphi(\theta)|}{g(\theta)}$$

are phase spaces. Moreover, in these cases $K(s)=1$ for $s\geqslant 0$.

Let $1\leqslant p<\infty$, $0\leqslant r<\infty$, and let $g(\cdot)$ be a Borel nonnegative measurable function on $(-\infty,r)$ which satisfies the conditions (g-5)–(g-6) in the terminology of [27]. This means that $g(\cdot)$ is locally integrable on $(-\infty,-r)$ and there exists a nonnegative and locally bounded function G on $(-\infty,0]$ such that $g(\xi+\theta)\leqslant G(\xi)g(\theta)$ for all $\xi\leqslant 0$ and $\theta\in(-\infty,-r)\setminus N_\xi$, where $N_\xi\subset(-\infty,-r)$ is a set with Lebesgue measure 0. Let $\mathcal{D}:=\mathcal{PC}_r\times L^p(g,E)$, $r\geqslant 0$, $p>1$, be the space formed by all classes of functions $\varphi\colon(-\infty,0]\to E$ such that $\varphi|_{[-r,0]}\in PC([-r,0],E)$, $\varphi(\cdot)$ is Lebesgue measurable on $(-\infty,-r]$ and $g|\varphi|^p$ is Lebesgue integrable on $(-\infty,-r]$. The seminorm in $\|\cdot\|_{\mathcal{D}}$ is defined by

$$\|\varphi\|_{\mathcal{D}}:=\sup_{\theta\in[-r,0]}|\varphi(\theta)|+\biggl(\int_{-\infty}^{-r}g(\theta)|\varphi(\theta)|^p\,\mathrm{d}\theta\biggr)^{1/p}.$$

Proceeding as in the proof of ([27], Theorem 1.3.8), we conclude that $\mathcal{D}$ is a phase space which satisfies Axioms (A) and (B). Moreover, for $r=0$ and $p=2$ this space coincides (see [27]) with $C_0\times L^2(g,E)$, $H=1$, $M(t)=G(-t)^{1/2}$ and

$$K(t)=1+\biggl(\int_{-t}^{0}g(s)\,\mathrm{d}s\biggr)^{1/2},\quad\text{for } t\geqslant 0.$$

5.2. An example

As an application of our results we consider the impulsive partial functional differential inclusion of the form

$$\begin{aligned}
(8)\qquad & \frac{\partial}{\partial t}\Big[v(t,\xi)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,\xi))\,\mathrm{d}\theta\Big] \\
&\in \frac{\partial^2}{\partial \xi^2}\Big[v(t,\xi)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,\xi))\,\mathrm{d}\theta\Big] \\
&\quad + \int_{-\infty}^{0}K_2(\theta)[Q_1(t,v(t+\theta,\xi)),Q_2(t,v(t+\theta,\xi))]\,\mathrm{d}\theta \\
&\qquad\qquad \text{for } \xi\in[0,\pi],\ t\in J\setminus\{t_1,t_2,\ldots,t_m\},
\end{aligned}$$

$$(9)\quad v(t_k^+,\xi)-v(t_k^-,\xi)\in[-b_k|v(t_k^-,\xi)|,b_k|v(t_k^-,\xi)|],\quad \xi\in[0,\pi],\ k=1,\ldots,m,$$

$$(10)\qquad v(t,0)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,0))\,\mathrm{d}\theta=0\quad\text{for } t\in J,$$

$$(11)\qquad v(t,\pi)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,\pi))\,\mathrm{d}\theta=0\quad\text{for } t\in J,$$

$$(12)\qquad v(\theta,\xi)=v_0(\theta,\xi)\quad\text{for } -\infty<\theta\leqslant 0\ \text{and}\ \xi\in[0,\pi],$$

where $b_k>0$, $k=1,\ldots,m$, $K_1\colon(-\infty,0]\to\mathbb{R}$, $K_2\colon(-\infty,0]\to\mathbb{R}$ and $g_1,g_2\colon J\times\mathbb{R}\to\mathbb{R}$ and $v_0\colon(-\infty,0]\times[0,\pi]\to\mathbb{R}$ are continuous functions, $0=t_0<t_1<t_2<\ldots<t_m<t_{m+1}=b$, $v(t_k^+,\xi)=\lim\limits_{(h,\xi)\to(0^+,\xi)}v(t_k+h,\xi)$, $v(t_k^-,\xi)=\lim\limits_{(h,\xi)\to(0^-,\xi)}v(t_k+h,\xi)$, $Q_1,Q_2\colon J\times\mathbb{R}\to\mathbb{R}$, are given functions. We assume that for each $t\in J$, $Q_1(t,\cdot)$ is lower semi-continuous (i.e, the set $\{y\in\mathbb{R}\colon Q_1(t,y)>\mu\}$ is open for each $\mu\in\mathbb{R}$), and assume that for each $t\in J$, $Q_2(t,\cdot)$ is upper semi-continuous (i.e. the set $\{y\in\mathbb{R}\colon Q_2(t,y)<\mu\}$ is open for each $\mu\in\mathbb{R}$).

Let

$$\begin{gathered}
y(t)(\xi)=v(t,\xi),\quad t\in J,\ \xi\in[0,\pi],\\
I_k(y(t_k^-))(\xi)=[-b_k|v(t_k^-,\xi)|,b_k|v(t_k^-,\xi)|],\quad \xi\in[0,\pi],\quad k=1,\ldots,m,\\
F(t,\varphi)(\xi)=\int_{-\infty}^{0}K_2(\theta)[Q_1(t,\varphi(\theta,\xi)),Q_2(t,\varphi(\theta,\xi))]\,\mathrm{d}\theta,\quad \theta\in(-\infty,0],\ \xi\in[0,\pi],\\
h(t,\varphi)(\xi)=\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,\xi))\,\mathrm{d}\theta
\end{gathered}$$

and

$$\varphi(\theta)(\xi)=\varphi(\theta,\xi),\quad \theta\in(-\infty,0],\ \xi\in[0,\pi].$$

Take $E = L^2[0, \pi]$ and define $A\colon\ D(A) \subset E \to E$ by $Aw = w''$ with the domain

$$D(A) = \{w \in E,\ w, w' \text{ are absolutely continuous, } w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A)$$

where $(\cdot, \cdot)$ is the inner product in L^2 and $w_n(s) = \sqrt{2/\pi} \sin ns$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors in A. It is well known (see [39]) that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \in (0, b]$, in E and

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in E.$$

Since the analytic semigroup $T(t)$, $t \in (0, b]$ is compact, there exists a constant $M \geqslant 1$ such that

$$\|T(t)\|_{B(E)} \leqslant M.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [17]). Assume that there are $p \in C(J, \mathbb{R}^+)$ and $\psi\colon\ [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\max(|Q_1(t, y)|, |Q_2(t, y)|) \leqslant p(t)\psi(|y|), \quad t \in J \text{ and } y \in \mathbb{R}.$$

Assume that there exist functions $\tilde{l_1}, \tilde{l_2} \in L^1(J, \mathbb{R}^+)$ such that

$$|Q_1(t, w) - Q_1(t, \overline{w})| \leqslant \tilde{l_1}(t)|w - \overline{w}|, \quad t \in J,\ w, \overline{w} \in \mathbb{R},$$

and

$$|Q_2(t, w) - Q_2(t, \overline{w})| \leqslant \tilde{l_2}(t)|w - \overline{w}|, \quad t \in J,\ w, \overline{w} \in \mathbb{R}.$$

We can show that problem (1)–(3) is an abstract formulation of problem (8)–(12). Since all conditions of Theorem 3.1 are satisfied, the problem (8)–(12) has a solution z on $(-\infty, b] \times [0, \pi]$.

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