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# TRIANGULAR REPUNIT—THERE IS BUT 1

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*Abstract.* In this paper, we demonstrate that 1 is the only integer that is both triangular and a repunit.

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A *repunit* is any integer that can be written in decimal notation as a string of 1's. Examples include $1, 11, 111, 1111, 11111, \ldots$ Furthermore, if we denote $r_n$ to be the $n$th repunit then it follows that

$$r_n = \frac{10^n - 1}{9}. \tag{1}$$

*Triangular numbers* are those integers that can be represented by the number of dots evenly arranged in an equilateral triangle. More specifically, they are the numbers $1, 3, 6, 10, 15, \ldots$ It follows that the $n$th triangular number, $t_n$ is the sum of the first $n$ consecutive integers beginning with 1. In particular, for $n \geqslant 1$

$$t_n = \frac{n(n+1)}{2}. \tag{2}$$

Now, the number 1 is both triangular and a repunit. It is the objective of this note to illustrate that 1 is the only such number. To this end, we state and prove the following lemma. It incorporates a result that [1] asserts had been known by the early Pythagoreans. Appearing in *Platonic Questions*, Plutarch states that eight times any triangular number plus one is a square. The result is actually necessary and sufficient. We demonstrate it as such here.

**Lemma.** *The integer $n$ is triangular if and only if $8n+1$ is a square.*

Proof. If $n = t_n$, then by (2),

$$8\left[\frac{n(n+1)}{2}\right] + 1 = 4n^2 + 4n + 1 = (2n+1)^2.$$

On the other hand, if $8n+1$ is a square, then $8n+1 = x^2$, for some odd positive integer $x$. Hence, $x^2 - 1$ is even, and so

$$8n = x^2 - 1 = (x+1)(x-1) = (2k+2)(2k),$$

for some positive integer $k$, from which it follows that $n = \frac{1}{2}k(k+1)$. □

The previous result may also be observed geometrically by letting an $n \times (n+1)$ rectangle represent twice $t_n$. Thus, four such rectangles plus a unit square comprise a square with sides $2n+1$.

**Theorem.** *The only triangular repunit is* 1.

Proof. In light of (1) and the previous lemma, it suffices to show that $8 \times (10^n - 1)/9 + 1$ is square only for $n = 1$.

To this end, we shall determine all $n$ for which

$$8\left[\frac{10^n - 1}{9}\right] + 1 = (2k+1)^2,$$

for some positive integer $k$. Thus,

$$10^n = (2 \cdot 5)^n = \frac{(3k+2)(3k+1)}{2};$$

whence,

$$5^n = \frac{(3k+2)(3k+1)}{2^{n+1}}. \tag{3}$$

Now, $5 \mid 3k+2$ or $5 \mid 3k+1$ but not both. Similarly, $2 \mid 3k+2$ or $2 \mid 3k+1$ but not both.

**Case 1.** Suppose $5 \mid 3k+2$. If $k$ is odd, then $3k+2$ is odd. As $3k+1$ is even, in light of (3) we have $3k+2 = 5^n$ and $3k+1 = 2^{n+1}$. So, $2^{n+1} + 1 = 5^n$. But this can only occur when $n = 1$ (i.e., for $t_n = 1$.) This is seen upon noting that for $n > 2$, $5^n > 2^{n+1} + 1$. On the other hand, if $k$ is even then $3k+1$ is odd, implying that 5 divides both $3k+1$ and $3k+2$, which is impossible.

**Case 2.** Suppose $5 \mid 3k+1$. If $k$ is odd, then $3k+1$ is even, which implies that both 5 and 2 divide $3k+1$. Hence, there must exist an integer different from 5 and 2 that is a factor of $3k+2$, which is impossible. Hence, $k$ must be even. It follows that $3k+1$ is odd, and so $3k+1 = 5^n$ and $3k+2 = 2^{n+1}$. A triangular repunit then results when $5^n + 1 = 2^{n+1}$. But the only solution is $n = 0$, as $n > 0$ implies $5^n + 1 > 2^{n+1}$. However, $t_0 = 0$ is neither triangular nor a repunit. $\square$

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*References*

[1] *J. J. Tattersall*: Elementary Number Theory in Nine Chapters, 2nd ed. Cambridge University Press, Cambridge, 2005.

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