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ELASTOPLASTIC REACTION OF A CONTAINER TO WATER FREEZING

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Abstract. The paper deals with a model for water freezing in a deformable elastoplastic container. The mathematical problem consists of a system of one parabolic equation for temperature, one integrodifferential equation with a hysteresis operator for local volume increment, and one differential inclusion for the water content. The problem is shown to admit a unique global uniformly bounded weak solution.

Keywords: phase transition, water, ice, energy, entropy, elastoplastic boundary

MSC 2010: 80A22, 35K85, 47J40

1. Introduction

Phase transition problems are quite popular in mathematical literature, cf. e.g. the books [1], [2], [11]. Only few publications, however, take into account different mass densities/specific volumes of the phases. In [3], the authors proposed to interpret a phase transition process in terms of a balance equation for macroscopic motions, and to include the possibility of voids. Well-posedness of an initial-boundary value problem for the resulting PDE system is proved there, and the case of two different densities for the substances undergoing phase transitions has been pursued in [4].

In [8] and [9], a model has been proposed to explain the occurrence of high stresses due to the difference between the specific volumes of the solid and liquid phases, assuming that the speed of sound and the specific heat are the same in the solid and in the liquid. The results there include the existence and uniqueness of global solutions, as well as their convergence to equilibria in the cases that the container is elastic or rigid, with or without gravity. In reality, the specific heat in water is about

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the double, while the speed of sound in water is less than one half of the one in ice. This leads to new mathematical and modeling difficulties that are discussed in [10].

Here, the model of [8] is extended to the case of elastoplastic boundary. This results in the occurrence of hysteresis operators in the mechanical equilibrium equation, as well as in the heat source term due to plastic dissipation on the boundary. We prove the existence, uniqueness and global boundedness of the solution. Due to strong memory effects, the question of convergence to equilibria is much more challenging here than in [8], [9], and we leave it open.

The remaining text has two parts. In Section 2, we briefly describe the model, and in Section 3, we state and prove the main existence and uniqueness Theorem 3.1.

2. The model

As the reference state, we consider a bounded connected container $\Omega \subset \mathbb{R}^3$ with Lipschitzian boundary, filled with water. For modeling details, see [8], [9]. Here, we only recall that the state variables are the absolute temperature $\theta > 0$, the displacement $\mathbf{u} \in \mathbb{R}^3$, and the phase variable $\chi \in [0,1]$. The value $\chi = 0$ means a solid, $\chi = 1$ means a liquid, $\chi \in (0,1)$ is a mixture of the two. We define the strain $\boldsymbol{\varepsilon}$ as an element of the space $\mathbb{T}^{3\times 3}_{\mathrm{sym}}$ of symmetric tensors by the formula

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \tag{2.1}$$

Let $\boldsymbol{\delta} \in \mathbb{T}^{3\times 3}_{\mathrm{sym}}$ denote the Kronecker tensor. We consider the specific free energy f in the form

$$\begin{aligned} f = {} & c_0\theta\Big(1 - \log\Big(\frac{\theta}{\theta_c}\Big)\Big) + \frac{\lambda}{2\varrho_0}((\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}(\chi)) : \boldsymbol{\delta})^2 - \frac{\beta}{\varrho_0}(\theta - \theta_c)\boldsymbol{\varepsilon} : \boldsymbol{\delta} \\ & + L_0\Big(\chi\Big(1 - \frac{\theta}{\theta_c}\Big) + I(\chi)\Big), \end{aligned} \tag{2.2}$$

where I is the indicator function of the interval $[0,1]$, and

$$\tilde{\boldsymbol{\varepsilon}}(\chi) = \frac{\alpha}{3}(1 - \chi)\boldsymbol{\delta} \tag{2.3}$$

is the strain component due to phase transition. The process is described in Lagrangian coordinates, hence the mass density ϱ_0 is constant. The coefficients α (relative specific volume increment), c_0 (specific heat capacity), λ (bulk elasticity modulus), β (thermal expansion coefficient), L_0 (latent heat) are assumed constant and positive, and $\theta_c > 0$ is the melting temperature at standard pressure.

The stress tensor $\boldsymbol{\sigma}$ is decomposed into the sum $\boldsymbol{\sigma}^v + \boldsymbol{\sigma}^e$ of the viscous component $\boldsymbol{\sigma}^v$ and the elastic component $\boldsymbol{\sigma}^e$. The state functions $\boldsymbol{\sigma}^v$, $\boldsymbol{\sigma}^e$, s (specific entropy), and e (specific internal energy) are given by the formulas

$$\boldsymbol{\sigma}^v = \nu(\boldsymbol{\varepsilon}_t : \boldsymbol{\delta})\boldsymbol{\delta}, \tag{2.4}$$

$$\boldsymbol{\sigma}^e = \varrho_0 \frac{\partial f}{\partial \boldsymbol{\varepsilon}} = (\lambda(\boldsymbol{\varepsilon} : \boldsymbol{\delta} - \alpha(1-\chi)) - \beta(\theta - \theta_c))\,\boldsymbol{\delta}, \tag{2.5}$$

$$s = -\frac{\partial f}{\partial \theta} = c_0 \log\left(\frac{\theta}{\theta_c}\right) + \frac{L_0}{\theta_c}\chi + \frac{\beta}{\varrho_0}\boldsymbol{\varepsilon} : \boldsymbol{\delta}, \tag{2.6}$$

$$e = f + \theta s = c_0\theta + \frac{\lambda}{2\varrho_0}(\boldsymbol{\varepsilon} : \boldsymbol{\delta} - \alpha(1-\chi))^2 + \frac{\beta}{\varrho_0}\theta_c \boldsymbol{\varepsilon} : \boldsymbol{\delta} + L_0(\chi + I(\chi)), \tag{2.7}$$

where $\nu > 0$ is the volume viscosity coefficient. The scalar quantity

$$p := -\nu\boldsymbol{\varepsilon}_t : \boldsymbol{\delta} - \lambda(\boldsymbol{\varepsilon} : \boldsymbol{\delta} - \alpha(1-\chi)) + \beta(\theta - \theta_c) \tag{2.8}$$

is the *pressure*, and the stress has the form $\boldsymbol{\sigma} = -p\boldsymbol{\delta}$. The process is governed by the balance equations

$$\operatorname{div} \boldsymbol{\sigma} = 0 \qquad \text{(mechanical equilibrium)}, \tag{2.9}$$

$$\varrho_0 e_t + \operatorname{div} \mathbf{q} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_t \qquad \text{(energy balance)}, \tag{2.10}$$

$$-\gamma_0 \chi_t \in \partial_\chi f \qquad \text{(phase relaxation law)}, \tag{2.11}$$

where ∂_χ is the partial subdifferential with respect to χ, and $\mathbf{q}$ is the heat flux vector that we assume in the form

$$\mathbf{q} = -\kappa \nabla\theta \tag{2.12}$$

with a constant heat conductivity $\kappa > 0$. The equilibrium equation (2.9) can be rewritten in the form $\nabla p = 0$, hence

$$p(x,t) = P(t) \tag{2.13}$$

with a function P of time only, which is to be determined. Recall that in the reference state $\boldsymbol{\varepsilon} : \boldsymbol{\delta} = \boldsymbol{\varepsilon}_t : \boldsymbol{\delta} = 0$, $\chi = 1$, and at the standard pressure P_{stand}, the freezing temperature is θ_c. We thus see from (2.8) that $P(t)$ is in fact the deviation from the standard pressure. We assume also the external pressure in the form $P_{\text{ext}} = P_{\text{stand}} + p_0$ with a constant deviation p_0. The normal force acting on the boundary is $(P(t) - p_0)\mathbf{n}$, where $\mathbf{n}$ denotes the unit outward normal vector.

The response of the boundary $\partial\Omega$ to pressure changes is assumed to be elastoplastic according to the Prager hardening model represented in Figure 1. We use the operator formalism introduced in [5], cf. also [1], [7], [11], and decompose the normal displacement $\mathbf{u} \cdot \mathbf{n}$ into the sum $\mathbf{u} \cdot \mathbf{n} = u^e + u^p$ of an elastic component u^e and a plastic component u^p. The pressure difference $P_0(t) = P(t) - p_0$ is also decomposed into the sum $P_0(t) = p^h(x,t) + p^b(x,t)$ of a kinematic hardening component p^h and a backstress p^b. We further assume that the heat transfer through the boundary is proportional to the inner and outer temperature difference. The boundary conditions on $\partial\Omega$ for $\mathbf{u}$ and θ then read

$$P_0(t) = k(x)u^e(x,t), \tag{2.14}$$

$$p^h(x,t) = b(x)u^p(x,t), \tag{2.15}$$

$$|p^b(x,t)| \leqslant r(x) \text{ a.e.}, \tag{2.16}$$

$$\frac{\partial u^p}{\partial t}(p^b(x,t) - y) \geqslant 0 \quad \text{a.e.} \quad \forall y \in [-r(x), r(x)], \tag{2.17}$$

$$\mathbf{q} \cdot \mathbf{n} = h(x)(\theta - \theta_\Gamma) - p^b \frac{\partial u^p}{\partial t}, \tag{2.18}$$

with given positive measurable functions k (elasticity of the boundary), b (hardening coefficient), r (yield stress), h (heat transfer coefficient), and a constant $\theta_\Gamma > 0$ (external temperature). The term $p^b \partial u^p/\partial t = r(x)|\partial u^p/\partial t|$ is the (nonnegative) plastic dissipation rate as a boundary heat source in the energy balance.

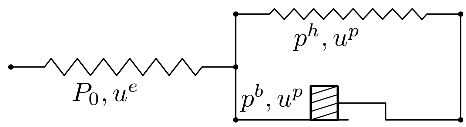

Figure 1. A rheological model for the boundary behavior

We rewrite (2.17) as

$$b(x)\frac{\partial u^p(x,t)}{\partial t}(P_0(t) - b(x)u^p(x,t) - y) \geqslant 0 \quad \text{a.e.} \quad \forall y \in [-r(x), r(x)], \tag{2.19}$$

which is precisely the variational inequality which defines the so-called *play operator*

$$b(x)u^p(x,t) = p^h(x,t) = \mathfrak{p}_{r(x)}[P_0](t) \tag{2.20}$$

with threshold $r(x)$, provided we choose the initial condition

$$b(x)u^p(x,0) = \min\{P_0(0) + r(x), \max\{0, P_0(0) - r(x)\}\} \tag{2.21}$$

corresponding to the initially undeformed state, see Figure 2.

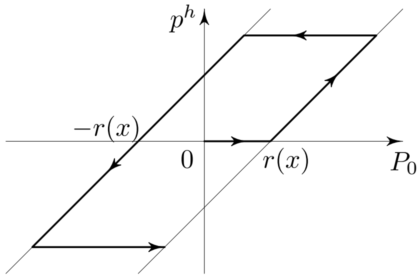

Figure 2. A diagram of the play operator $P_0 \mapsto p^h = \mathfrak{p}_{r(x)}[P_0]$ with threshold $r(x)$

This enables us to find an explicit relation between $\mathbf{u}$ and P. From (2.14) and (2.20) it follows that

$$\mathbf{u}\cdot\mathbf{n} \;=\; \frac{1}{k(x)}P_0(t) + \frac{1}{b(x)}\mathfrak{p}_{r(x)}[P_0](t). \tag{2.22}$$

Assuming that $1/k$ and $1/b$ belong to $L^1(\partial\Omega)$, we set

$$\frac{1}{K_\Gamma} = \int_{\partial\Omega} \frac{1}{k(x)}\,\mathrm{d}S(x), \tag{2.23}$$

and obtain by Gauss' Theorem that

$$\begin{aligned} U_\Omega(t) := \int_\Omega \operatorname{div}\mathbf{u}(x,t)\,\mathrm{d}x = \mathcal{F}[P_0](t) := \frac{1}{K_\Gamma}P_0(t) \\ + \int_{\partial\Omega} \frac{1}{b(x)}\mathfrak{p}_{r(x)}[P_0](t)\,\mathrm{d}S(x). \end{aligned} \tag{2.24}$$

Under the small strain hypothesis, the function $\operatorname{div}\mathbf{u}$ describes the local relative volume increment. Hence, Eq. (2.24) establishes a hysteresis relation between the relative pressure $P_0(t)$ and the total relative volume increment $U_\Omega(t)$. The mapping $\mathcal{F}$ defined in (2.24) is a *Prandtl-Ishlinskii operator.* If the function $r(x)$ is nonconstant, plastic yielding occurs at different pressures at different points of the boundary, which produces a global multiyield character of the model resulting in a smooth hysteresis diagram as in Figure 3.

Some analytical properties of Prandtl-Ishlinskii operators are listed below in Subsection 3.1. Here, we just point out that both $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are Lipschitz continuous in the space $C[0,T]$ of continuous functions as well as in the space $W^{1,1}(0,T)$ of absolutely continuous functions from $[0,T]$ to $\mathbb{R}$. The hysteresis loops of $\mathcal{F}$ are oriented counterclockwise, the loops of $\mathcal{F}^{-1}$ are oriented clockwise. Figure 3 illustrates the situation when the pressure difference P_0 increases from zero to some maximal value and then decreases to zero again (the thick part of the diagram). We see that a remanent deformation U^* persists even if the inner and outer pressure are in equilibrium.

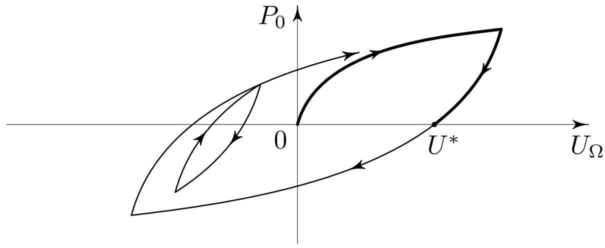

Figure 3. A diagram of the inverse Prandtl-Ishlinskii operator $\mathcal{F}^{-1}$

We have $\boldsymbol{\varepsilon} : \boldsymbol{\delta} = \operatorname{div} \mathbf{u}$, and thus the mechanical equilibrium equation (2.13), due to (2.8) and (2.24), reads

$$\nu \operatorname{div} \mathbf{u}_t + \lambda(\operatorname{div} \mathbf{u} - \alpha(1-\chi)) - \beta(\theta - \theta_c) = -p_0 - \mathcal{F}^{-1}[U_\Omega]. \tag{2.25}$$

As a consequence of (2.2), the energy balance and the phase relaxation equation in (2.10)–(2.11) have the form

$$\varrho_0 c_0 \theta_t - \kappa \Delta \theta = \nu (\operatorname{div} \mathbf{u}_t)^2 - \beta \theta \operatorname{div} \mathbf{u}_t - \big(\alpha\lambda(\operatorname{div} \mathbf{u} - \alpha(1-\chi)) + \varrho_0 L_0\big)\chi_t, \tag{2.26}$$

$$-\varrho_0 \gamma_0 \chi_t \in \alpha\lambda(\operatorname{div} \mathbf{u} - \alpha(1-\chi)) + \varrho_0 L_0 \Big(1 - \frac{\theta}{\theta_c} + \partial I(\chi)\Big), \tag{2.27}$$

where ∂ denotes the subdifferential. For simplicity, we now set

$$c := \varrho_0 c_0, \quad \gamma := \varrho_0 \gamma_0, \quad L := \varrho_0 L_0. \tag{2.28}$$

For the unknown absolute temperature θ, local relative volume increment $U = \operatorname{div} \mathbf{u}$, and liquid fraction χ, we have the evolution system (note that mathematically, $\partial I(\chi)$ is the same as $L\partial I(\chi)$)

$$c\theta_t - \kappa\Delta\theta = \nu U_t^2 - \beta\theta U_t - \big(\alpha\lambda(U - \alpha(1-\chi)) + L\big)\chi_t, \tag{2.29}$$

$$\nu U_t + \lambda U = \alpha\lambda(1-\chi) + \beta(\theta - \theta_c) - p_0 - \mathcal{F}^{-1}[U_\Omega], \tag{2.30}$$

$$-\gamma\chi_t \in \alpha\lambda(U - \alpha(1-\chi)) + L\Big(1 - \frac{\theta}{\theta_c}\Big) + \partial I(\chi), \tag{2.31}$$

with the boundary condition (2.18), (2.12), that is,

$$\kappa \nabla\theta \cdot \mathbf{n} + h(x)(\theta - \theta_\Gamma) = \frac{r(x)}{b(x)} \big|\mathfrak{p}_{r(x)}[P_0]_t\big|. \tag{2.32}$$

In terms of the new variables, the energy e and the entropy s can be written as

$$e = c_0\theta + \frac{\lambda}{2\varrho_0}(U - \alpha(1-\chi))^2 + \frac{\beta}{\varrho_0}\theta_c U + L_0(\chi + I(\chi)), \tag{2.33}$$

$$s = c_0 \log\Big(\frac{\theta}{\theta_c}\Big) + \frac{L_0}{\theta_c}\chi + \frac{\beta}{\varrho_0}U. \tag{2.34}$$

The boundary energy term has the form

$$E_\Gamma(t) \;=\; \frac{1}{2K_\Gamma}P_0^2(t) + \frac{1}{2}\int_{\partial\Omega}\frac{1}{b(x)}\mathfrak{p}_{r(x)}^2[P_0](t)\,\mathrm{d}S(x) + p_0\mathcal{F}[P_0](t) + C_\Gamma \tag{2.35}$$

with $P_0 = \mathcal{F}^{-1}[U_\Omega]$, and with a constant C_Γ which ensures that $E_\Gamma(t) \geqslant 0$. We may take for example

$$C_\Gamma = \frac{p_0^2}{2}\bigg(\frac{1}{K_\Gamma} + \int_{\partial\Omega}\frac{1}{b(x)}\,\mathrm{d}S(x)\bigg). \tag{2.36}$$

The energy and entropy balance equations then read

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega \varrho_0 e(x,t)\,\mathrm{d}x + E_\Gamma(t)\bigg) = \int_{\partial\Omega} h(x)(\theta_\Gamma - \theta)\,\mathrm{d}S(x), \tag{2.37}$$

$$\varrho_0 s_t + \operatorname{div}\frac{\mathbf{q}}{\theta} = \frac{\kappa|\nabla\theta|^2}{\theta^2} + \frac{\gamma}{\theta}\chi_t^2 + \frac{\nu}{\theta}U_t^2 \geqslant 0, \tag{2.38}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_0 s(x,t)\,\mathrm{d}x &= \int_\Omega \Big(\frac{\kappa|\nabla\theta|^2}{\theta^2} + \frac{\gamma}{\theta}\chi_t^2 + \frac{\nu}{\theta}U_t^2\Big)\,\mathrm{d}x \\ &+ \int_{\partial\Omega}\Big(\frac{h(x)}{\theta}(\theta_\Gamma - \theta) + \frac{r(x)}{\theta b(x)}\big|\mathfrak{p}_{r(x)}[P_0]_t\big|(t)\Big)\,\mathrm{d}S(x).\end{aligned} \tag{2.39}$$

The entropy balance (2.38) says that the entropy production on the right hand side is nonnegative in agreement with the second principle of thermodynamics. Also the plastic dissipation produces a positive contribution to the entropy in (2.39). The system is not closed, and the energy supply through the boundary is given by the right hand side of (2.37).

3. Existence and uniqueness of solutions

We prescribe the initial conditions

$$\theta(x,0) = \theta^0(x), \tag{3.1}$$

$$U(x,0) = U^0(x), \tag{3.2}$$

$$\chi(x,0) = \chi^0(x) \tag{3.3}$$

for $x \in \Omega$, and construct the solution of (2.30)–(2.31) by a fixed point argument. The method of proof is independent of the actual values of the material constants, and we choose for simplicity

$$L = 2, \quad c = \theta_c = \alpha = \beta = \gamma = \kappa = \lambda = \nu = 1. \tag{3.4}$$

System (2.29)–(2.31) with boundary condition (2.18) then reads

$$\begin{aligned} \int_\Omega \theta_t w(x)\,\mathrm{d}x + \int_\Omega \nabla\theta \cdot \nabla w(x)\,\mathrm{d}x = &-\int_{\partial\Omega} h(x)(\theta - \theta_\Gamma)w(x)\,\mathrm{d}S(x) \\ &+ \int_\Omega (U_t^2 - \theta U_t - (U + \chi + 1)\chi_t)w(x)\,\mathrm{d}x \\ &+ \int_{\partial\Omega} \frac{r(x)}{b(x)}\big|\mathfrak{p}_{r(x)}[P_0]_t\big|(t)w(x)\,\mathrm{d}S(x), \end{aligned} \tag{3.5}$$

$$U_t + U + \chi + \mathcal{F}^{-1}[U_\Omega] = \theta - p_0, \tag{3.6}$$

$$\chi_t + U + \chi + \partial I(\chi) \ni 2\theta - 1, \tag{3.7}$$

where (3.5) is to be satisfied for all test functions $w \in W^{1,2}(\Omega)$ and a.e. $t > 0$, while (3.6)–(3.7) are supposed to hold a.e. in $\Omega_\infty := \Omega \times (0, \infty)$.

The main existence and uniqueness result reads as follows.

Theorem 3.1. *Let $0 < \theta_* \leqslant \theta_\Gamma \leqslant \theta^*$, $B^* > 0$, and $p_0 \in \mathbb{R}$ be given constants, let b, r, h be positive functions in $L^\infty(\partial\Omega)$ such that*

$$r(x) \leqslant B^* b(x) h(x) \qquad \text{a.e.}, \tag{3.8}$$

and let the data satisfy the conditions

$$\begin{gathered} \theta^0 \in L^\infty(\Omega), \quad \theta_* \leqslant \theta^0(x) \leqslant \theta^* \quad \text{a.e.}, \\ U^0, \chi^0 \in L^\infty(\Omega), \quad 0 \leqslant \chi^0(x) \leqslant 1 \quad \text{a.e.} \end{gathered}$$

Then there exists a unique solution (θ, U, χ) to (3.5)–(3.7), (3.1)–(3.3) such that $\theta > 0$ a.e., $\chi \in [0, 1]$ a.e., $U, U_t, \chi_t, \theta, 1/\theta \in L^\infty(\Omega_\infty)$, $\theta_t \in L^2_{\mathrm{loc}}(0, \infty; (W^{1,2}(\Omega))')$, $\nabla\theta, U_t, \chi_t \in L^2(\Omega_\infty)$.

Remark 3.2. Condition (3.8) is certainly not optimal. A possible relation between mechanical and thermal characteristics of $\partial\Omega$ deserves further investigation.

The proof of Theorem 3.1 will be carried out in the following subsections. Notice first that the term $U_t^2 - \theta U_t - (U + \chi + 1)\chi_t$ on the right hand side of (3.5) can be rewritten alternatively, using (3.7) and (3.6), as

$$\begin{aligned} U_t^2 - \theta U_t - (U + \chi + 1)\chi_t &= U_t^2 - \theta U_t + \chi_t^2 - 2\theta\chi_t \\ &= -(\chi + U + p_0 + \mathcal{F}^{-1}[U_\Omega])U_t - (U + \chi + 1)\chi_t. \end{aligned} \tag{3.9}$$

We now fix a constant $R > 0$, a function $\hat\theta$ to be specified below, and construct the solution for the truncated system

$$
\begin{aligned}
&(3.10)\quad \int_\Omega (\theta_t w(x) + \nabla\theta\cdot\nabla w(x))\,\mathrm{d}x = \int_\Omega (U_t^2 + \chi_t^2 - Q_R(\hat\theta)(U_t + 2\chi_t))w(x)\,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega}\Big(\frac{r(x)}{b(x)}\big|\mathfrak{p}_{r(x)}[P_0]_t\big|(t) - h(x)(\theta-\theta_\Gamma)\Big)w(x)\,\mathrm{d}S(x) \quad \forall w\in W^{1,2}(\Omega),\\
&(3.11)\qquad U_t + U + \chi + \mathcal{F}^{-1}[U_\Omega] = Q_R(\hat\theta) - p_0,\\
&(3.12)\qquad \chi_t + U + \chi + \partial I(\chi) \ni 2Q_R(\hat\theta) - 1
\end{aligned}
$$

first in a bounded domain $\Omega_T := \Omega\times(0,T)$ for any given $T > 0$, where Q_R is the cutoff function $Q_R(z) = \min\{z^+, R\}$. Then we define a norm in a suitable space of admissible functions $\hat\theta$ such that the mapping $\hat\theta\mapsto\theta$ is a contraction. Eventually, we derive upper and lower bounds for θ independent of R and T, so that the fixed point $\theta = \hat\theta$ of (3.10)–(3.12) is also a global solution of (3.5)–(3.7) if R is sufficiently large.

3.1. Prandtl-Ishlinskii operators. We give here a survey of known properties of Prandtl-Ishlinskii operators that are needed in the sequel. The proofs can be found in [7, Chapter II]. We restrict ourselves to the case (2.24), that is,

$$
(3.13)\qquad U_\Omega(t) \;=\; \mathcal{F}[P_0](t) := \frac{1}{K_\Gamma}P_0(t) + \int_{\partial\Omega}\frac{1}{b(x)}\mathfrak{p}_{r(x)}[P_0](t)\,\mathrm{d}S(x).
$$

For monotone input functions P_0, the operator $\mathcal{F}$ can be represented by a superposition (Nemytskii) operator. In particular, there exists a function $\varphi_{\mathcal{F}}$ (the so-called *initial loading curve*) given by the formula

$$
(3.14)\qquad \varphi_{\mathcal{F}}(z) = \frac{z}{K_\Gamma} + \int_{\partial\Omega}\frac{1}{b(x)}(z - r(x))^+\,\mathrm{d}S(x) \quad \text{for } \ z > 0
$$

such that for every $t > 0$ the following implications hold:

$$
\begin{aligned}
&(3.15)\quad (\forall\tau\in[0,t]\colon\ P_0(t)\geqslant\max\{0,P_0(\tau)\}) \Longrightarrow \mathcal{F}[P_0](t) = \varphi_{\mathcal{F}}(P_0(t)),\\
&(3.16)\quad (\forall\tau\in[0,t]\colon\ P_0(t)\leqslant\min\{0,P_0(\tau)\}) \Longrightarrow \mathcal{F}[P_0](t) = -\varphi_{\mathcal{F}}(-P_0(t)),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&(3.17)\quad (\forall\tau\in[0,t]\colon\ U_\Omega(t)\geqslant\max\{0,U_\Omega(\tau)\}) \Longrightarrow \mathcal{F}^{-1}[U_\Omega](t) = \varphi_{\mathcal{F}^{-1}}(U_\Omega(t)),\\
&(3.18)\quad (\forall\tau\in[0,t]\colon\ U_\Omega(t)\leqslant\min\{0,U_\Omega(\tau)\}) \Longrightarrow \mathcal{F}^{-1}[U_\Omega](t) = -\varphi_{\mathcal{F}^{-1}}(-U_\Omega(t)),
\end{aligned}
$$

where $\varphi_{\mathcal{F}^{-1}} = \varphi_{\mathcal{F}}^{-1}$. We have

$$
(3.19)\qquad \varphi'_{\mathcal{F}}(z) = \frac{1}{K_\Gamma} + \int_{\partial\Omega}\frac{1}{b(x)}H(z - r(x))\,\mathrm{d}S(x),
$$

where H is the Heaviside function, hence $\varphi_{\mathcal{F}}$ is increasing and convex, $\varphi_{\mathcal{F}^{-1}}$ is increasing and concave, $\varphi_{\mathcal{F}^{-1}}(0) = \varphi_{\mathcal{F}}(0) = 0$, and $(\varphi_{\mathcal{F}^{-1}})'(0) = 1/\varphi'_{\mathcal{F}}(0) = K_\Gamma$.

Proposition 3.3. *Let the hypotheses of Theorem* 3.1 *hold. Then for all* $U_\Omega^1, U_\Omega^2 \in W^{1,1}(0,T)$ *such that* $U_\Omega^1(0) = U_\Omega^2(0)$ *and for (almost) all* $t \in (0,T)$ *we have, denoting* $P_0^i = \mathcal{F}^{-1}[U_\Omega^i]$, $i = 1, 2$, *the following inequalities:*

$$|\mathcal{F}^{-1}[U_\Omega^1](t) - \mathcal{F}^{-1}[U_\Omega^2](t)| \leqslant 2K_\Gamma \max_{\tau\in[0,t]} |U_\Omega^1(\tau) - U_\Omega^2(\tau)|, \tag{3.20}$$

$$\begin{aligned}(\mathcal{F}^{-1}[U_\Omega^1](t) - \mathcal{F}^{-1}[U_\Omega^2](t))(\dot U_\Omega^1(t) - \dot U_\Omega^2(t)) \\ \geqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{1}{K_\Gamma}(P_0^1 - P_0^2)^2(t) \\ + \int_{\partial\Omega} \frac{1}{b(x)}(\mathfrak{p}_{r(x)}[P_0^1](t) - \mathfrak{p}_{r(x)}[P_0^1](t))^2\,\mathrm{d}S(x)\biggr),\end{aligned} \tag{3.21}$$

$$|\mathfrak{p}_{r(x)}[P_0^i]_t(t)| \leqslant |\dot P_0^i| \leqslant K_\Gamma |\dot U_\Omega^i|, \quad i = 1, 2, \tag{3.22}$$

$$\begin{aligned}\int_0^t |\mathfrak{p}_{r(x)}[P_0^1]_t(\tau) - \mathfrak{p}_{r(x)}[P_0^2]_t(\tau)|\,\mathrm{d}\tau \\ \leqslant \int_0^t |\dot P_0^1 - \dot P_0^2|(\tau)\,\mathrm{d}\tau \leqslant 2K_\Gamma \int_0^t |\dot U_\Omega^1 - \dot U_\Omega^2|(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{3.23}$$

3.2. Gradient flow. Integrating (3.11) over Ω yields

$$\dot U_\Omega + U_\Omega + |\Omega|\mathcal{F}^{-1}[U_\Omega] = \int_\Omega (Q_R(\hat\theta) - p_0 - \chi)\,\mathrm{d}x. \tag{3.24}$$

System (3.11)–(3.12) is a gradient flow in $L^2(\Omega) \times L^2(\Omega)$ of the form

$$\dot v(t) + \partial\psi(v(t)) \ \ni\ f(v,t), \quad v(0) = v^0, \tag{3.25}$$

with a convex functional ψ, where

$$v = \begin{pmatrix} U \\ \chi \end{pmatrix}, \tag{3.26}$$

$$\psi(v) = \int_\Omega \Bigl(\frac{1}{2}(U+\chi)^2 + I(\chi)\Bigr)\,\mathrm{d}x, \tag{3.27}$$

$$f(v,t) = \begin{pmatrix} Q_R(\hat\theta) - p_0 - \mathcal{F}^{-1}[U_\Omega] \\ 2Q_R(\hat\theta) - 1 \end{pmatrix}. \tag{3.28}$$

The initial condition v^0 is given by (3.2), (3.3). We will prove the following result.

Proposition 3.4. *Let the hypotheses of Theorem* 3.1 *hold, and let a function* $\hat\theta \in L^2_{\mathrm{loc}}(0,\infty; L^1(\Omega))$ *be given. Then there exist a unique solution* $(U,\chi) \in (L^\infty(\Omega_\infty))^2$ *to* (3.24)–(3.28), *and a constant* C_0 *independent of* x, t *and* R, *such that*

$$|U(x,t)| + |U_t(x,t)| + |\chi_t(x,t)| \leqslant C_0(1+R) \tag{3.29}$$

a.e. in Ω_∞. *Furthermore, there exists an increasing function* $\mu\colon (0,\infty) \to (0,\infty)$ *such that whenever* $\hat\theta_1, \hat\theta_2 \in L^2_{\mathrm{loc}}(0,\infty; L^1(\Omega))$ *are given functions and* $(U_1,\chi_1), (U_2,\chi_2)$ *are the corresponding solutions to* (3.24)–(3.28), *then the differences* $\hat\theta_d = \hat\theta_1 - \hat\theta_2$, $U_d = U_1 - U_2$, $\chi_d = \chi_1 - \chi_2$ *satisfy for every* $t \geqslant 0$ *and a.e.* $x \in \Omega$ *the inequality*

$$\int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x,\tau)\,\mathrm{d}\tau \leqslant 9\int_0^t |\hat\theta_d(x,\tau)|\,\mathrm{d}\tau + \mu(t)\biggl(\int_0^t |\hat\theta_d(\tau)|_1^2\,\mathrm{d}\tau\biggr)^{1/2}. \tag{3.30}$$

where the symbol $|\cdot|_p$ *for* $p \in [1,\infty]$ *stands for the norm in* $L^p(\Omega)$.

In what follows, we denote by $C_1, C_2, \ldots$ any constants independent of x, t and R.

P r o o f. The right-hand side f of (3.25) is Lipschitz continuous in v, and we easily obtain the existence and uniqueness of solutions to (3.25)–(3.28) by a contraction argument.

Eq. (3.24) is an ODE with a Lipschitz continuous nonlinearity, and given initial condition, hence for every given χ and $\hat\theta$, it admits a unique Lipschitz continuous solution U_Ω. It is easy to see that both U_Ω and $\dot U_\Omega$ remain globally bounded: Let U_Ω attain at some point $t > 0$ the maximum of its absolute value, that is, $U_\Omega(t) = \max\limits_{\tau\in[0,t]} |U_\Omega(\tau)|$. Then $\dot U_\Omega(t) \geqslant 0$, and (3.17) and (3.24) imply

$$U_\Omega(t) + \varphi_{\mathcal{F}^{-1}}(U_\Omega(t)) \leqslant C_1(1+R).$$

The argument is similar if $U_\Omega(t) = -\max\limits_{\tau\in[0,t]} |U_\Omega(\tau)|$, and we conclude that

$$|\dot U_\Omega(t)| + |U_\Omega(t)| \leqslant C_2(1+R). \tag{3.31}$$

Equation (3.11) now has a right hand side bounded by a multiple of $1+R$, hence $|U_t| + |U| \leqslant C_3(1+R)$ a.e. To get the same bound for $|\chi_t|$, it suffices to multiply (3.12) by χ_t. This completes the proof of (3.29).

Consider now two different inputs. As above, we denote the differences $\{\}_1 - \{\}_2$ by $\{\}_d$ for all symbols $\{\}$. Testing the difference of Eqs. (3.24) by $\dot U_{\Omega d}$ and using (3.21), we obtain after integration

$$\int_0^t |\dot U_{\Omega d}(\tau)|^2\,\mathrm{d}\tau + |U_{\Omega d}(t)|^2 \leqslant \int_0^t (|\hat\theta_d|_1 + |\chi_d|_1)^2(\tau)\,\mathrm{d}\tau \qquad \forall t > 0. \tag{3.32}$$

To prove (3.30), we rewrite (3.25) as two scalar gradient flows

$$U_t + \partial\psi_1(U) = a, \tag{3.33}$$

$$\chi_t + \partial\psi_2(\chi) \ni b, \tag{3.34}$$

where $\psi_1(U) = \frac{1}{2}U^2$, $\psi_2 = \frac{1}{2}\chi^2 + I(\chi)$, $a = Q_R(\hat\theta) - \chi - p_0 - \mathcal{F}^{-1}[U_\Omega]$, $b = 2Q_R(\hat\theta) - 1 - U$.

By [6, Theorem 1.12], we have for all $t > 0$ and a.e. $x \in \Omega$ that

$$\int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x,\tau)\,\mathrm{d}\tau \leqslant 2\int_0^t (|a_d| + |b_d|)(x,\tau)\,\mathrm{d}\tau. \tag{3.35}$$

We multiply the difference of (3.33) by $\operatorname{sign}(U_d)$, the difference of (3.34) by $\operatorname{sign}(\chi_d)$, and sum them up. Using the monotonicity of ∂I and the elementary inequality $(p+q)(\operatorname{sign}(p) + \operatorname{sign}(q)) \geqslant 0$, we obtain that

$$|U_d|_t + |\chi_d|_t \leqslant 3|\hat\theta_d| + |\mathcal{F}_d^{-1}| \quad \text{a.e.}, \tag{3.36}$$

where $\mathcal{F}_d^{-1} := \mathcal{F}^{-1}[U_\Omega^1] - \mathcal{F}^{-1}[U_\Omega^2]$. By (3.32) and Proposition 3.3, we thus have

$$|U_d|_t + |\chi_d|_t \leqslant 3|\hat\theta_d| + 2K_\Gamma\left(\int_0^t (|\hat\theta_d|_1 + |\chi_d|_1)^2(\tau)\,\mathrm{d}\tau\right)^{1/2} \quad \text{a.e.} \tag{3.37}$$

Integrating (3.37) over Ω and using Gronwall's inequality we find an increasing function $\mu_1(t)$ such that

$$\left(|\mathcal{F}_d^{-1}(t)| + |U_d|_1(t) + |\chi_d|_1(t)\right)^2 \leqslant \mu_1(t)\int_0^t |\hat\theta_d(\tau)|_1^2\,\mathrm{d}\tau \qquad \forall t > 0. \tag{3.38}$$

Integrating (3.36) with respect to t yields

$$\left(|U_d| + |\chi_d|\right)(x,t) \leqslant 3\int_0^t |\hat\theta_d(x,\tau)|\,\mathrm{d}\tau + \left(\mu_1(t)\int_0^t |\hat\theta_d(\tau)|_1^2\,\mathrm{d}\tau\right)^{1/2}. \tag{3.39}$$

From (3.35) we now immediately get the desired inequality

$$\int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x,\tau)\,\mathrm{d}\tau \leqslant 9\int_0^t |\hat\theta_d(x,\tau)|\,\mathrm{d}\tau + 3\left(\mu_1(t)\int_0^t |\hat\theta_d(\tau)|_1^2\,\mathrm{d}\tau\right)^{1/2} \tag{3.40}$$

for a.e. $x \in \Omega$ and all $t \geqslant 0$, with $\mu(t) = 3\sqrt{\mu_1(t)}$. This completes the proof. □

3.3. Fixed point argument. Let now a final time $T > 0$ be fixed. For every given $\hat{\theta}$, Eq. (3.10) is a linear heat equation with a given right hand side and boundary and initial conditions, hence it admits a unique solution θ with the regularity

$$\theta \in C([0,T];L^2(\Omega)), \quad \theta_t \in L^2((0,T;(W^{1,2}(\Omega))'), \quad \nabla\theta \in L^2(0,T;L^2(\Omega)). \tag{3.41}$$

By Proposition 3.3 we have $|\mathfrak{p}_{r(x)}[P_0]_t|(t) \leqslant K_\Gamma|\dot{U}_\Omega(t)|$ a.e. Testing (3.10) by $w = \theta$, we may use Proposition 3.4 to find a constant $M(T,R)$ depending on T and R such that

$$\sup_{t\in(0,T)}\operatorname{ess} \int_\Omega \theta^2(x,t)\,\mathrm{d}x + \int_0^T \left(\int_\Omega |\nabla\theta|^2\,\mathrm{d}x + \int_{\partial\Omega} h(x)\theta^2\,\mathrm{d}S(x)\right)\,\mathrm{d}t \leqslant M(T,R). \tag{3.42}$$

We can define the mapping that associates with $\hat{\theta}$ the solution θ of (3.10)–(3.12) with initial conditions (3.1)–(3.3). We now show that it is a contraction on the set

$$\Xi_{T,R} := \{\theta \in L^2(\Omega_T)\colon \text{ conditions (3.1) and (3.41)–(3.42) hold}\}. \tag{3.43}$$

Let $\hat{\theta}_1, \hat{\theta}_2$ be two functions in $\Xi_{T,R}$, and let (θ_1, U_1, χ_1), (θ_2, U_2, χ_2) be the corresponding solutions to (3.10)–(3.12) with the same initial conditions θ^0, U^0, χ^0. We see from (3.42) that θ_1, θ_2 belong to $\Xi_{T,R}$. We test the difference of Eqs. (3.10) for θ_1 and θ_2 by $w = \operatorname{sign}(\theta_d)$ obtaining

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\theta_d(x,t)|\,\mathrm{d}x + \int_{\partial\Omega} h(x)|\theta_d(x,t)|\,\mathrm{d}S(x) \\ &\qquad \leqslant C_4(1+R)\int_\Omega (|(U_d)_t| + |(\chi_d)_t| + |\hat{\theta}_d|)(x,t)\,\mathrm{d}x \\ &\qquad\quad + \int_{\partial\Omega} \frac{r(x)}{b(x)}|\mathfrak{p}_d(x,t)|\,\mathrm{d}S(x) \quad \text{a.e.,} \end{aligned} \tag{3.44}$$

where we set $\mathfrak{p}_d(x,t) := |\mathfrak{p}_{r(x)}[P_0^1]_t| - |\mathfrak{p}_{r(x)}[P_0^2]_t|(t)$. By Proposition 3.3 we have for a.e. $x \in \partial\Omega$ that

$$\int_0^t |\mathfrak{p}_d(x,\tau)|\,\mathrm{d}\tau \leqslant 2K_\Gamma \int_0^t |\dot{U}_{\Omega d}(\tau)\,\mathrm{d}\tau \leqslant 2K_\Gamma \int_0^t \int_\Omega |(U_d)_t|(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Integrating (3.44) with respect to t and using Proposition 3.4 we find an increasing function $\mu_R(t)$ depending also on R such that

$$|\theta_d|_1(t) \leqslant \mu_R(t)\left(\int_0^t |\hat{\theta}_d|_1^2(\tau)\,\mathrm{d}\tau\right)^{1/2} \quad \text{a.e.} \tag{3.45}$$

Set $\Theta^2(t) = \int_0^t |\theta_d|_1^2(\tau)\,\mathrm{d}\tau$, $\widehat{\Theta}^2(t) = \int_0^t |\hat\theta_d|_1^2(\tau)\,\mathrm{d}\tau$, $\hat\mu_R(t) = \int_0^t \mu_R^2(\tau)\,\mathrm{d}\tau$. Let us introduce in $L^\infty(0,T)$ the norm

$$\|w\|_C := \sup_{\tau\in[0,T]} \mathrm{e}^{-\hat\mu_R(t)}|w(\tau)|.$$

Then $\|\Theta\|_C^2 \leqslant \frac{1}{2}\|\widehat{\Theta}\|_C^2$, hence the mapping $\hat\theta \mapsto \theta$ is a contraction in $L^2(0,T;L^1(\Omega))$ with respect to the norm induced by $\|\cdot\|_C$. The set $\Xi_{T,R}$ is a closed subset of $L^2(0,T;L^1(\Omega))$. This implies the existence of a fixed point $\theta \in \Xi_{T,R}$, which is indeed a solution to (3.10)–(3.12). Since T has been chosen arbitrarily, the solution is global in Ω_∞.

3.4. Estimates. A positive lower bound for θ follows from the maximum principle. Let us introduce an auxiliary function $\theta^\flat(t) = \theta_*/(1+2\theta_* t)$. On the right hand side of (3.10) with $\theta = \hat\theta$ we have $U_t^2 + \chi_t^2 - Q_R(\theta)(U_t + 2\chi_t) \geqslant -2(\theta^+)^2$. For every nonnegative test function w and a.e. $t \in (0,T)$ we thus obtain

$$\int_\Omega \theta_t w(x)\,\mathrm{d}x + \int_\Omega \nabla\theta\cdot\nabla w(x)\,\mathrm{d}x + \int_{\partial\Omega} h(x)(\theta-\theta_\Gamma)w(x)\,\mathrm{d}S(x) \geqslant -2\int_\Omega (\theta^+)^2 w(x)\,\mathrm{d}x, \tag{3.46}$$

$$\int_\Omega \theta^\flat_t w(x)\,\mathrm{d}x + \int_\Omega \nabla\theta^\flat\cdot\nabla w(x)\,\mathrm{d}x + \int_{\partial\Omega} h(x)(\theta^\flat-\theta_\Gamma)w(x)\,\mathrm{d}S(x) \leqslant -2\int_\Omega (\theta^\flat)^2 w(x)\,\mathrm{d}x. \tag{3.47}$$

We subtract (3.46) from (3.47) and test by $w = (\theta^\flat - \theta)^+$, which yields $\theta(x,t) \geqslant \theta^\flat(t)$ a.e. In particular, the temperature remains positive for all times $t > 0$.

The energy e_R and the entropy s_R corresponding to the fixed point $\theta = \hat\theta$ of (3.10)–(3.12) have the form

$$\varrho_0 e_R = \theta + \frac{1}{2}(U+\chi-1)^2 + U + 2(\chi + I(\chi)), \tag{3.48}$$

$$\varrho_0 s_R = l_R(\theta) + 2\chi + U, \tag{3.49}$$

where $l_R(\theta) = \int_1^\theta (1/Q_R(\theta')\,\mathrm{d}\theta'$, that is, $l_R(\theta) = \log\theta$ for $\theta < R$, $l_R(\theta) = \log R + (1/R)(\theta - R)$ for $\theta \geqslant R$.

Let E_Γ be given by (2.35). We compute from (3.48)–(3.49) the initial values e^0, E_Γ^0, and s^0 for specific energy, boundary energy, and entropy, respectively. Let $E^0 = \int_\Omega \varrho_0 e^0\,\mathrm{d}x$, $S^0 = \int_\Omega \varrho_0 s^0\,\mathrm{d}x$ denote the total initial energy and the entropy, respectively. From the energy end entropy balance equations (2.37), (2.39) we derive the following crucial balance equation for the "extended" energy $\varrho_0(e_R - \theta_\Gamma s_R)$:

$$
\begin{aligned}
(3.50)\qquad & \int_\Omega \Big(\theta + \frac{1}{2}(U+\chi-1))^2 + U + 2\chi\Big)(x,t)\,\mathrm{d}x + E_\Gamma(t) \\
&+ \theta_\Gamma \int_0^t \int_\Omega \Big(\frac{|\nabla Q_R(\theta)|^2}{Q_R^2(\theta)} + \frac{\chi_t^2}{Q_R(\theta)} + \frac{U_t^2}{Q_R(\theta)}\Big)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ \int_0^t \int_{\partial\Omega} \frac{h(x)}{Q_R(\theta)}(\theta_\Gamma - \theta)(Q_R(\theta_\Gamma) - Q_R(\theta))(x,\tau)\,\mathrm{d}S(x)\,\mathrm{d}\tau \\
&+ \int_0^t \int_{\partial\Omega} \frac{\theta_\Gamma r(x)}{Q_R(\theta) b(x)} \big|\mathfrak{p}_{r(x)}[P_0]_t\big|(t)\,\mathrm{d}S(x)\,\mathrm{d}\tau \\
&= E^0 + E_\Gamma^0 - \theta_\Gamma S^0 + \theta_\Gamma \int_\Omega \left(l_R(\theta) + 2\chi + U\right)(x,t)\,\mathrm{d}x.
\end{aligned}
$$

It is easy to see that there exists $C_5 > 0$ independent of R such that $\theta_\Gamma l_R(\theta) \leqslant \theta/2 + C_5$ if R is sufficiently large. Indeed, assuming e.g. that

$$R > 2\theta_\Gamma(1 + \log R),$$

we have $\theta_\Gamma l_R(\theta) - \theta/2 \leqslant 0$ if $\theta \geqslant R$, and $\theta_\Gamma l_R(\theta) - \theta/2 \leqslant \theta_\Gamma(\log(2\theta_\Gamma) - 1)$ if $\theta < R$. We conclude that there exists a constant $C_6 > 0$ independent of t and R such that for all $t > 0$ we have

$$
(3.51)\qquad \int_\Omega \left(\theta + U^2\right)(x,t)\,\mathrm{d}x + \int_0^t \int_\Omega \Big(\frac{|\nabla\theta|^2}{\theta^2} + \frac{\chi_t^2}{\theta} + \frac{U_t^2}{\theta}\Big)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \;\leqslant\; C_6.
$$

In particular, the right hand side of (3.24) with $\hat\theta = \theta$ is uniformly bounded independently of R, and the argument of the proof of Proposition 3.4 entails that

$$
(3.52)\qquad |\dot U_\Omega(t)| + |U_\Omega(t)| + |\dot P_0| + |P_0| + |\mathfrak{p}_{r(x)}[P_0]_t| \leqslant C_7 \qquad \text{a.e.}
$$

For every nonnegative test function w and a.e. $t \in (0,T)$, the fixed point θ of (3.10)–(3.12) satisfies the inequality

$$
\begin{aligned}
(3.53)\; & \int_\Omega \theta_t w(x)\,\mathrm{d}x + \int_\Omega \nabla\theta\cdot\nabla w(x)\,\mathrm{d}x \\
&+ \int_{\partial\Omega} h(x)(\theta - \theta_\Gamma - C_7 B^*) w(x)\,\mathrm{d}S(x) \leqslant C_8(1+R)^2 \int_\Omega w(x)\,\mathrm{d}x
\end{aligned}
$$

with a suitable constant $C_8 > 0$. Let us define another auxiliary function

$$\theta^\sharp(t) = \theta^* + C_7 B^* + C_8(1+R)^2 t.$$

For every nonnegative test function w and a.e. $t \in (0,T)$ we then have

$$(3.54) \qquad \int_\Omega \theta^\sharp_t w(x)\,\mathrm{d}x + \int_\Omega \nabla\theta^\sharp\cdot\nabla w(x)\,\mathrm{d}x \\ + \int_{\partial\Omega} h(x)(\theta^\sharp - \theta_\Gamma - C_7 B^*)w(x)\,\mathrm{d}S(x) \geqslant C_8(1+R)^2\int_\Omega w(x)\,\mathrm{d}x.$$

We now subtract (3.54) from (3.53) and test by $w = (\theta-\theta^\sharp)^+$, which yields the pointwise bound $\theta(x,t) \leqslant \theta^\sharp(t)$. We thus have the inequalities

$$(3.55) \qquad \theta^\flat(t) \leqslant \theta(x,t) \leqslant \theta^\sharp(t) \quad \text{a.e.}$$

3.5. Uniform global bounds. The unique solution fixed point (θ, U, χ) to the system (3.10)–(3.12), (3.1)–(3.3) exists globally in the whole domain Ω_∞. We now show that θ remains globally bounded independently of t and R if R is sufficiently large. Take first for instance any $R > 2\theta^* + C_7B^*$. By (3.55), we know that θ remains smaller than R in a nondegenerate interval $(0,T)$ with $T > \theta^*/(C_8(1+R)^2)$. Let $(0,T_0)$ be the maximal interval in which θ is bounded by R. Then, in $(0,T_0)$, the solution constructed in Subsection 3.3 is also a solution of the original problem (3.5)–(3.7). Moreover, due to estimate (3.51), we know that θ admits a bound in $L^\infty(0,T_0;L^1(\Omega))$ independent of R. In order to prove that $T_0 = +\infty$ if R is sufficiently large, we refer to the following statement, which is proved in detail in [8] by a variant of the Moser iteration technique.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitzian boundary. Given nonnegative functions $h \in L^1(\partial\Omega)$ and $r \in L^\infty(0,\infty;L^q(\Omega))$ with a fixed $q > N/2$, an initial condition $v^0 \in L^\infty(\Omega)$, and a boundary datum $v_\Gamma \in L^\infty(\partial\Omega\times(0,\infty))$, consider the problem*

$$(3.56) \qquad v_t - \Delta v + v = r(x,t)\mathcal{H}[v] \qquad \text{a.e. in } \Omega_\infty,$$

$$(3.57) \qquad \nabla v\cdot\mathbf{n} = -h(x)\,(f(x,t,v(x,t)) - v_\Gamma(x,t)) \quad \text{a.e. on } \partial\Omega\times(0,\infty),$$

$$(3.58) \qquad v(x,0) = v^0 \qquad \text{a.e. in } \Omega,$$

under the assumption that there exist positive constants $H_0, C_f, V, V_\Gamma, E_0$ such that the following assertions hold:

(i) *The mapping $\mathcal{H}\colon L^\infty_{\mathrm{loc}}(\Omega_\infty) \to L^\infty_{\mathrm{loc}}(\Omega_\infty)$ satisfies for every $v \in L^\infty_{\mathrm{loc}}(\Omega_\infty)$ and a.e. $(x,t) \in \Omega_\infty$ the inequality*

$$v(x,t)\mathcal{H}[v](x,t) \leqslant H_0|v(x,t)|\left(1 + |v(x,t)| + \int_0^t \xi(t-\tau)|v(x,\tau)|\,\mathrm{d}\tau\right),$$

where $\xi \in W^{1,1}(0,\infty)$ is a given nonnegative function such that

$$\dot{\xi}(t) \leqslant -\xi(0)\xi(t) \quad \text{a.e.} \tag{3.59}$$

(ii) *f is a Carathéodory function on $\Omega \times (0,\infty) \times \mathbb{R}$ such that $f(x,t,v)v \geqslant C_f v^2$ a.e. for all $v \in \mathbb{R}$.*

(iii) *$|v^0(x)| \leqslant V$ a.e. in Ω.*

(iv) *$|v_\Gamma(x,t)| \leqslant V_\Gamma$ a.e. on $\partial\Omega \times (0,\infty)$.*

(v) *System (3.56)–(3.58) admits a solution $v \in W^{1,2}_{\mathrm{loc}}(0,\infty;(W^{1,2})'(\Omega)) \cap L^2_{\mathrm{loc}}(0,\infty; W^{1,2}(\Omega)) \cap L^\infty_{\mathrm{loc}}(\Omega_\infty)$ satisfying the estimate*

$$\int_\Omega |v(x,t)|\,\mathrm{d}x \leqslant E_0 \quad \text{a.e. in } (0,\infty).$$

Then there exists a positive constant C^* depending only on $|h|_{L^1(\partial\Omega)}$, C_f, H_0, $\xi(0)$, and $r^* := |r|_{L^\infty(0,\infty;L^q(\Omega))}$, such that

$$|v(t)|_{L^\infty(\Omega)} \leqslant C^* \max\{1, V, V_\Gamma, E_0\} \quad \text{for a.e. } t > 0. \tag{3.60}$$

Remark 3.6. As a consequence of (3.59), we have $\xi(t) \leqslant \xi(0)\mathrm{e}^{-\xi(0)t}$ for all $t \geqslant 0$, hence $\int_0^\infty \xi(t)\,\mathrm{d}t \leqslant 1$. As a typical function satisfying (3.59), let us mention for example

$$\xi(t) = \frac{m_1}{\sum\limits_{k=1}^n r_k} \sum_{k=1}^n r_k \mathrm{e}^{-m_k t} \tag{3.61}$$

with any $0 < m_1 \leqslant \ldots \leqslant m_n$ and $r_k > 0$, $k = 1,\ldots,n$.

We now complete the proof of Theorem 3.1 by showing that T_0 introduced at the beginning of this subsection is $+\infty$ if R is sufficiently large. By (3.51)–(3.52), we obtain directly from (3.6)–(3.7) that

$$|U(x,t)| + |U_t(x,t)| + |\chi_t(x,t)| \leqslant C_{10}\left(1 + \theta(x,t) + \int_0^t \mathrm{e}^{\tau-t}\theta(x,\tau)\,\mathrm{d}\tau\right) \quad \text{a.e.} \tag{3.62}$$

As in (3.9), we rewrite the right hand side of Eq. (3.5) as

$$-(\chi + U + p_0 + \mathcal{F}^{-1}[U_\Omega])U_t - (U + \chi + 1)\chi_t.$$

By (3.51), the function U is in $L^\infty(0,\infty;L^2(\Omega))$ and the bound does not depend on R. Eq. (3.5), with θ added to both the left and the right hand side, thus satisfies the hypotheses of Proposition 3.5 for $N = 3$ and $q = 2$. This enables us to conclude that $\theta(x,t)$ is uniformly bounded from above by a constant, independently of R, so that θ never reaches the value R if R is sufficiently large, which we wanted to prove. By (3.62), also U, U_t, and χ_t are uniformly bounded by a constant.

We proceed similarly to prove a uniform positive lower bound for θ. We denote $R_0 := \sup\theta$, and in the equation

$$\begin{aligned}(3.63)\quad \int_\Omega \theta_t w(x)\,\mathrm{d}x + \int_\Omega \nabla\theta\cdot\nabla w(x)\,\mathrm{d}x &= \int_\Omega \big(U_t^2 + \chi_t^2 - \theta(U_t + 2\chi_t)\big)\,w(x)\,\mathrm{d}x \\ &+ \int_{\partial\Omega}\Big(\frac{r(x)}{b(x)}|\mathfrak{p}_{r(x)}[P_0]_t|(t) - h(x)(\theta-\theta_\Gamma)\Big)w(x)\,\mathrm{d}S(x) \quad \forall w\in W^{1,2}(\Omega)\end{aligned}$$

set $w = -\tilde w/\theta$ with an arbitrary $\tilde w \in W^{1,2}(\Omega)$. For a new (nonnegative) variable $v(x,t) := \log R_0 - \log\theta(x,t)$ we obtain the equation

$$\begin{aligned}(3.64)\quad \int_\Omega v_t\tilde w(x)\,\mathrm{d}x + \int_\Omega \nabla v\cdot\nabla\tilde w(x)\,\mathrm{d}x + \int_{\partial\Omega} h(x)\Big(\frac{B(x,t)}{\theta} - 1\Big)\tilde w(x)\,\mathrm{d}S(x) \\ = \int_\Omega\Big(-\frac{U_t^2+\chi_t^2}{\theta} - \frac{|\nabla\theta|^2}{\theta^2} + U_t + 2\chi_t\Big)\tilde w(x)\,\mathrm{d}x\end{aligned}$$

with $B(x,t) = \theta_\Gamma + r(x)/(b(x)h(x))|\mathfrak{p}_{r(x)}[P_0]_t|(t)$. For

$$\mathcal{H}[v] := \operatorname{sign}(v)\Big(-\frac{U_t^2+\chi_t^2}{\theta} - \frac{|\nabla\theta|^2}{\theta^2} + U_t + 2\chi_t\Big)$$

we check that the hypotheses of Proposition 3.5 are satisfied with the choice $f(x,t,v) = (B(x,t)/R_0)(\mathrm{e}^v - 1)$, $v_\Gamma = 1 - B(x,t)/R_0$, $r \equiv 1$, and $v\mathcal{H}[v] \leqslant 3C_7|v|$. Hence, v is bounded above by some v^*, which entails $\theta \geqslant R_0\mathrm{e}^{-v^*}$. This, together with (3.51), concludes the proof of Theorem 3.1.

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