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# GENERAL INTEGRATION AND EXTENSIONS I

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*Abstract.* A general concept of integral is presented in the form given by S. Saks in his famous book Theory of the Integral. A special subclass of integrals is introduced in such a way that the classical integrals (Newton, Riemann, Lebesgue, Perron, Kurzweil-Henstock...) belong to it.

A general approach to extensions is presented. The Cauchy and Harnack extensions are introduced for general integrals. The general results give, as a specimen, the Kurzweil-Henstock integration in the form of the extension of the Lebesgue integral.

*Keywords*: abstract integration, extension of integral, Kurzweil-Henstock integration

*MSC 2010*: 26A39, 26A42

## 1. Introduction: Notation and preliminaries

Speaking about a function on a compact interval $E = [a, b]$, $-\infty < a < b < +\infty$ we have in mind real functions.

For $M \subset E$ and a function $f\colon E \to \mathbb{R}$ we denote

$$|f|_M = \sup\{|f(x)|;\ x \in M\}.$$

If $J \subset E$ is a closed interval in $E$, then we denote by $\operatorname{Sub}(J)$ the set of all closed subintervals of $J$.

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The editors learnt with great sadness that Professor Štefan Schwabik passed away on November 4, 2009. The galleys of this paper were therefore not proofread by the author, and the responsibility for any typesetting inaccuracies lies solely with the editors.

A *division* is a finite system $D = \{I_j;\ j \in \Gamma\}$ of intervals, where $\operatorname{Int}(I_j) \cap I_k = \emptyset$ for $j \neq k$ (the elements of a division do not overlap).

For a given set $M \subset E$ the division $D$ is called a *division in* $M$ if $M \supset \bigcup_{j\in\Gamma} I_j$, $D$ is a *division of* $M$ if $M = \bigcup_{j\in\Gamma} I_j$ and the division $D$ *covers* $M$ if $M \subset \bigcup_{j\in\Gamma} I_j$.

A map $\tau$ from $\operatorname{Sub}(E)$ into $E$ is called a *tag* if $\tau(I) \in I$ for $I \in \operatorname{Dom}(\tau)$.

A *tagged system* is a pair $(D, \tau)$, where $D = \{I_j;\ j \in \Gamma\}$ is a division and $\tau$ is a tag defined on the range of $D$, i.e. for all $I_j$, $j \in \Gamma$. In this case we write usually $\tau_j$ instead of $\tau(I_j)$.

The tagged system $(D, \tau)$ is called $M$*-tagged* for a set $M \subset E$ if $\tau_j \in M$ for $j \in \Gamma$.

A *gauge* is any function on $E$ with values in the set $\mathbb{R}^+$ of positive reals. The set of all gauges is denoted by $\Delta(E)$.

If $\delta \in \Delta(E)$, then a tagged system $(D, \tau)$, where $D = \{I_j;\ j \in \Gamma\}$, is called $\delta$*-fine* if $|I_j| < \delta(\tau_j)$ for $j \in \Gamma$.

Assume that $F\colon E \to \mathbb{R}$ is a real function defined on $E$. For $I = [c, d] \in \operatorname{Sub}(E)$, $c \leqslant d$, define, as usual, the interval function

$$F[I] = F(d) - F(c).$$

Denote by $C(E)$ the set of all continuous real-valued functions on $E$.

The *oscillation* $\omega(F, I)$ of $F \in C(E)$ on an interval $I \in \operatorname{Sub}(E)$ is

$$\omega(F, I) = \sup\{|F(x) - F(y)|;\ \ x, y \in I\} = \sup\{|F[J]|;\ J \in \operatorname{Sub}(I)\}.$$

In the sequel we will use a certain type of variational measure in our consideration (see e.g. [8] and [7]). Let us recall its definition.

**Definition 1.1.** For $F \in C(E)$ and a division $D = \{I_j;\ j \in \Gamma\}$ let us set

$$\Omega(F, D) = \sum_{j\in\Gamma} \omega(F, I_j).$$

If $F \in C(E)$ and $M \subset E$, then for any $\delta \in \Delta(E)$ set

$$W_\delta(F, M) = \sup\{\Omega(F, D);\ D \text{ is } \delta\text{-fine}, M\text{-tagged}\}.$$

Put

$$W_F(M) = \inf\{W_\delta(F, M);\ \delta \in \Delta(E)\}.$$

$W_F$ is the full variational measure generated by the interval function $\omega(F, I)$ for $I \in \operatorname{Sub}(E)$ (see [8] and [7]; in [7] a detailed survey of this particular concept is given).

**Definition 1.2.** By $C^*(E)$ we denote the set of all continuous functions on $E$ for which $W_F(N) = 0$ on sets of Lebesgue measure zero, i.e.

$$C^*(E) = \{F \in C(E);\ W_F(N) = 0 \text{ whenever } \mu(N) = 0\}.$$

This is Definition 2.11 in [7] and it turns out by the results given in [7] that $C^*(E)$ is the set of continuous functions which are of negligible variation on sets of Lebesgue measure zero (see Lemma 2.9 in [7]).

## 2. The Saks class $\mathfrak{S}$ of integrals and its subclass $\mathfrak{T}$

A *functional $S$ in $E$* is a mapping from a set of functions on $E$ into $\mathbb{R}$, i.e. $S$ is a set of pairs $(f, \gamma)$ ($f$ being a function $f\colon E \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ the value of the functional $S$) and it is assumed that $\gamma$ is uniquely determined by $f$. We write $\gamma = S(f)$.

The set of all $f$ for which the functional $S$ is defined will be denoted by $\mathrm{Dom}(S)$.

Denote the *characteristic function* of a set $M \subset E$ by $\chi(M)$, i.e. $\chi(M) = 1$ on $M$ and $\chi(M) = 0$ on $E \setminus M$. The characteristic function of the empty set $\emptyset$ may be denoted simply by 0.

If the product $f \cdot \chi(M)$ belongs to $\mathrm{Dom}(S)$, then $S(f, M)$ denotes the value of the functional $S$ for $f \cdot \chi(M)$, i.e. $S(f, M) = S(f \cdot \chi(M))$ and of course $S(f, E) = S(f)$ provided $f \in \mathrm{Dom}(S)$.

**Definition 2.1.** A functional $S$ in $E$ is called *additive* if the following two conditions hold:

A) $0 \in \mathrm{Dom}(S)$ and $S(0) = 0$,

B) if $c \in [a, b] = E$ and $I_1 = [a, c], I_2 = [c, b]$, then $f \in \mathrm{Dom}(S)$ if and only if $f \cdot \chi(I_1), f \cdot \chi(I_2) \in \mathrm{Dom}(S)$ and

$$S(f) = S(f, I_1) + S(f, I_2).$$

**Remark.** Let us note that B) means the following two conditions:

B1) if $f \in \mathrm{Dom}(S)$ then for every $c \in [a, b] = E$ and $I_1 = [a, c]$, $I_2 = [c, b]$ we have

$$f \cdot \chi(I_1), f \cdot \chi(I_2) \in \mathrm{Dom}(S),$$

B2) if there is a $c \in [a, b] = E$ such that
$f \cdot \chi(I_1), f \cdot \chi(I_2) \in \mathrm{Dom}(S)$ then $f \in \mathrm{Dom}(S)$,
and the equality

$$S(f) = S(f, I_1) + S(f, I_2) \tag{2.1}$$

holds in both cases.

Further, note that if $S$ is additive and $f \in \mathrm{Dom}(S)$, then

$$f \cdot \chi(I) \in \mathrm{Dom}(S) \tag{2.2}$$

for every $I \in \mathrm{Sub}(E)$.

Indeed, if e.g. $I = [c,d]$, then by B) we have $f \cdot \chi([a,d]) \in \mathrm{Dom}(S)$ and also $f \cdot \chi(I) = f \cdot \chi([a,d]) \cdot \chi([c,b]) \in \mathrm{Dom}(S)$.

If $I_1, I_2 \in \mathrm{Sub}(E)$ are such that $I_1 \cap \mathrm{Int}(I_2) = \emptyset$ ($I_1, I_2$ are non-overlapping intervals in $E$), then

$$f \cdot \chi(I_1 \cup I_2) \in \mathrm{Dom}(S) \text{ provided } f \cdot \chi(I_1), f \cdot \chi(I_2) \in \mathrm{Dom}(S)$$

and

$$S(f, I_1 \cup I_2) = S(f, I_1) + S(f, I_2). \tag{2.3}$$

For a given $I \in \mathrm{Sub}(E)$ denote by $l(I)$, $r(I)$ the left and right endpoints of $I$, respectively.

For showing (2.3) assume that $r(I_1) \leqslant l(I_2)$.

Take $c \in [r(I_1), l(I_2)]$. Then

$$\begin{aligned} f \cdot \chi(I_1 \cup I_2) \cdot \chi([a,c]) &= f \cdot \chi(I_1) \in \mathrm{Dom}(S), \\ f \cdot \chi(I_1 \cup I_2) \cdot \chi([c,b]) &= f \cdot \chi(I_2) \in \mathrm{Dom}(S) \end{aligned}$$

and by (2.1) we get $f \cdot \chi(I_1 \cup I_2) \in \mathrm{Dom}(S)$ as well as the equality $S(f, I_1 \cup I_2) = S(f, I_1 \cup [a,c]) + S(f, I_2 \cup [c,b]) = S(f, I_1) + S(f, I_2)$.

Denote by $\mathrm{Alg}(E)$ the set of all finite unions of closed subintervals of $E$ (i.e. unions of elements of all finite systems).

If $S$ is additive, $M \in \mathrm{Alg}(E)$ and $\{I_j;\ j \in \Gamma\}$ is a division of $M$, then $f \cdot \chi(M) \in \mathrm{Dom}(S)$ if and only if $f \cdot \chi(I_j) \in \mathrm{Dom}(S)$ for all $j \in \Gamma$ and

$$S(f, M) = \sum_{j \in \Gamma} S(f, I_j).$$

Moreover, $f \in \mathrm{Dom}(S)$ if and only if $f \cdot \chi(M), f \cdot \chi(\overline{E \setminus M}) \in \mathrm{Dom}(S)$ for every $M \in \mathrm{Alg}(E)$ and

$$S(f) = S(f, M) + S(f, \overline{E \setminus M}).$$

**Definition 2.2.** If $S$ is an additive functional in $E$ and $f \in \mathrm{Dom}(S)$, then a function $F\colon E \to \mathbb{R}$ is called an *$S$-primitive function* to $f$ provided

$$F[I] = S(f, I)$$

holds for every $I \in \mathrm{Sub}(E)$.

An $S$-primitive function to $f \in \mathrm{Dom}(S)$ always exists (e.g. $F(x) = S(f, [a, x])$ for $x \in E = [a, b]$ is an $S$-primitive to $f$) and it is determined uniquely up to a constant.

**Definition 2.3.** An additive functional $S$ in $E$ is called an *integral* in $E$ if all $S$-primitive functions to $f \in \mathrm{Dom}(S)$ are continuous in $E$.

Denote *the set of all integrals in* $E$ by $\mathfrak{S}$.

If $S \in \mathfrak{S}$ and $f \in \mathrm{Dom}(S)$, then $f$ is called $S$-integrable.

If $S \in \mathfrak{S}$ and $M \subset E$, then a function $f$ is said to be *$S$-integrable on $M$* if $f \cdot \chi(M) \in \mathrm{Dom}(S)$.

**Remark.** The properties (2.2) and (2.3) presented in the previous remark together with Definition 2.3 of an integral show that an integral in our sense is also an integral in the sense of S. Saks [6, VIII, § 4]. It is also easy to show that if $S$ satisfies the conditions presented by Saks, then $S$ is additive in the sense of Definition 2.1 and the continuity property of Definition 2.3 is satisfied, too. The general integral in the sense of Saks is equivalent to the integral in the sense of our Definition 2.3.

**Definition 2.4.** If $T, S \in \mathfrak{S}$ then $T$ *includes* $S$, and we write $S \sqsubset T$, provided $\mathrm{Dom}(S) \subset \mathrm{Dom}(T)$ and for $f \in \mathrm{Dom}(S)$ and every $I \in \mathrm{Sub}(E)$ the equality $T(f, I) = S(f, I)$ is satisfied ($f \cdot \chi(I) \in \mathrm{Dom}(S)$ holds by (2.2)).

The concept of $S \sqsubset T$ for $S, T \in \mathfrak{S}$ in the above definition follows the setting given in the book of S. Saks [6, VIII, § 4], see also [4].

By definition it can be checked easily that the following proposition holds.

**Proposition 2.5.** *If $R, S, T \in \mathfrak{S}$, then $R \sqsubset R$ (reflexivity); if $R \sqsubset S$ and $S \sqsubset T$ then $R \sqsubset T$ (transitivity). This means that by the relation $\sqsubset$ a partial ordering in $\mathfrak{S}$ is given.*

*Moreover, if $S \sqsubset T$ and $T \sqsubset S$ then $T = S$ (antisymmetry).*

The binary relation $\sqsubset$ on $\mathfrak{S}$ is an order and $(\mathfrak{S}, \sqsubset)$ is an ordered set.

In the sequel we restrict ourselves to a certain subclass of $\mathfrak{S}$.

**Definition 2.6.** $\mathfrak{T}$ denotes the set of all integrals $S \in \mathfrak{S}$ fulfilling the following conditions (2.4)–(2.8) ($N, A \subset E$, $\mu(A)$ is the Lebesgue measure of a set $A$, $f$ is a function on $E$ and $F$ is an $S$-primitive function to $f$):

$$\text{If } \mu(N) = 0, \text{ then } f \cdot \chi(N) \in \mathrm{Dom}(S) \text{ and } S(f, N) = 0. \tag{2.4}$$

$$\text{If } f \in \mathrm{Dom}(S), \text{ then } F \in C^*(E). \tag{2.5}$$

(For $C^*(E)$ see Definition 1.2).

$$\text{If } f \in \mathrm{Dom}(S), \text{ then } f \text{ is measurable.} \tag{2.6}$$

There exists $\lambda < \infty$ such that

$$W_F(A) \leqslant \lambda |f|_A \tag{2.7}$$

if $f \in \mathrm{Dom}(S)$ and $A$ is a closed set ($W_F(\cdot)$ is the full variational measure generated by the oscillation $\omega(F, I)$ for $I \in \mathrm{Sub}(E)$, see Definition 1.1).

$$S \text{ is a linear functional.} \tag{2.8}$$

The linearity of $S$ mentioned in (2.8) means that if $f, g \in \mathrm{Dom}(S)$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \mathrm{Dom}(S)$ and

$$S(\alpha f + \beta g) = \alpha S(f) + \beta S(g).$$

We close this section by a few simple consequences and remarks concerning the conditions (2.4)–(2.8).

The Lebesgue integral $L$ satisfies (2.4).

If $T, S \in \mathfrak{S}$, $S \sqsubset T$ while $T \in \mathfrak{T}$, then also $S \in \mathfrak{T}$.

**Definition 2.7.** $K$ denotes the set of all pairs $(f, \gamma)$, where $f$ is a function on $E$ and $\gamma \in \mathbb{R}$, such that for any $\varepsilon > 0$ there exists a gauge $\delta$ such that

$$\left| \sum_{j \in \Gamma} f(\tau_j)|I_j| - \gamma \right| < \varepsilon$$

for any $\delta$-fine division $(\{I_j\,;\ j \in \Gamma\}, \tau)$ of the interval $E$.

The value $\gamma \in \mathbb{R}$ is called the *Kurzweil-Henstock integral* of $f$ over $E$ and it will be denoted by $K(f)$ or $(K)\int_E f$.

Taking into account the definition of the Kurzweil-Henstock integral $K$ as presented in Definition 3.1 in [7] and its properties described especially by Proposition 3.3 and Lemma 3.7 from [7], we get immediately the following statement.

**Theorem 2.8.** *The Kurzweil-Henstock integral $K$ belongs to the class $\mathfrak{T}$ of integrals, i.e. it fulfils the conditions* (2.4)–(2.8)

Since the Kurzweil-Henstock integral $K$ includes the Newton ($N$), the Riemann ($R$) and the Lebesgue ($L$) integrals (i.e. $N \sqsubset K$, $R \sqsubset L \sqsubset K$), all these known integrals also belong to the subclass $\mathfrak{T}$ of integrals given by Definition 2.6.

Let us mention at this point that condition (2.7) is not very usual in abstract integration theory. Nevertheless, the known reasonable integrals satisfy this condition because they are included in the integral $K$ as was mentioned before.

Condition (2.6) makes it possible to use results like the following version of Egoroff's theorem:

**Proposition 2.9.** *Let $(f_j)$, $j \in \mathbb{N}$ be a sequence of measurable functions which converges pointwise almost everywhere (a.e.) in $E$ to a function $f$, i.e.*

$$\lim_{j\to\infty} f_j(x) = f(x) \quad \text{for } x \in E \setminus M,$$

*where $M$ is measurable and $\mu(M) = 0$.*

*Then there exists a sequence $(A_k)$ of closed subsets of $E$ and a subsequence $(g_j)$ of $(f_j)$ such that $A_k \nearrow E \setminus N$,[1] where $N \subset E$ with $\mu(N) = 0$ and*

$$|f - g_j|_{A_k} \to 0 \quad \text{for } j \to \infty$$

*for any $k \in \mathbb{N}$.*

(See e.g. [2, Theorem 6.3] or [3, Theorem 2.13] for this known result.) This leads to a more general form of the condition (2.7) in comparison to the results of [7, Proposition 2.12 and Lemma 2.13].

**Lemma 2.10.** *If $S \in \mathfrak{S}$ satisfies* (2.5) *and* (2.7), *then for every $f \in \mathrm{Dom}(S)$ the inequality*

$$W_F(M) \leqslant \lambda |f|_M \tag{2.9}$$

*holds provided the set $M \subset E$ is measurable and $F$ is an $S$-primitive function to $f$.*

**Remark.** Note that if $S \in \mathfrak{T}$, then (2.9) holds for any measurable $M \subset E$.

## 3. Extension of integrals

First of all, we recollect some basic notions on partially ordered systems.

A non-empty set $\mathscr{E}$ with a relation $\sqsubset$ on $\mathscr{E}$, for which

(a) $a \sqsubset b$ and $b \sqsubset c$ implies $a \sqsubset c$,

(b) $a \sqsubset a$

for $a, b, c \in \mathscr{E}$, is a *partially ordered system* $(\mathscr{E}, \sqsubset)$ with the *order relation* $\sqsubset$.

If $\mathscr{F}$ is a subset of the partially ordered system $(\mathscr{E}, \sqsubset)$, then an element $M \in \mathscr{E}$ ($m \in \mathscr{E}$) is called a *majorant* (*minorant*) for $\mathscr{F}$ if $T \sqsubset M$ ($m \sqsubset T$) for every $T \in \mathscr{F}$.

A majorant $\overline{M} \in \mathscr{E}$ (minorant $\underline{m} \in \mathscr{E}$) of $\mathscr{F}$ is said to be the *least upper bound* (*greatest lower bound*) of $\mathscr{F}$ if every majorant $M$ (minorant $m$) of $\mathscr{F}$ has the property $\overline{M} \sqsubset M$ ($m \sqsubset \underline{m}$).

---

[1] If $M \subset E$ then $A_k \nearrow M$ means that $A_k \subset A_{k+1} \subset M$ for $k \in \mathbb{N}$ and that for every $c \in M$ there exists $k_0 \in \mathbb{N}$ such that $c \in A_{k_0}$.

A subset $\mathscr{F}$ of $(\mathscr{E}, \sqsubset)$ is totally ordered if for every pair $T_1, T_2 \in \mathscr{F}$ we have either $T_1 \sqsubset T_2$ or $T_2 \sqsubset T_1$.

Theorem 5 from Section I.2 in [1] reads as follows:

**Theorem 3.1.** *Assume that $P\colon \mathscr{E} \to \mathscr{E}$ is a mapping such that $T \sqsubset P(T)$ where $(\mathscr{E}, \sqsubset)$ is a nonempty partially ordered set which satisfies:*

(a) *If $R, T \in \mathscr{E}$, $R \sqsubset T$ and $T \sqsubset R$, then $R = T$.*

(b) *Every totally ordered subset of $\mathscr{E}$ possesses a least upper bound.*

*Then in $\mathscr{E}$ there is an element $T$ for which $P(T) = T$.*

An element $M \in \mathscr{E}$ ($m \in \mathscr{E}$) is called *maximal (minimal)* if the relation $M \sqsubset T$ ($T \sqsubset m$) for $T \in \mathscr{E}$ implies $T \sqsubset M$ ($m \sqsubset T$), i.e. $T = M$ ($T = m$).

We will use the following well known statement (see e.g. Theorem 7 from Section I.2 and Exercise 1. from Section I.3 in [1]).

**Lemma 3.2** (Zorn's Lemma). *If every totally ordered subset of a partially ordered set $(\mathscr{E}, \sqsubset)$ has a majorant (minorant), then there is a maximal (minimal) element in $\mathscr{E}$.*

Let us start with some necessary concepts and definitions concerning integrals.

**Definition 3.3.** A mapping $Q\colon \mathfrak{S} \to \mathfrak{S}$ defined on $\mathrm{Dom}(Q) \subset \mathfrak{S}$ is called an *extension* if for every $S \in \mathrm{Dom}(Q)$ we have $S \sqsubset Q(S)$, $Q(S) \in \mathrm{Dom}(Q)$ and, moreover, if $S_1, S_2 \in \mathrm{Dom}(Q) \subset \mathfrak{S}$ with $S_1 \sqsubset S_2$, then $Q(S_1) \sqsubset Q(S_2)$.

The extension $Q$ is called *effective* if $Q^2 = Q$, i.e. if $Q(Q(S)) = Q(S)$ for every $S \in \mathrm{Dom}(Q)$.

Let us mention that, by definition, an extension can be applied repeatedly and that it is "monotone".

**Definition 3.4.** An integral $S$ is called *invariant with respect to an extension $Q$* (shortly *$Q$-invariant*) if $S \in \mathrm{Dom}(Q)$ and $Q(S) = S$.

**Remarks.**

a) Note that $S$ is invariant with respect to an extension $Q$ if and only if $Q(S) \sqsubset S$ since the relation $S \sqsubset Q(S)$ is required by Definition 3.3.

b) If $Q$ is an effective extension, then for $S \in \mathrm{Dom}(Q)$ the image $Q(S)$ is $Q$-invariant.

c) Assume that $S, T \in \mathfrak{S}$ are integrals such that $S \sqsubset T$ and let $Q$ be an extension.

Asserting $T \sqsubset Q(S)$ we get in fact a statement which gives necessary conditions for a function $f$ to be $T$-integrable in terms of its $S$-integrability only. The relation $S \sqsubset T$ represents a simple sufficient condition for $T$-integrability of a function (if $f$ is

$S$-integrable then it is $T$-integrable). If, in addition, we have $Q(S) \sqsubset T$, then this relation gives also sufficient conditions for $T$-integrability of a function $f$ in terms of the integral $S$ and these conditions can, in some sense, be better than the conditions coming from $S \sqsubset T$ only.

d) Assume we have two extensions $Q_1$, $Q_2$ such that $Q_2(R) \sqsubset Q_1(R)$ for all $R \in \mathrm{Dom}(Q_1) \cap \mathrm{Dom}(Q_2)$.

If $S, T \in \mathfrak{S}$, $S \sqsubset T$ with $S \in \mathrm{Dom}(Q_1) \cap \mathrm{Dom}(Q_2)$, then the statement $T \sqsubset Q_2(S)$ is stronger than $T \sqsubset Q_1(S)$ and it may produce in a certain sense better conditions for $T$-integrability of a function $f$ even if $Q_1(S) = Q_2(S)$.

Statements of the type $Q_j(S) \sqsubset T$, $j = 1, 2$ give sufficient conditions for $T$-integrability in terms of $S$-integrability. In our situation the condition $Q_1(S) \sqsubset T$ is considered to be "better" than $Q_2(S) \sqsubset T$.

Let us mention some possibilities how to examine the validity of the relations $Q(S) \sqsubset T$ or $T \sqsubset Q(S)$ for the case $S \sqsubset T$.

The relation $Q(S) \sqsubset T$ is valid e.g. in the case that $T$ is $Q$-invariant. Indeed, $S \sqsubset T$ implies that $Q(S) \sqsubset Q(T) = T$.

Moreover, the equality $Q(S) = T$ can be often regarded as a certain convergence theorem for the integral $T$. More precisely, an appropriate convergence result for the integral $T$ may give sufficient conditions for $T$- integrability in terms of the integral $S$.

For discussing the relation $T \sqsubset Q(S)$ the following concept will be useful.

**Definition 3.5.** Suppose $\mathfrak{R} \subset \mathfrak{S}$ is some set of integrals.

If $S \in \mathfrak{R}$ and $\mathscr{P}$ is a set of extensions defined on the whole $\mathfrak{R}$ ($\mathrm{Dom}(P) = \mathfrak{R}$ for every $P \in \mathscr{P}$), then $\mathrm{Min}(\mathscr{P}; S)$ denotes the *minimal $\mathscr{P}$-invariant integral containing* $S$ (if it exists, of course), i.e. $T = \mathrm{Min}(\mathscr{P}; S)$ if and only if

(i) $S \sqsubset T$

and

(ii) if $R \in \mathfrak{R}$ is such that $S \sqsubset R$ and $P(R) = R$ for all $P \in \mathscr{P}$, then $T \sqsubset R$ holds.

If the set $\mathscr{P}$ consists of only one extension $P$, then we denote in some situations $\mathrm{Min}(\mathscr{P}; S) = \mathrm{Min}(P; S) = \overline{P}(S)$.

Given an extension $Q$ we can use Definition 3.5 to state that the relation $T \sqsubset Q(S)$ holds for $T = \mathrm{Min}(\mathscr{P}; S)$, if $Q(S)$ is $P$-invariant for $P \in \mathscr{P}$, i.e. $P(Q(S)) = Q(S)$ for $P \in \mathscr{P}$.

If $Q$ is effective, $P(R) \sqsubset Q(R)$ for $R \in \mathrm{Dom}(Q)$ and $P \in \mathscr{P}$, then $P(Q(S)) = Q(S)$ holds. In fact, in this situation we have $P(Q(S)) \sqsubset Q^2(S) = Q(S)$ and $P(Q(S)) = Q(S)$.

**Lemma 3.6.** *Suppose $\mathfrak{R} \subset \mathfrak{S}$ is some set of integrals.*

*Assume that $\mathscr{P} = (P_1, P_2, \ldots, P_n)$, where $P_j$, $j = 1, \ldots, n$ are extensions defined on $\mathfrak{R}$ ($\mathrm{Dom}(P_j) = \mathfrak{R}$, $j = 1, \ldots, n$) and that $S \in \mathfrak{R}$ is given.*

*Then there exists $T_0 \in \mathfrak{R}$ with $S \sqsubset T_0$ such that $T_0$ is $P_j$-invariant for all $j = 1, \ldots, n$.*

P r o o f. The relation $\sqsubset$ represents a partial ordering in $\mathfrak{S}$ and this ordering has the property that $R, T \in \mathfrak{S}$, $T \sqsubset R$, $R \sqsubset T$ implies $R = T$, see Proposition 2.5. For a subset $\mathfrak{R} \subset \mathfrak{S}$ this partial ordering is inherited from $\mathfrak{S}$, i.e. the partial ordering in subsets of $\mathfrak{R}$ is induced by the partial ordering $\sqsubset$ given in $\mathfrak{S}$.

Let us set

$$\mathscr{E} = \{T \in \mathfrak{R};\ S \sqsubset T\}.$$

We have $\mathscr{E} \neq \emptyset$ because $S \in \mathscr{E}$.

Assume that $\mathscr{M} \subset \mathscr{E}$ is an arbitrary totally ordered subset in $\mathscr{E}$, i.e. if $T_1, T_2 \in \mathscr{M}$ then either $T_1 \sqsubset T_2$ or $T_2 \sqsubset T_1$.

Put

$$\mathrm{Dom}(R) = \bigcup_{T \in \mathscr{M}} \mathrm{Dom}(T)$$

and assume that $f \in \mathrm{Dom}(R)$. Then, by definition, there is a $T_1 \in \mathscr{M}$ such that $f \in \mathrm{Dom}(T_1)$.

Define

$$R(f) = T_1(f) = \gamma.$$

This value is determined uniquely. Indeed, if $f \in \mathrm{Dom}(T_2)$, $T_2 \in \mathscr{M}$, $T_1 \neq T_2$ then we have either $T_1 \sqsubset T_2$ and then $\mathrm{Dom}(T_1) \subset \mathrm{Dom}(T_2)$ and $T_1(f) = T_2(f) = \gamma$, or $T_2 \sqsubset T_1$ and then $\mathrm{Dom}(T_2) \subset \mathrm{Dom}(T_1)$ and again $T_2(f) = T_1(f) = \gamma$.

It is a matter of routine to show that $R$ is a functional satisfying the additivity conditions A), B) from Definition 1.1 and that any $R$-primitive function to $f \in \mathrm{Dom}(R)$ coincides with some $T_1$-primitive function $f \in \mathrm{Dom}(T_1)$ which is continuous (see Definition 2.3) because $T_1 \in \mathfrak{R}$. Hence $R \in \mathfrak{S}$ and of course $S \sqsubset R$ since $\mathscr{M} \subset \mathscr{E}$.

Since $T \sqsubset R$ for every $T \in \mathscr{M}$, $R$ is a majorant to $\mathscr{M}$. If $T^*$ is another majorant to $\mathscr{M}$ then $T \sqsubset T^*$ for every $T \in \mathscr{M}$, i.e.

$$\mathrm{Dom}(T) \subset \mathrm{Dom}(T^*) \quad \text{for every } T \in \mathscr{M}$$

and therefore

$$\mathrm{Dom}(R) = \bigcup_{T \in \mathscr{M}} \mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$$

and

$$R(f, I) = T^*(f, I) \quad \text{for any } I \in \mathrm{Sub}(E).$$

This yields $R \sqsubset T^*$ and therefore $R$ is a least upper bound of $\mathscr{M}$.

All the assumptions being satisfied, the statement of Theorem 3.1 can be used for asserting that there is a $T \in \mathscr{E}$ such that $P(T) = T$. This yields the statement of Lemma 3.6 for $n = 1$.

For $T \in \mathscr{E}$ define further

$$P(T) = P_1(P_2(\ldots P_n(T)\ldots)).$$

For this composition of extensions defined on the whole $\mathfrak{R}$ we have

$$T \sqsubset P_n(T) \sqsubset P_{n-1}(P_n(T)) \sqsubset \ldots \sqsubset P_1(P_2(\ldots P_n(T)\ldots)) = P(T)$$

for any $T \in \mathscr{E}$. Clearly $\mathrm{Dom}(P) = \mathfrak{R}$ and $S \sqsubset P(T)$, i.e. $P(T) \in \mathscr{E}$.

It is easy to show that if $T_1 \sqsubset T_2$, then $P(T_1) \sqsubset P(T_2)$ and therefore $P\colon \mathscr{E} \to \mathscr{E}$ is an extension. Hence, by the result recalled above, there is a $T_0 \in \mathscr{E}$ such that $P(T_0) = T_0$.

For this $T_0 \in \mathscr{E}$ we have

$$T_0 \sqsubset P_n(T_0) \sqsubset P_{n-1}(P_n(T_0)) \sqsubset \ldots \sqsubset P_1(P_2(\ldots P_n(T_0)\ldots)) = P(T_0) = T_0,$$

which implies $P_n(T_0) = T_0$ and

$$T_0 \sqsubset P_{n-1}(T_0) \sqsubset P_{n-2}(P_{n-1}(T_0)) \sqsubset \ldots \sqsubset P_1(P_2(\ldots P_{n-1}(T_0)\ldots)) = P(T_0) = T_0.$$

In this way we obtain successively $P_{n-1}(T_0) = T_0, \ldots, P_1(T_0) = T_0$ and this completes the proof. □

Lemma 3.6 and the "dual" version of Zorn's Lemma 3.2 leads to the following

**Corollary 3.7.** *Suppose $\mathfrak{R} \subset \mathfrak{S}$ is some set of integrals.*

*Assume that $\mathscr{P} = (P_1, P_2, \ldots, P_n)$ where $P_j$, $j = 1, \ldots, n$ are extensions defined on $\mathfrak{R}$ ($\mathrm{Dom}(P_j) = \mathfrak{R}$, $j = 1, \ldots, n$) and that $S \in \mathfrak{R}$ is given.*

*Then there exists $T_* \in \mathfrak{R}$ such that $S \sqsubset T_*$, $P_j(T_*) = T_*$ for $j = 1, \ldots, n$ and for every $T \in \mathfrak{R}$ such that $S \sqsubset T$ and $P_j(T) = T$ for $j = 1, \ldots, n$ we have*

$$\mathrm{Min}(\mathscr{P}; S) = T_* \sqsubset T$$

*(cf. Definition 3.5).*

P r o o f. Define

$$\mathscr{E} = \{T \in \mathfrak{R};\ S \sqsubset T,\ P_j(T) = T \text{ for } j = 1, \ldots, n\}.$$

By Lemma 3.6 the set $\mathscr{E}$ is nonempty. If $\mathscr{M} \subset \mathscr{E}$ is totally ordered then we set

$$\operatorname{Dom}(R_*) = \bigcap_{T \in \mathscr{M}} \operatorname{Dom}(T).$$

If $f \in \operatorname{Dom}(R_*)$ we have $f \in \operatorname{Dom}(T)$ for all $T \in \mathscr{M}$ and we define

$$R_*(f) = T_1(f) = \gamma,$$

where $T_1$ is an arbitrary element of $\mathscr{M}$. The value $R_*(f)$ is determined uniquely, $R_* \in \mathfrak{R}$ and $R_*$ is a minorant of $\mathscr{M}$.

Let us check these facts. Assume that $T_2 \in \mathscr{M}$, $T_1 \neq T_2$, where $T_1 \in \mathscr{M}$ determines $R_*$. Then either $T_1 \sqsubset T_2$ or $T_2 \sqsubset T_1$. In both cases we obtain by definition that for $f \in \operatorname{Dom}(R_*)$ we have $T_1(f) = T_2(f) = \gamma$ and therefore $(f, \gamma) \in R_*$.

The properties A), B) from Definition 2.1 as well as the fact that every $R_*$-primitive function to $f \in \operatorname{Dom}(R_*)$ is continuous can be checked easily. Hence $R_* \in \mathfrak{R}$.

Since $\operatorname{Dom}(R_*) \sqsubset \operatorname{Dom}(T)$ for every $T \in \mathscr{M}$ and for $f \in \operatorname{Dom}(R_*)$ we have $R_*(f) = T(f)$, we get $R_* \sqsubset T$ for every $T \in \mathscr{M}$ and $R_*$ is a minorant to the totally ordered set $\mathscr{M}$.

Using Zorn's lemma 3.2 we obtain that there is a minimal element $T_*$ in $\mathscr{E}$, i.e. for every $T \in \mathfrak{R}$ such that $S \sqsubset T$ and $P_j(T) = T$ for $j = 1, \ldots, n$ we have $T_* \sqsubset T$ and the statement is proved. It can be seen immediately that $T_* = \operatorname{Min}(\mathscr{P}; S)$ according to Definition 3.5. □

## 4. Cauchy and Harnack extensions

**Definition 4.1.** If $f$ is a function on $E$ and $S \in \mathfrak{S}$, then $x \in E$ is called an *$S$-regular point of $f$* if there is an $I \in \operatorname{Sub}(E)$ such that $x \in \operatorname{Int}(I)$ (the interior of $I$) and $f \cdot \chi(I) \in \operatorname{Dom}(S)$.

The set of all $S$-regular points of $f$ is denoted by $\varrho(f, S)$.

The complement $\sigma(f, S) = E \setminus \varrho(f, S)$ of $\varrho(f, S)$ in $E$ is called the set of *$S$-singular points of the function $f$.*

If $I \in \operatorname{Sub}(E)$ contains one or both endpoints of $E$, then we consider them as points belonging to $\operatorname{Int}(I)$.

The set $\sigma(f, S)$ is closed because $\varrho(f, S)$ is evidently open by definition. Moreover, $\sigma(f, S) = \emptyset$ if and only if $f \in \operatorname{Dom}(S)$. (See also [2, 9.1 Theorem].)

**Definition 4.2.** For $S \in \mathfrak{S}$ denote by $S_C$ the set of all pairs $(f, \gamma)$, where $f$ is a function on $E$ and $\gamma \in \mathbb{R}$, such that $\sigma(f, S)$ is a finite set for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$ and for every $I \subset \varrho(f, S)$ we have $f \cdot \chi(I) \in \operatorname{Dom}(S)$ and $F[I] = S(f, I)$.

For $I \in \operatorname{Sub}(E)$ put $S_C(f, I) = F[I]$.

The set $\{(S, S_C);\ S \in \mathfrak{S},\ S_C \text{ exists}\}$ is denoted by $P_C$.

It is easy to see that $S_C \in \mathfrak{S}$ and that if $S \in \mathfrak{T}$, then $S_C \in \mathfrak{T}$.

Denoting $\operatorname{Dom}(P_C) = \{S \in \mathfrak{S};\ S_C \text{ exists}\}$ $P_C$ can be interpreted as a map $\mathfrak{S} \to \mathfrak{S}$ and it can be shown easily that $P_C$ is an extension (i.e. if $S \in \operatorname{Dom}(P_C)$ then $S_C \in \mathfrak{S}$, $S \sqsubset S_C$ for $S \in \mathfrak{S}$ and if $S_1 \sqsubset S_2$ then $P_C(S_1) \sqsubset P_C(S_2)$).

The map $P_C$ is called the *Cauchy extension.* Our setting of this concept is in good accordance with the similar concept presented in the book [6] of S. Saks.

The function $F$ occurring in Definition 4.2 is an $S_C$-primitive function to $f$.

Note that an integral $S$ is $P_C$-invariant (i.e. $P_C(S) = S_C \sqsubset S$) if and only if for the integral $S$ the following statement holds.

**Theorem 4.3** (Hake's Theorem). *If $I = [c, d] \in \operatorname{Sub}(E)$ then $f \cdot \chi(I) \in \operatorname{Dom}(S)$ if and only if for every $c < \alpha < \beta < d$ we have $f \cdot \chi_{[\alpha,\beta]} \in \operatorname{Dom}(S)$ and*

$$\lim_{\alpha \to c+,\, \beta \to d-} S(f \cdot \chi_{[\alpha,\beta]}) = A \in \mathbb{R}.$$

*In this case $S(f, I) = A$.*

Note further that Hake's theorem holds for the Kurzweil-Henstock integral $K$ (see Theorem 3.6 in [7] where the result of Theorem 9.21 from [3] is recalled) and therefore we get the known fact that the Kurzweil-Henstock integral $K$ is $P_C$-invariant.

For example $\operatorname{Min}(P_C; L) = \overline{P_C}(L)$, with $L$ the Lebesgue integral, can be characterized as follows.

$f \in \operatorname{Dom}(\overline{P_C}(L))$ if and only if $\sigma(f, L)$ is at most countable and there is a function $F \in C(E)$ such that $F[I] = L(f, I)$ for every $I \in \operatorname{Sub}(E)$ for which $I \subset \varrho(f, L)$.

The function $F$ is the $\overline{P_C}(L)$-primitive to $f$ in this case.

If $I \in \operatorname{Sub}(E)$ and $A \subset E$ is closed then denote by $\operatorname{Comp}(I, A)$ *the set of all (maximal and nonempty) connected components* of the set $I \setminus A$.

Note that for a function $f$ and some $S \in \mathfrak{S}$ the set $\operatorname{Comp}(E, \sigma(f, S))$ of maximal connected components of the complement $E \setminus \sigma(f, S)$ either consists of one element (in the case that $f \in \operatorname{Dom}(S)$ and $\sigma(f, S) = \emptyset$) or it has an at most countable number of elements consisting of open intervals contiguous to the closed set $\sigma(f, S)$.

Now we define the Harnack extension presented in [6] for the use with the special Denjoy (Denjoy-Perron) integral which is known to be equivalent to the Kurzweil-Henstock integral $K$ (see [3]).

**Definition 4.4.** For $S \in \mathfrak{S}$ denote by $S_H$ the set of all pairs $(f, \gamma)$, where $f$ is a function on $E$ and $\gamma \in \mathbb{R}$, for which $f \cdot \chi(\sigma(f, S)) \in \mathrm{Dom}(S)$, $f \cdot \chi(U_j) \in \mathrm{Dom}(S)$ for $j \in \Gamma$, where $\{U_j;\ j \in \Gamma\} = \mathrm{Comp}(E, \sigma(f, S))$ and for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$,

$$\sum_{U \in \mathrm{Comp}(E, \sigma(f,S))} \omega(F, \overline{U}) = \sum_{j \in \Gamma} \omega(F, \overline{U}) < \infty$$

and

$$F[I] = S(f, I \cap \sigma(f, S)) + \sum_{j \in \Gamma} S(f, I \cap \overline{U_j})$$

for any $I \in \mathrm{Sub}(E)$. ($\omega(F, \overline{U})$ is the oscillation of $F$ over the interval $\overline{U}$.)

The set $\{(S, S_H);\ S \in \mathfrak{S},\ S_H \text{ exists}\}$ is denoted by $P_H$.

As before, $P_H$ is a map $\mathfrak{S} \to \mathfrak{S}$. Let us call it the *Harnack extension.*

In [6] also the following concept is presented.

**Definition 4.5.** For $S \in \mathfrak{S}$ denote by $S_{\mathscr{H}}$ the set of all pairs $(f, \gamma)$, where $f$ is a function on $E$ and $\gamma \in \mathbb{R}$, for which there is a closed set $Q \subset E$ such that $f \cdot \chi(Q) \in \mathrm{Dom}(S)$ and $f \cdot \chi(U_j) \in \mathrm{Dom}(S)$ for $j \in \mathbb{N}$, $\{U_j;\ j \in \mathbb{N}\} = \mathrm{Comp}(E, Q)$, where $\sum\limits_{j \in \mathbb{N}} |S(f, \overline{U_j})| < \infty$ and for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$ and

$$F[I] = S(f, I \cap Q) + \sum_{j \in \Gamma} S(f, I \cap \overline{U_j})$$

for any $I \in \mathrm{Sub}(E)$, while

$$\omega(S(f, \overline{U_j} \cap [a, x]), \overline{U_j}) = \omega(F, \overline{U_j}) \to 0 \quad \text{for } j \to \infty.$$

The set $\{(S, S_{\mathscr{H}});\ S \in \mathfrak{S}\}$ is denoted by $P_{\mathscr{H}}$.

$P_{\mathscr{H}}$ is a map $\mathfrak{S} \to \mathfrak{S}$. It is an extension called the *wide Harnack extension.*

It is known (see e.g. [4]) that $\mathrm{Min}(\{P_C, P_{\mathscr{H}}\}, L) = D$, where $D$ is the general Denjoy integral.

In [6] Saks deals with both types of Harnack extensions $P_H$, $P_{\mathscr{H}}$ while we use the first and simpler one only.

Finally, let us mention also the following concept.

**Definition 4.6.** For $S \in \mathfrak{S}$ denote by $\widetilde{S_H}$ the set of all pairs $(f, \gamma)$ where $f$ is a function on $E$ and $\gamma \in \mathbb{R}$, for which there is a closed set $Q \subset E$ such that $f \cdot \chi(Q) \in$

$\mathrm{Dom}(S)$ and $f \cdot \chi(U_j) \in \mathrm{Dom}(S)$ for $j \in \Gamma$ where $\{U_j\,;\ j \in \Gamma\} = \mathrm{Comp}(E, Q)$ and for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$,

$$\sum_{U \in \mathrm{Comp}(E,Q)} \omega(F, \overline{U}) < \infty$$

and

$$F[I] = S(f, I \cap Q) + \sum_{j \in \Gamma} S(f, I \cap \overline{U_j})$$

for any $I \in \mathrm{Sub}(E)$.

The set $\{(S, \widetilde{S_H})\,;\ S \in \mathfrak{S},\ \widetilde{S_H} \text{ exists}\}$ is denoted by $\widetilde{P_H}$.

$\widetilde{P_H}$ is a map $\mathfrak{S} \to \mathfrak{S}$. It is the so-called *(general) Harnack extension.*

Note that for the set $Q$ in Definition 4.6 we have $\mathrm{Comp}(E, Q) \subset \varrho(f, S)$ and therefore $\sigma(f, S) \subset Q$ for $f \in \mathrm{Dom}(S_H)$.

For Cauchy and Harnack extensions see also [5, Section 7 in Chapter 2] or [2, Chapter 9].

We recall the following result presented by Gordon in [3, Theorem 9.22] for the Kurzweil-Henstock integral $K$.

**Theorem 4.7.** *Let $Q \subset E$ be a closed set, $\{U_j\,;\ j \in \Gamma\} = \mathrm{Comp}(E, Q)$. Let $f\colon E \to \mathbb{R}$ be such that $f \cdot \chi(Q) \in \mathrm{Dom}(K)$ and $f \cdot \chi(\overline{U_j}) \in \mathrm{Dom}(K)$ for $j \in \Gamma$. If*

$$\sum_{j \in \Gamma} \omega(K(f, \overline{U_j} \cap [a, x]), \overline{U_j}) < \infty,$$

*then $f \in \mathrm{Dom}(K)$ and*

$$K(f) = K(f \cdot \chi(Q)) + \sum_{j \in \Gamma} K(f, \overline{U_j}).$$

Let us mention that if $I \in \mathrm{Sub}(E)$ then we also have

$$K(f, I) = K(f \cdot \chi(Q), I) + \sum_{j \in \Gamma} K(f, \overline{U_j} \cap I)$$

in the situation of Theorem 4.7. Looking at this result in connection with Definition 4.6 of the (general) Harnack extension $\widetilde{P_H}$, we can see immediately that $\widetilde{P_H}(K) \sqsubset K$ and this yields the well known fact that the Kurzweil-Henstock integral $K$ is $\widetilde{P_H}$-invariant. Putting $Q = \sigma(f, K)$ in Theorem 4.7 we can see also that the Kurzweil-Henstock integral $K$ is $P_H$-invariant.

**Lemma 4.8.** *Assume that $S \in \mathfrak{S}$ is $P_C$ invariant, i.e. $P_C(S) = S$.*

*Assume further that for every $x \in \operatorname{Int}(E)$ there is a $J_x \in \operatorname{Sub}(E)$ with $x \in \operatorname{Int}(J_x)$ such that if $f \cdot \chi(J_x) \in \operatorname{Dom}(K)$, then $f \cdot \chi(J_x) \in \operatorname{Dom}(S)$ while for any $I \subset J_x$ the equality $K(f, I) = S(f, I)$ holds.*

*Then $K \sqsubset S$ (i.e. $\operatorname{Dom}(K) \subset \operatorname{Dom}(S)$ and for every $I \in \operatorname{Sub}(E)$ the equality $K(f, I) = S(f, I)$ is valid for $f \in \operatorname{Dom}(K)$).*

Proof. Assume that $f \in \operatorname{Dom}(K)$ and that $I \in \operatorname{Sub}(E)$, $I \in \operatorname{Int}(E)$. Then by (2.2) we have $f \cdot \chi(I) \in \operatorname{Dom}(K)$. By the assumption and by the Borel covering theorem there is a finite sequence of $J_{x_k} \in \operatorname{Sub}(E)$, $k = 1, \ldots, n$, $x_k \neq x_l$ for $k \neq l$ such that $I \subset \bigcup_{k=1}^{n} J_{x_k}$ and $f \cdot \chi(J_{x_k}) \in \operatorname{Dom}(S)$, $k = 1, \ldots, n$. Without loss of generality it can be assumed that the intervals $J_{x_k}$, $k = 1, \ldots, n$ are non-overlapping. Therefore $f \cdot \chi\Big(\bigcup_{k=1}^{n} J_{x_k}\Big) \in \operatorname{Dom}(S)$ and

$$K\Big(f, \bigcup_{k=1}^{n} J_{x_k}\Big) = \sum_{k=1}^{n} K(f, J_{x_k}) = \sum_{k=1}^{n} S(f, J_{x_k}) = S\Big(f, \bigcup_{k=1}^{n} J_{x_k}\Big).$$

Hence $f \cdot \chi(I) \in \operatorname{Dom}(S)$ and we have $K(f, I) = S(f, I)$ for an arbitrary $I \in \operatorname{Int}(E)$.

Since we assume that $S = P_C(S)$ and we know that the Kurzweil-Henstock integral $K$ is also $P_C$-invariant we obtain by Hake's Theorem 4.3 immediately that $f \in \operatorname{Dom}(S)$, $K(f) = S(f)$ and by the result presented before also $K(f, I) = S(f, I)$ for an arbitrary $I \in \operatorname{Int}(E)$. Hence we have obtained $K \sqsubset S$. $\square$

In [6, (1.4) Theorem on p. 244] Saks presents the following result for the Denjoy-Perron integral.

*If a function $f$ is Denjoy-Perron integrable on $E$, then every closed subset of $E$ contains a portion $P$ such that the function $f$ is summable on $\overline{P}$ and such that the series of the oscillations of the indefinite Denjoy-Perron integrals of $f$ over the intervals contiguous to $\overline{P}$ is convergent.*

Reformulating this, using our notation and the known fact that the Denjoy-Perron integral coincides with the Kurzweil-Henstock integral $K$ (see e.g. [3]), we get the following

**Theorem 4.9.** *If $f \in \operatorname{Dom}(K)$, then for every closed set $Q \subset E$ there exists an interval $J \in \operatorname{Sub}(E)$ with $\operatorname{Int}(J) \cap Q \neq \emptyset$ such that $f \cdot \chi(J \cap Q) = f \cdot \chi(J) \cdot \chi(Q) \in \operatorname{Dom}(L)$ ($L$ is the Lebesgue integral) and*

$$\sum_{U \in \operatorname{Comp}(E, J \cap Q)} \omega(F, \overline{U}) < \infty,$$

*where $F \in C(E)$ is a $K$-primitive function to $f$.*

**Theorem 4.10.** *Assume that $S \in \mathfrak{S}$, where $L \sqsubset S$ and $P_C(S) = P_H(S) = S$. Then $K \sqsubset S$.*

In other words, the Kurzweil-Henstock integral $K$ is contained in every integral which contains the Lebesgue integral $L$ and which is both $P_C$- and $P_H$-invariant.

Proof. Assume that $f \in \operatorname{Dom}(K)$.

Denote $R = \varrho(f, S)$. The set $R \subset E$ is open. (See Definition 4.1.)

Let us set $Q = E \setminus R = \sigma(f, S)$. $Q \subset E$ is a closed set.

By Theorem 4.9 there is an interval $J \in \operatorname{Sub}(E)$ such that $\operatorname{Int}(J) \cap Q \neq \emptyset$, $f \cdot \chi(J \cap Q) = f \cdot \chi(J) \cdot \chi(Q) \in \operatorname{Dom}(L)$ and

$$\sum_{U \in \operatorname{Comp}(E, J \cap Q)} \omega(F, \overline{U}) < \infty, \tag{4.1}$$

where $F \in C(E)$ is a $K$-primitive function to $f$.

Since $L \sqsubset S$ and $L \sqsubset K$ we have $f \cdot \chi(J \cap Q) \in \operatorname{Dom}(S)$ and also $f \cdot \chi(J \cap Q) \in \operatorname{Dom}(K)$ and of course

$$K(f \cdot \chi(J \cap Q)) = L(f \cdot \chi(J \cap Q)) = S(f \cdot \chi(J \cap Q)).$$

Assume that $\operatorname{Comp}(J, Q) = \{U_j ;\ j \in \Gamma\}$. Using the definition of the set $Q$ and the fact that $U_j \cap Q = \emptyset$ for all $j \in \Gamma$, we know that for every $x \in U_j$ there is an interval $J_x \subset U_j$ such that $f \cdot \chi(J_x) = f \cdot \chi(U_j) \cdot \chi(J_x) \in \operatorname{Dom}(S)$ and for every $I \in \operatorname{Sub}(J_x)$ we have $K(f, I) = S(f, I)$. Hence by Lemma 4.8 we have $K(f, U_j) = S(f, U_j)$ for $j \in \Gamma$. Using (4.1) we can see by Definition 4.6 that $f \cdot \chi(J) \in \operatorname{Dom}(P_H(K))$ and at the same time also $f \cdot \chi(J) \in \operatorname{Dom}(P_H(S))$. Since $S$ is assumed to be $P_H$-invariant and $K$ is also $P_H$-invariant (see Theorem 4.7) we obtain

$$\begin{aligned} S(f \cdot \chi(J), I) &= S(f \cdot \chi(J), I \cap Q) + \sum_{j \in \Gamma} S(f \cdot \chi(J), I \cap \overline{U_j}) \\ &= K(f \cdot \chi(J), I \cap Q) + \sum_{j \in \Gamma} K(f \cdot \chi(J), I \cap \overline{U_j}) \\ &= K(f \cdot \chi(J), I) \end{aligned}$$

for any $I \in \operatorname{Sub}(J)$ and this contradicts the relation $\operatorname{Int}(J) \cap Q \neq \emptyset$. Hence the set $Q$ is empty and therefore $R = E$. By Lemma 4.8 this yields the relation $K \sqsubset S$. □

**Remark.** The proof of Theorem 4.10 follows closely the ideas presented by Y. Kubota in [4]. In [4] a detailed proof is presented for the case when the extension $P_H$ is replaced by the wide Harnack type extension $P_{\mathscr{H}}$, which leads to the general Denjoy integral.

Since the extensions $P_C$ and $P_H$ are defined on $\mathfrak{S}$, Corollary 3.7 can be used for the following statement.

*There exists* $\operatorname{Min}(\{P_C, P_H\}; L)$, *the minimal integral which is both* $P_C$*- and* $P_H$*-invariant and contains the Lebesgue integral* $L$.

Since the Kurzweil-Henstock integral is $P_C$- and $P_H$-invariant and $L \sqsubset K$, we have

$$\operatorname{Min}(\{P_C, P_H\}; L) \sqsubset K.$$

Theorem 4.10 states the converse relation

$$K \sqsubset \operatorname{Min}(\{P_C, P_H\}; L)$$

and therefore we get the following

**Theorem 4.11.** $K = \operatorname{Min}(\{P_C, P_H\}; L)$.

This result is in fact a reformulation of the result presented by S. Saks in [6, pp. 258–259], where the constructive definition of the Denjoy integrals is given using transfinite induction. Our result in Theorem 4.11 concerns the special Denjoy integral which is known to be equivalent to the Kurzweil-Henstock one.

**Lemma 4.12.** *Assume that* $A \subset E$ *is a closed set,* $\operatorname{Comp}(E, A) = \{U_j;\ j \in \Gamma\}$ *the set of all (maximal and nonempty) connected components of the set* $E \setminus A$. *Let* $F \in C(E)$ *be such that*

$$\sum_{j \in \Gamma} \omega(F, \overline{U_j}) < \infty$$

*and*

$$F(x) = F(a) + \sum_{j \in \Gamma_x} F[\overline{U_j}] \quad \text{for } x \in A,$$

*where* $\Gamma_x = \{j \in \Gamma;\ [a, x] \cap \overline{U_j} = \overline{U_j}\}$.

*Then* $W_F(A) = 0$.

Note that a function $F \in C(E)$ satisfying the latter property is not "varying" at points belonging to $A$.

P r o o f. Assume that $\Gamma = \mathbb{N}$ (the case when $\Gamma$ is finite is easy) and let $\varepsilon > 0$ be given. Then, by the assumption, there is a $k \in \mathbb{N}$ such that

$$\sum_{j=k+1}^{\infty} \omega(F, \overline{U_j}) < \varepsilon.$$

Define a gauge $\delta\colon E\to(0,+\infty)$ such that

$$\delta(x)=1 \quad \text{for } x\in E\setminus A=\operatorname{Comp}(E,A)=\bigcup_{j=1}^{\infty}U_j,$$

$$\delta(x)<\operatorname{dist}\Bigl(x,\bigcup_{j=1}^{k}\overline{U_j}\Bigr) \quad \text{for } x\in A_1=A\setminus\bigcup_{j=1}^{k}\overline{U_j}.$$

This means that for $x\in A_1=A\setminus\bigcup\limits_{j=1}^{k}\overline{U_j}$ the interval $(x-\delta(x),x+\delta(x))$ does not intersect $\bigcup\limits_{j=1}^{k}\overline{U_j}$.

For the remaining at most $2k$ points of the set $x\in A_2=A\cap\bigcup\limits_{j=1}^{k}\overline{U_j}$ let $\delta(x)>0$ be such that $|F(y)-F(x)|<\varepsilon/(2k)$ for $y\in(x-\delta(x),x+\delta(x))$; this is possible since $F\in C(E)$.

Let $(\{V_i;\ i\in\widetilde{\Gamma}\},\tau)=\{(V_i,\tau_i);\ i\in\widetilde{\Gamma}\}$ be an arbitrary $\delta$-fine and $A$-tagged division. Then

$$\sum_{i\in\widetilde{\Gamma}}\omega(F,V_i)=\sum_{i\in\widetilde{\Gamma},\,\tau_i\in A_1}\omega(F,V_i)+\sum_{i\in\widetilde{\Gamma},\,\tau_i\in A_2}\omega(F,V_i).$$

First of all, it is clear that

$$\sum_{i\in\widetilde{\Gamma},\,\tau_i\in A_2}\omega(F,V_i)<2k\cdot\frac{\varepsilon}{2k}=\varepsilon$$

by the definition of the gauge $\delta$ on $A_2$.

Observe that if $x<y$, $x,y\in E$, then $\Gamma_x\subset\Gamma_y$ and

$$\sum_{j\in\Gamma_y}F[\overline{U_j}]=\sum_{j\in\Gamma_x}F[\overline{U_j}]+\sum_{j\in\Gamma_y\setminus\Gamma_x}F[\overline{U_j}].$$

By the properties of $F$ we get

$$F(y)-F(x)=\sum_{j\in\Gamma_y\setminus\Gamma_x}F[\overline{U_j}]=\sum_{j\in\Gamma_y,\,U_j\cap[x,y]\neq\emptyset}F[\overline{U_j}].$$

Hence, if $x,y\in V_i$, $x<y$ while $\tau_i\in A_1$, we have

$$\begin{aligned}|F(y)-F(x)|&\leqslant\sum_{j\in\Gamma_y,\,U_j\cap[x,y]\neq\emptyset}|F[\overline{U_j}]|\\&\leqslant\sum_{j>k,\,U_j\cap V_i\neq\emptyset}|F[\overline{U_j}]|\leqslant\sum_{j>k,\,U_j\cap V_i\neq\emptyset}\omega(F,\overline{U_j})\end{aligned}$$

and, consequently,

$$\omega(F,V_i) \leqslant \sum_{j>k,\, U_j\cap V_i\neq\emptyset} \omega(F,\overline{U_j}).$$

Since the intervals $V_i$ are non-overlapping, observe that for a given $V_i$ with $\tau_i \in A_1$ there are at most two $U_j$, $j > k$ such that $U_j \cap V_i \neq \emptyset$ while $U_j \subsetneq V_i$, and one $U_j$, $j > k$ can have common points with at most one $V_i$. Since

$$\bigcup_{i\in\widetilde{\Gamma},\, \tau_i\in A_1} V_i \cap \bigcup_{j=1}^{k} \overline{U_j} = \emptyset,$$

we get

$$\sum_{i\in\widetilde{\Gamma},\, \tau_i\in A_1} \omega(F,V_i) \leqslant \sum_{i\in\widetilde{\Gamma},\, \tau_i\in A_1} \ \sum_{j>k,\, U_j\cap V_i\neq\emptyset} \omega(F,\overline{U_j}) \leqslant 2 \sum_{j=k+1}^{\infty} \omega(F,\overline{U_j}) < 2\varepsilon.$$

In this way we obtain the estimate

$$\sum_{i\in\widetilde{\Gamma}} \omega(F,V_i) = \sum_{i\in\widetilde{\Gamma},\, \tau_i\in A_1} \omega(F,V_i) + \sum_{i\in\widetilde{\Gamma},\, \tau_i\in A_2} \omega(F,V_i) < 3\varepsilon$$

and therefore $W_F(A) = 0$. $\square$

**Corollary 4.13.** *If $S \in \mathfrak{T}$, $f \in \mathrm{Dom}(S_H)$ then the function $F \in C(E)$ from Definition 4.4 belongs to $C^*(E)$.*

Proof. If $f \in \mathrm{Dom}(S_H)$ then $f \cdot \chi(\sigma(f,S)) \in \mathrm{Dom}(S)$ by Definition 4.4 of the Harnack extension. Let us set $\widetilde{F}(x) = S(f \cdot \chi(\sigma(f,S)) \cap [a,x])$ for $x \in E$. We have $\widetilde{F} \in C^*(E)$ because $\widetilde{F}$ is an $S$-primitive function to $f \cdot \chi(\sigma(f,S))$ (see (2.5)).

Put $G(x) = F(x) - \widetilde{F}(x)$ for $x \in E$. The function $G$ is continuous and

$$\begin{aligned} G(x) &= S(f,[a,x]\cap\sigma(f,S)) + \sum_{j\in\Gamma} S(f,[a,x]\cap\overline{U_j}) - S(f,[a,x]\cap\sigma(f,S)) \\ &= \sum_{j\in\Gamma} S(f,[a,x]\cap\overline{U_j}) \end{aligned}$$

where $\{U_j;\ j\in\Gamma\} = \mathrm{Comp}(E,\sigma(f,S))$ is the set of all (maximal and nonempty) connected components of the set $E \setminus \sigma(f,S)$ and $\sigma(f,S)$ is closed.

Then

$$G(x) = \sum_{j\in\Gamma_x} F[\overline{U_j}] = \sum_{j\in\Gamma_x} G[\overline{U_j}]$$

for $x \in \sigma(f,S)$, where $\Gamma_x = \{j\in\Gamma;\ [a,x]\cap\overline{U_j} = \overline{U_j}\}$.

For $I \in \text{Sub}(E)$ such that $I \subset U_k$ we have $G[I] = F[I] = S(F, I)$. This implies $\omega(G, \overline{U_k}) = \omega(F, \overline{U_k})$ for all $k \in \Gamma$ and

$$\sum_{U \in \text{Comp}(E, \sigma(f,S))} \omega(G, \overline{U}) = \sum_{U \in \text{Comp}(E, \sigma(f,S))} \omega(F, \overline{U}) < \infty.$$

Hence by Lemma 4.12 we have $W_G(\sigma(f, S)) = 0$.

Since $F = \widetilde{F} + G$ we get the inequality $W_F(M) \leqslant W_{\widetilde{F}}(M) + W_G(M)$ for any $M \subset E$ and if $M \subset \sigma(f, S)$ then $W_G(M) = 0$.

If $M \subset E$ with $\mu(M) = 0$, then $W_{\widetilde{F}}(M) = 0$ since $\widetilde{F} \in C^*(E)$.

For $M \subset E$, $\mu(M) = 0$ we have further

$$\begin{aligned} W_G(M) &\leqslant W_G(M \cap \sigma(f, S)) + W_G(M \setminus \sigma(f, S)) \\ &\leqslant W_G(\sigma(f, S)) + W_G(M \setminus \sigma(f, S)) = W_G(M \setminus \sigma(f, S)). \end{aligned}$$

We also have

$$W_G(M \setminus \sigma(f, S)) \leqslant \sum_{j \in \Gamma} W_G(U_j \cap M).$$

Since $G$ is an $S$-primitive function to $f \cdot \chi(U_j)$ for every $j \in \Gamma$, we get $W_G(U_j \cap M) = 0$ for $j \in \Gamma$ and $W_G(M \setminus \sigma(f, S)) = 0$.

Therefore $G \in C^*(E)$ and also $F = \widetilde{F} + G \in C^*(E)$. $\square$

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