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REMARKS ON THE UNIQUENESS OF SECOND ORDER ODEs*

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Dedicated to Professor K. R. Rajagopal on the occasion of his 60th birthday

Abstract. We are concerned with the uniqueness problem for solutions to the second order ODE of the form $x'' + f(x,t) = 0$, subject to appropriate initial conditions, under the sole assumption that f is non-decreasing with respect to x, for each t fixed. We show that there is non-uniqueness in general; on the other hand, several types of reasonable additional assumptions make the problem uniquely solvable.

The interest in this problem comes, among other, from the study of oscillations of lumped parameter systems with implicit constitutive relations.

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1. Introduction

As is well known from the basic ODE theory, the problem of uniqueness of solutions is intimately related to the continuity properties of the nonlinearities with respect to the unknown variable. Consider the *first order* equation

$$(1.1) \qquad\qquad x' + f(x,t) = 0.$$

The solution is uniquely determined by the initial condition provided f is locally Lipschitz continuous in x. This is the most commonly used sufficient criterion of uniqueness. Simple as it may seem, it cannot be pushed much further. If we assume, more generally, that

$$|f(x,t) - f(y,t)| \leqslant \omega(|x-y|),$$

then uniqueness is guaranteed provided the modulus of continuity satisfies

$$\int_0^\delta \frac{\mathrm{d}\eta}{\omega(\eta)} = +\infty;$$

and this condition is actually optimal. In particular: if f is only α-Hölder continuous, $\alpha < 1$, then there is non-uniqueness. For example, the problem

$$\text{(1.2)} \qquad\qquad x' - 3\sqrt[3]{x^2} = 0, \quad x(0) = 0,$$

is satisfied by $x \equiv 0$; but the function $x = t^3$ is another solution. Indeed, one can construct infinity of such solutions, branching at arbitrary points $t_0 > 0$.

A different line of thought aims at employing the *monotonicity* properties of f. In particular, we have

Theorem 1. *Assume that $f(x,t)$ is non-decreasing[1] in x for any t fixed. Then the solutions to (1.1) are forward unique, meaning that if two solutions x, y coincide at some t_0, then $x \equiv y$ for all $t \geqslant t_0$.*

P r o o f. Subtract the equations for x, y and multiply by $2(x-y)$, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}|x-y|^2 \leqslant 0.$$

The conclusion follows immediately. □

Let us remark that the theorem requires no other properties (continuity or integrability) of f. The fact that we only deduce forward uniqueness makes no harm, and is actually in agreement with physical applications. In fact, we cannot expect backward uniqueness by the example (1.2) above. Equivalently, there is certainly non-uniqueness should one require that f is non-increasing in x.

It is of some interest to ask whether a conclusion analogous to Theorem 1 holds also for the *second order* equation

$$\text{(1.3)} \qquad\qquad x'' + f(x,t) = 0.$$

More precisely, we ask whether forward uniqueness is guaranteed for (1.3) provided f is non-decreasing in x, for any t fixed. This is the main issue to be addressed in the present paper. We will see that the answer is in the negative for the general non-autonomous problem. The affirmative answer can be given either for the autonomous problem, or if f satisfies additional structural assumptions.

The paper is organized as follows. In Section 2, we discuss the physical motivation to our problem. Section 3 is devoted to the case when (1.3) is linear. Section 4 studies the autonomous problem. Finally, Section 5 is devoted to the general case.

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[1] In vectorial case, this amounts to $\langle f(x,t) - f(y,t), x-y \rangle \geqslant 0$, for any t fixed.

R e m a r k. Throughout the paper, a *solution* means an absolutely continuous function satisfying the equation almost everywhere. In the case of a second order problem, we require that the first derivative is absolutely continuous, too.

2. Physical motivation

One of the basic motivations for the study of second order ODEs comes from oscillations of lumped parameter systems. Consider a mass m, attached to a spring and a (parallel) dashpot, subject to an external force $F(t)$. The corresponding equation reads

$$(2.1) \qquad mx'' + F_\mathrm{d} + F_\mathrm{s} = F(t),$$
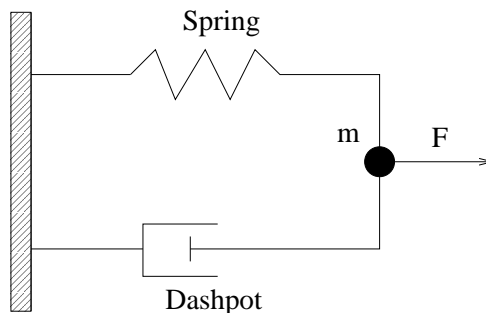
where x is the unknown displacement, and F_d, F_s are the forces in the dashpot and the spring, respectively. It is customary to close the system by constitutive relations of the form

$$(2.2) \qquad F_\mathrm{d} = f(x'),$$
$$F_\mathrm{s} = g(x),$$

which leads to a second order ODE for x. If the functions f, g are regular enough, one obtains a well-posed problem. It might be of interest, however, to reverse the relations and write

$$(2.3) \qquad x' = \hat{f}(F_\mathrm{d}),$$
$$x = \hat{g}(F_\mathrm{s}).$$

Note that this is indeed very sensible philosophically: kinematical quantities (velocity, displacement) are a consequence, hence a function of the forces. One can think of still more general constitutive relations

$$\tilde{f}(x', F_{\mathrm{d}}) = 0,$$

(2.4)

$$\tilde{g}(x, F_{\mathrm{s}}) = 0.$$

There are actually meaningful examples of materials that can only be modeled by general implicit relations like (2.4), see [4]. The most prominent example is the Coulomb friction, see for example [2].

The general problem (2.1), (2.4) is a differential-algebraic system of three equations and three unknowns x, F_{d}, F_{s}. Surely enough, the first step towards its analysis is to ask whether, at least locally or in certain regimes, one can replace (2.4) by (2.2). While this is often possible, it would lead to nonlinearities that are not sufficiently regular to guarantee uniqueness. If, for example, one has

$$x = F_{\mathrm{s}}^3,$$

it is indeed equivalent to

$$F_{\mathrm{s}} = \sqrt[3]{x};$$

but we have seen above that Hölder continuity is not enough for uniqueness.

We emphasize that the problem of uniqueness is very important in particular in such situations when the problem is attacked by an ad-hoc method of "patching up" solutions obtained by solving the equation in different regimes. On the other hand, we see that the monotonicity type of the relation is naturally preserved.

These are the problems that motivate the analysis given in the present paper. More specifically, we neglect the influence of damping here, since the relation between F_{d} and x'' is essentially that of the first order equations, where the situation is much easier to understand (see also Introduction). Thus, we are naturally lead to the problem

$$mx'' + h(x) = F(t),$$

where h lacks sufficient continuity properties (not being Lipschitz in particular), but is a non-decreasing function. We will come back to this equation in Section 5.

3. Linear problem

This section is devoted to the analysis of the linear variant of problem (1.3), that is

$$(3.1) \qquad x'' + Q(t)x = 0.$$

The assumption that f is non-decreasing in x is tantamount to

$$(3.2) \qquad Q(t) \geqslant 0,$$

while the lack of continuity of f with respect to x is reflected in the fact that in general, we do not assume that $Q(t)$ is integrable in time.

In view of linearity, the problem is uniquely solvable for arbitrary initial condition, if and only if $x \equiv 0$ is the only (forward) solution subject to the initial condition

$$(3.3) \qquad x(0) = x'(0) = 0.$$

We will first show that (3.2) is not enough to guarantee forward uniqueness of solutions. Consider a function u_1, defined on $[-9\pi/4, 0]$ as follows:

$$u_1(t) = \begin{cases} \sin t, & t \in \left[-\frac{1}{2}\pi, 0\right], \\ \sin \frac{1}{2}\left(t - \frac{1}{2}\pi\right), & t \in \left[-\frac{3}{2}\pi, -\frac{1}{2}\pi\right], \\ \frac{1}{4}\sin 2(t + \pi), & t \in \left[-\frac{7}{4}\pi, -\frac{3}{2}\pi\right], \\ \frac{1}{4}\sin\left(t + \frac{1}{4}\pi\right), & t \in \left[-\frac{9}{4}\pi, -\frac{7}{4}\pi\right]. \end{cases}$$

Observe that u_1, u_1' are continuous, $u_1'(0-) = 1$, $u_1'(-9\pi/4+) = 1/4$, and (3.1) holds with $Q(t)$ equal to 1, 1/2, 2 and 1 on the respective intervals. We will now repeat the same pattern, shrinking by factor 2 in horizontal and by factor 8 in vertical direction. We set

$$l = \frac{9}{4}\pi,$$
$$t_0 = 0,$$
$$t_{k+1} = t_k - \frac{l}{2^k},$$
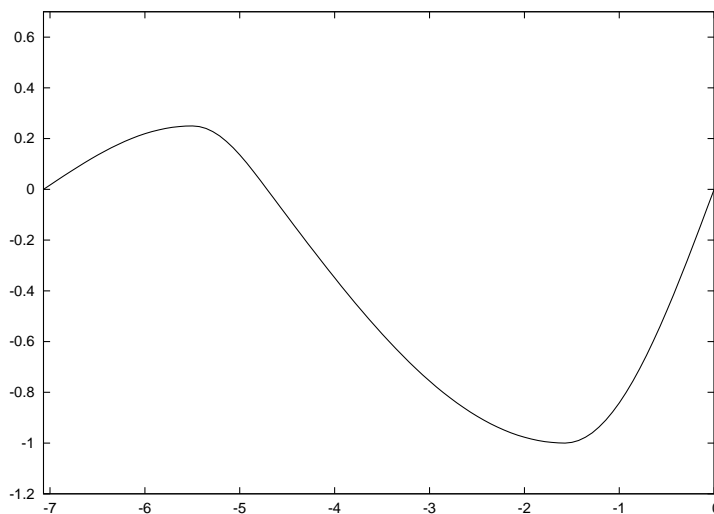$$I_k = [t_{k+1}, t_k],$$

Figure 1. Function u_1.

and

$$u(t) = \frac{1}{8^k} u_1(2^k(t + t_k)), \quad t \in I_k.$$

Observe that by moving from I_k to I_{k+1}, the amplitude decreases by the factor 8, while the derivative is 4times smaller (in particular, it is smoothly joined at the point t_{k+1}). It follows that $Q(t)$ (the ratio of u and u'') is 4times larger.

One deduces easily that

$$t_\infty = \lim_{k \to \infty} t_k = -\frac{9}{2}\pi,$$
$$u(t_\infty+) = u'(t_\infty+) = 0.$$

Hence, we have constructed a nontrivial solution to (3.1), subject to zero initial conditions at t_∞.

The function $Q(t)$ in the above example is not integrable on $[t_\infty, t_\infty + \delta)$, for otherwise the uniqueness would follow by standard theory. On the other hand, there exist a non-integrable majorant for $Q(t)$ that still guarantees the (forward) unique solvability of (3.1).

Theorem 2. *Assume that*

$$0 \leqslant Q(t) \leqslant \frac{1}{4t^2}, \quad t \in [0, \delta].$$

Then $x \equiv 0$ is the only solution to (3.1) on $[0, \delta]$ satisfying the zero initial condition (3.3).

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P r o o f.   We proceed by contradiction. Let x be a nontrivial solution on $[0, \delta]$. Our claim is that x has infinity of zeroes.

Indeed, let $x(t_0) > 0$, $t_0 \in (0, \delta]$. Set

$$t_1 = \inf\{\tau \in [0, t_0]: \ x > 0 \text{ on } [\tau, t_0]\}.$$

By continuity, we have $x(t_1) \geqslant 0$, and $x(t_1) > 0$ is only possible if $t_1 = 0$, which, however, contradicts (3.3). Thus $x(t_1) = 0$ and $x \geqslant 0$ on $[t_1, t_0]$, hence by (3.1), x is concave on this interval, from which we deduce that $x'(t_1) > 0$. Consequently, $t_1 > 0$, and x is negative on its left neighborhood. A symmetric argument yields the existence of $t_2 \in (0, t_1)$ such that $x(t_2) = 0$, $x'(t_2) < 0$. We proceed by induction to find an infinite number of zeroes in an arbitrary right neighborhood of 0.

By the Sturm separation theorem (see [1, XI, Corollary 3.1]), any nontrivial solution to

$$u'' + \frac{1}{4t^2} u = 0$$

has infinity of zeroes in $(0, \delta)$. Yet, this is an Euler equation, with the general solution in the form

$$u = \sqrt{t}(a + b \ln t),$$

a contradiction. □

Coming back to the counterexample above, one notices that $Q(t)$ is non-monotone. We can prove that this is indeed necessary for the non-uniqueness to occur.

Theorem 3. *Let $Q(t) \geqslant 0$ be monotone on $(0, \delta)$. Then $x \equiv 0$ is the only solution to (3.1), (3.3) on $[0, \delta]$.*

P r o o f.   If Q is non-decreasing, it is bounded and hence integrable on $(0, \delta')$ for any $\delta' < \delta$. The conclusion follows by standard theory.

Assume that Q is non-increasing. Multiplying (3.1) by $2x'$, we have

(3.4) $$\frac{\mathrm{d}}{\mathrm{d}t}(x')^2 + Q(t)\frac{\mathrm{d}}{\mathrm{d}t}x^2 = 0.$$

Integration by parts gives

$$\int_{t_1}^{t_0} Q(t)\frac{\mathrm{d}}{\mathrm{d}t}x^2(t)\,\mathrm{d}t = [Q(t)x^2(t)]_{t=t_1}^{t=t_0} - \int_{t_1}^{t_0} x^2(t)\,\mathrm{d}Q(t).$$

The integral on the right is taken in the sense of Lebesgue-Stieltjes. Note that it is non-positive ($t_1 < t_0$ here) by our assumptions.

Now, if x is a non-trivial solution, we have seen in the proof of Theorem 2 that it has an infinity of zeroes $t_k \to 0+$ such that $x'(t_k) \neq 0$. Integrating (3.4) between $t_{k+1} < t_k$ gives

$$(x')^2(t_k) - (x')^2(t_{k+1}) \leqslant 0,$$

which contradicts the fact that $x'(t_k) \to x'(0) = 0$ for $k \to \infty$. $\qquad\square$

4. AUTONOMOUS PROBLEM

This section is devoted to the (forward) uniqueness of the problem

(4.1)
$$x'' + h(x) = 0,$$
$$x(0) = x_0, \quad x'(0) = x_1,$$

where h is non-decreasing.

Theorem 4. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous, non-decreasing function. Then the solutions of (4.1) are forward unique.*

P r o o f.   We want to show that for an arbitrary initial condition, there is at most one solution on $[0, \delta]$ with some $\delta > 0$. The key step is to employ the first integral:

$$(x'(t))^2 + H(x(t)) = C, \quad t \geqslant 0,$$

where $H = 2 \int h$ and $C = x_1^2 + H(x_0)$. If $x_1 \neq 0$, then

$$(x'(t))^2 = C - H(x(t)) > 0 \quad \text{on } [0, \delta]$$

with a suitable $\delta > 0$; hence

$$x' = \sqrt{C - H(x)}$$

or

$$x' = -\sqrt{C - H(x)},$$

depending on the sign of x_1. These equations determine the solution uniquely, since the right-hand sides are Lipschitz continuous with respect to x.

If $x_1 = h(x_0) = 0$, then

$$(x'(t))^2 + H(x(t)) = H(x_0), \quad t \geqslant 0.$$

Yet $h(x_0) = 0$ together with monotonicity of h implies that H has global minimum at x_0, hence

$$(x'(t))^2 \leqslant 0, \quad t \geqslant 0,$$

and thus $x \equiv x_0$ is the unique forward solution. It remains to treat the case

$$x_1 = 0, \quad h(x_0) \neq 0.$$

Assume on the contrary that we have two distinct solutions x, y with the same initial condition; moreover, for the sake of definiteness, let $h(x_0) < 0$. It follows that $x''(0) = y''(0) > 0$, hence x', $y' > 0$ on $(0, \delta)$. Now, $z = x - y$ satisfies

$$z'' + Q(t)z = 0,$$

where

$$(4.2) \qquad Q(t) = \begin{cases} 0, & x(t) = y(t), \\ \dfrac{h(x(t)) - h(y(t))}{x(t) - y(t)}, & x(t) \neq y(t). \end{cases}$$

Note that $Q(t) \geqslant 0$, and the zero initial conditions

$$z(0) = z'(0) = 0$$

hold. By the argument of Theorem 2, if z is not identically zero, it has infinitely many zeroes. Taking into account also the fact that x, y are strictly increasing and concave on $[0, \delta]$, a simple geometrical argument shows the existence of τ such that $x(t)$ and $y(t - \tau)$ have the same function value and the same derivative at some $\tilde{t}$ between the consecutive zeroes of z.

Since the problem is autonomous, we have two distinct solutions with the same initial condition, of which the derivative part is not zero. This contradicts the uniqueness, proved under these circumstances above. $\qquad \square$

Remark. It is not necessary to assume continuity of h in the above theorem. Indeed, a much more general case can be treated along similar lines; see [3, Theorem 2].

5. Non-autonomous undamped oscillator

The last section is devoted to the analysis of the problem

$$(5.1) \qquad\qquad\qquad x'' + h(x) = g(t),$$

subject to the initial condition

$$x(t_0) = x_0, \quad x'(t_0) = x_1.$$

The standing assumption is that h is a continuous, non-decreasing function, while we want to address the problem of (forward) uniqueness of solutions. We remark that the uniqueness follows by the standard theory, provided h is Lipschitz on some neighborhood of x_0. To deliberately exclude such a situation, we will impose additional structural assumptions. Observe first that, by means of a simple transformation, we can have

$$(5.2) \qquad\qquad 0 = t_0 = x_0 = h(x_0).$$

Hence, we restrict ourselves to the initial condition

$$(5.3) \qquad\qquad x(0) = 0, \quad x'(0) = x_1,$$

keeping in mind that x_1 and $g(0)$ can be non-zero. Now, we will additionally require that

$$(5.4) \qquad\qquad h'(0) = \infty,$$
$$h \text{ is concave on } [0, \eta) \text{ and convex on } (-\eta, 0].$$

Once again, it will be useful to note that if x, y solve (5.1) with the same initial condition, then $z = x - y$ solves the *linear* problem

$$(5.5) \qquad\qquad z'' + Q(t)z = 0,$$
$$z(0) = z'(0) = 0,$$

where $Q(t) \geqslant 0$ is given by (4.2). We can thus employ the results from Section 3.

Theorem 5. *Assume that (5.4) holds. Let any solution to (5.1), (5.3) be non-decreasing (or non-increasing) on $[0, \delta]$. Then there is at most one solution on $[0, \delta]$.*

P r o o f. Let x, y be two solutions. By our assumption, they have the same monotonicity type on some $[0, \delta]$. More precisely, both the functions either increase from 0 or decrease to 0 for $t \searrow 0$.

By (5.4), h' exists and is monotone on $(-\eta, 0)$ and on $(0, \eta)$. In either case,

$$Q(t) = \int_0^1 h'\big(\sigma x(t) + (1 - \sigma)y(t)\big) \, d\sigma$$

is non-increasing on $[0, \delta]$. By Theorem 3, this entails that $x \equiv y$. $\qquad\square$

Corollary 1. *Let $x_1 \neq 0$ or $g(0) \neq 0$. Then the problem (5.1), (5.4) is uniquely forward solvable.*

P r o o f.   These assumptions guarantee, in view of (5.2), that any solution has either $x'(0) \neq 0$, or $x'(0) = 0$ and $x''(0) \neq 0$. This implies a definite sign of x' at some $[0, \delta]$, and we conclude by Theorem 5. □

Still another approach to ensuring monotonicity is based on additional growth estimates of h and g.

Corollary 2. *Let*

(5.6)
$$|h(x)| \leqslant c|x|^a,$$
$$g(t) \sim t^b, \quad t \to 0+,$$

where $a \in (0, 1)$ and

(5.7)
$$0 < b < \frac{2a}{1-a}.$$

Then any solution to (5.1) subject to zero initial condition $x(0) = x'(0) = 0$ is increasing on $[0, \delta]$. Hence, the problem is uniquely solvable.

P r o o f.   It follows from the equation that $x''(0) = 0$; hence

$$x''(t) = o(1),$$
$$x(t) = o(t^2)$$

for $t \to 0+$. Substituting again, we have

$$x''(t) = g(t) + o(t^{2a}).$$

If $b \leqslant 2a$, we see that $x'' > 0$ on some $[0, \delta]$, hence x is increasing and uniqueness follows by Theorem 5.

If $b > 2a$, we improve the estimates on $x(t)$ iteratively. Set $\gamma_0 = 2$. Assume we have

$$x(t) = o(t^{\gamma_k}),$$
$$x''(t) = g(t) + o(t^{a\gamma_k}).$$

If, moreover,

(5.8)
$$b > a\gamma_k,$$

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we deduce

$$x''(t) = o(t^{a\gamma_k}),$$
$$x(t) = o(t^{\gamma_{k+1}}), \quad \gamma_{k+1} = a\gamma_k + 2.$$

Observe that $\gamma_k \nearrow 2/(1-a)$ as long as the induction can be continued. However, by (5.7), after a finite number of steps (5.8) will not hold anymore. This implies that $x'' > 0$ on $[0, \delta]$ and we have monotonicity (and hence uniqueness) as above. $\qquad \square$

Concluding remarks. The results of this section prove (forward) uniqueness of (5.1) in a number of situations; yet the general problem still remains open. More precisely, it is not clear to us if the sole assumption that h is non-decreasing is enough.

Should there exist counterexamples, then necessarily one would need a zero initial condition, and very slowly increasing (possibly oscillating) right-hand side. Another possibility would be to drop the convexity/concavity assumption (5.4).

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