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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 4, 933–943

Persistent URL: <http://dml.cz/dmlcz/140794>

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LINEAR LIFTINGS OF SKEW SYMMETRIC TENSOR FIELDS OF
TYPE $(1, 2)$ TO WEIL BUNDLES

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(Received April 7, 2009)

Abstract. The paper contains a classification of linear liftings of skew symmetric tensor fields of type $(1, 2)$ on n -dimensional manifolds to tensor fields of type $(1, 2)$ on Weil bundles under the condition that $n \geq 3$. It complements author's paper "Linear liftings of symmetric tensor fields of type $(1, 2)$ to Weil bundles" (Ann. Polon. Math. 92, 2007, pp. 13–27), where similar liftings of symmetric tensor fields were studied. We apply this result to generalize that of author's paper "Affine liftings of torsion-free connections to Weil bundles" (Colloq. Math. 114, 2009, pp. 1–8) and get a classification of affine liftings of all linear connections to Weil bundles.

Keywords: natural operator, Weil bundle

MSC 2010: 58A32

Let A be a Weil algebra inducing the Weil functor T^A (see [15], [6], [12], [8]) and let n be a non-negative integer. We will denote by $\text{Te } M$ the vector space of all tensor fields of type $(1, 2)$ on a manifold M and by $\text{SkTe } M$ the subspace of $\text{Te } M$ consisting of all skew symmetric tensor fields.

A linear lifting of skew symmetric tensor fields of type $(1, 2)$ to tensor fields of type $(1, 2)$ on T^A is, by definition, a family of linear maps $L_M: \text{SkTe } M \rightarrow \text{Te } T^A M$ indexed by all n -dimensional manifolds and satisfying

$$(1) \quad L_M(\varphi^*t) = (T^A\varphi)^*(L_N t)$$

for all n -dimensional manifolds M, N , every embedding $\varphi: M \rightarrow N$ and every $t \in \text{SkTe } N$ (see [10] for the general theory of natural operators).

Our purpose is to describe explicitly all such liftings (for classifications of liftings of tensor fields of some other types to Weil bundles see for example [9], [13], [1], [2], [11]).

In [3] we have constructed six kinds of liftings of symmetric tensor fields. Quite similarly we can now construct six kinds of liftings of skew symmetric tensor fields. So, if $E \in A$, $F: A \rightarrow A$ is \mathbb{R} -linear and $G, H: A \times A \rightarrow A$ are \mathbb{R} -bilinear and such that

$$(2) \quad G(u, vw) = G(u, v)w + G(u, w)v,$$

$$(3) \quad H(u, vw) = H(uv, w) + H(uw, v)$$

for all $u, v, w \in A$, then there are unique linear liftings \overline{E} , \overline{F}^L , \overline{F}^R , \overline{G} , \overline{H}^L , \overline{H}^R of skew symmetric tensor fields of type $(1, 2)$ to tensor fields of type $(1, 2)$ on T^A such that

$$(4) \quad (\overline{E}_U t)_X^p(Y, Z) = E \cdot (T^A t_{qr}^p)(X) \cdot Y^q \cdot Z^r,$$

$$(5) \quad (\overline{F}_U^L t)_X^p(Y, Z) = F((T^A t_{qr}^q)(X) \cdot Y^r) \cdot Z^p,$$

$$(6) \quad (\overline{F}_U^R t)_X^p(Y, Z) = F((T^A t_{qr}^q)(X) \cdot Z^r) \cdot Y^p,$$

$$(7) \quad (\overline{G}_U t)_X^p(Y, Z) = \frac{1}{2} G \left(\left(T^A \left(\frac{\partial t_{qs}^q}{\partial x^r} - \frac{\partial t_{qr}^q}{\partial x^s} \right) \right) (X) \cdot Y^r \cdot Z^s, X^p \right),$$

$$(8) \quad (\overline{H}_U^L t)_X^p(Y, Z) = \frac{1}{2} H \left(\left(T^A \left(\frac{\partial t_{qs}^q}{\partial x^r} - \frac{\partial t_{qr}^q}{\partial x^s} \right) \right) (X) \cdot Y^r, X^s \right) \cdot Z^p,$$

$$(9) \quad (\overline{H}_U^R t)_X^p(Y, Z) = \frac{1}{2} H \left(\left(T^A \left(\frac{\partial t_{qs}^q}{\partial x^r} - \frac{\partial t_{qr}^q}{\partial x^s} \right) \right) (X) \cdot Z^r, X^s \right) \cdot Y^p$$

for every open subset U of \mathbb{R}^n , every $t \in \text{SkTe}U$, every $p \in \{1, \dots, n\}$, every $X \in T^A U$ and all $Y, Z \in A^n$.

We can now formulate our main result.

Theorem. *If $n \geq 3$, then for every linear lifting L of skew symmetric tensor fields of type $(1, 2)$ to tensor fields of type $(1, 2)$ on T^A there are unique $E \in A$, \mathbb{R} -linear $F, F': A \rightarrow A$, \mathbb{R} -bilinear $G: A \times A \rightarrow A$ satisfying (2) and \mathbb{R} -bilinear $H, H': A \times A \rightarrow A$ satisfying (3) such that*

$$(10) \quad L = \overline{E} + \overline{F}^L + \overline{F}'^R + \overline{G} + \overline{H}^L + \overline{H}'^R.$$

The above theorem is quite similar to that of [3], but its proof is not. It is based on the lemma below, which differs from that of [3]. Nevertheless, some parts of both the proofs coincide, so we will omit them and focus on those which are essentially different.

Lemma. Let L and L' be two linear liftings of skew symmetric tensor fields of type $(1, 2)$ to tensor fields of type $(1, 2)$ on T^A . If $n \geq 2$, then

$$L_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right) = L'_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right) \implies L = L'.$$

Proof. Suppose that

$$(11) \quad L_{\mathbb{R}^n} \left(x^1 (\partial/\partial x^1) dx^1 \wedge dx^2 \right) = 0.$$

Let $\alpha \in \mathbb{N}^n$, where \mathbb{N} denotes the set of all non-negative integers. According to the Peetre theorem (see [10]), it suffices to show that $L_{\mathbb{R}^n}(x^\alpha (\partial/\partial x^p) dx^q \wedge dx^r) = 0$ for all $p, q, r \in \{1, \dots, n\}$. Since for any permutation σ of $\{1, \dots, n\}$ we can take $\varphi: \mathbb{R}^n \ni x \rightarrow (x^{\sigma(1)}, \dots, x^{\sigma(n)}) \in \mathbb{R}^n$ in (1), the proof will be completed as soon as we can show that

$$(12) \quad L_{\mathbb{R}^n} \left(x^\alpha \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right) = 0,$$

$$(13) \quad L_{\mathbb{R}^n} \left(x^\alpha \frac{\partial}{\partial x^3} dx^1 \wedge dx^2 \right) = 0$$

(the latter, of course, only in the case $n \geq 3$).

From (11) and (1) with $\varphi: \mathbb{R}^n \ni x \rightarrow (x^1 + 1, x^2, \dots, x^n) \in \mathbb{R}^n$ and $t = x^1 (\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that $L_{\mathbb{R}^n}((\partial/\partial x^1) dx^1 \wedge dx^2) = 0$. If $n \geq 3$, then from this and (1) with $\varphi: \mathbb{R}^n \ni x \rightarrow (x^1, x^2, x^1 + x^3, x^4, \dots, x^n) \in \mathbb{R}^n$ and $t = (\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that $L_{\mathbb{R}^n}((\partial/\partial x^3) dx^1 \wedge dx^2) = 0$. Next, from this and (1) with $\varphi: U \ni x \rightarrow (x^1, x^2, \varphi^3(x), x^4, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^3 > 0, x^4 \neq 0, \dots, x^n \neq 0\}$ and

$$\varphi^3(x) = \begin{cases} \frac{(x^3)^{1-\alpha^3}}{(1-\alpha^3)(x^4)^{\alpha^4} \dots (x^n)^{\alpha^n}} & \text{if } \alpha^3 \neq 1, \\ \frac{\ln |x^3|}{(x^4)^{\alpha^4} \dots (x^n)^{\alpha^n}} & \text{if } \alpha^3 = 1, \end{cases}$$

and $t = (\partial/\partial x^3) dx^1 \wedge dx^2$ it follows that $L_U((x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} (\partial/\partial x^3) dx^1 \wedge dx^2) = 0$. The same conclusion can be drawn for $V = \{x \in \mathbb{R}^n: x^3 < 0, x^4 \neq 0, \dots, x^n \neq 0\}$, and so for \mathbb{R}^n , because $U \cup V$ is dense in \mathbb{R}^n . Finally, this and (1) with $\varphi: U \ni x \rightarrow ((x^1)^{\alpha^1+1}/(\alpha^1+1), (x^2)^{\alpha^2+1}/(\alpha^2+1), x^3, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^1 > 0, x^2 > 0\}$, and $t = (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} (\partial/\partial x^3) dx^1 \wedge dx^2$ imply (13).

From (11) and (1) with $\varphi: U \ni x \rightarrow (\varphi^1(x), x^2, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^1 > 0, x^3 \neq 0, \dots, x^n \neq 0\}$ and

$$\varphi^1(x) = \begin{cases} x^1 + (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} & \text{if } \alpha^1 = 0, \\ (x^1)^{\alpha^1} (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} & \text{if } \alpha^1 \neq 0, \end{cases}$$

and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that

$$L_U \left(\frac{\varphi^1}{\partial\varphi^1/\partial x^1} \frac{\partial}{\partial x^1} \left(\frac{\partial\varphi^1}{\partial x^1} dx^1 + \sum_{p=3}^n \frac{\partial\varphi^1}{\partial x^p} dx^p \right) \wedge dx^2 \right) = 0.$$

But for every $p \in \{3, \dots, n\}$

$$\frac{\varphi^1 \frac{\partial\varphi^1}{\partial x^p}}{\frac{\partial\varphi^1}{\partial x^1}} = \begin{cases} \varphi^1 \frac{\partial\varphi^1}{\partial x^p} & \text{if } \alpha^1 = 0, \\ \frac{x^1}{\alpha^1} \frac{\partial\varphi^1}{\partial x^p} & \text{if } \alpha^1 \neq 0, \end{cases}$$

is a polynomial. Hence from (13), which we have proved for every $\alpha \in \mathbb{N}^n$ with x^3, x^1 replaced by x^1, x^p (using (11) again in the case $\alpha^1 = 0$) we obtain $L_{\mathbb{R}^n}((x^1)^{\alpha^1} (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} (\partial/\partial x^1) dx^1 \wedge dx^2) = 0$. This and (1) with $\varphi: U \ni x \rightarrow (x^1, (x^2)^{\alpha^2+1}/(\alpha^2+1), x^3, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^2 > 0\}$, and $t = (x^1)^{\alpha^1} (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} (\partial/\partial x^1) dx^1 \wedge dx^2$ yield (12), which establishes the lemma. \square

Proof of the theorem. In the same manner as in [3] we can see that there are unique \mathbb{R} -trilinear maps $a, b, c, d, e, f, g, h, i, j, k, l: A \times A \times A \rightarrow A$ such that

$$(14) \quad L_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^1(Y, Z) = a(X^1, Y^1, Z^2) + b(X^1, Y^2, Z^1) + c(X^2, Y^1, Z^1),$$

$$(15) \quad L_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^2(Y, Z) = d(X^1, Y^2, Z^2) + e(X^2, Y^1, Z^2) + f(X^2, Y^2, Z^1),$$

$$(16) \quad L_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^p(Y, Z) = g(X^1, Y^2, Z^p) + h(X^1, Y^p, Z^2) + i(X^2, Y^1, Z^p) + j(X^2, Y^p, Z^1) + k(X^p, Y^1, Z^2) + l(X^p, Y^2, Z^1)$$

for every $p \in \{3, \dots, n\}$ and all $X, Y, Z \in A^n$. Also similarly to [3] (using (16) with $p = 3$ and (1) with $\varphi: U \ni x \rightarrow (x^1, x^2, (x^3)^2/2, x^4, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^3 > 0\}$ and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ and carrying out polarization if necessary) we get for all $w, x, y, z \in A$

$$(17) \quad wg(x, y, z) = g(x, y, wz),$$

$$(18) \quad wh(x, y, z) = h(x, wy, z),$$

$$(19) \quad wi(x, y, z) = i(x, y, wz),$$

$$(20) \quad wj(x, y, z) = j(x, wy, z),$$

$$(21) \quad wk(x, y, z) + xk(w, y, z) = k(wx, y, z).$$

From (16) with $p = 3$ and (1) with $\varphi: U \ni x \rightarrow (x^1, \frac{1}{2}(x^2)^2, x^3, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^2 > 0\}$, and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that

$$(22) \quad L_{\mathbb{R}^n} \left(x^1 x^2 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^3 (Y, Z) \\ = g(X^1, X^2 Y^2, Z^3) + h(X^1, Y^3, X^2 Z^2) + i \left(\frac{(X^2)^2}{2}, Y^1, Z^3 \right) \\ + j \left(\frac{(X^2)^2}{2}, Y^3, Z^1 \right) + k(X^3, Y^1, X^2 Z^2) + l(X^3, X^2 Y^2, Z^1).$$

On the other hand, from (16) with $p = 3$ and (1) with $\varphi: U \ni x \rightarrow (x^1 x^2, x^2, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^2 \neq 0\}$ and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that

$$(23) \quad L_{\mathbb{R}^n} \left(x^1 x^2 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^3 (Y, Z) \\ = g(X^1 X^2, Y^2, Z^3) + h(X^1 X^2, Y^3, Z^2) \\ + i(X^2, X^2 Y^1 + X^1 Y^2, Z^3) + j(X^2, Y^3, X^2 Z^1 + X^1 Z^2) \\ + k(X^3, X^2 Y^1 + X^1 Y^2, Z^2) + l(X^3, Y^2, X^2 Z^1 + X^1 Z^2).$$

Comparing (22) with (23) and carrying out polarization if necessary we get

$$(24) \quad g(x, wy, z) = g(xw, y, z) + i(w, xy, z),$$

$$(25) \quad h(x, y, wz) = h(xw, y, z) + j(w, y, xz),$$

$$(26) \quad i(wx, y, z) = i(w, xy, z) + i(x, wy, z),$$

$$(27) \quad j(wx, y, z) = j(w, y, xz) + j(x, y, wz),$$

$$(28) \quad k(x, y, wz) = k(x, wy, z),$$

$$(29) \quad 0 = k(x, wy, z) + l(x, y, wz).$$

Define $G(u, v) = 4k(v, u, 1)$ for all $u, v \in A$. (21) shows that G satisfies (2). From (28) we deduce that $k(x, y, z) = \frac{1}{4}G(yz, x)$ for all $x, y, z \in A$. In addition, by (29), $l = -k$. Therefore $k(X^p, Y^1, Z^2) + l(X^p, Y^2, Z^1) = \frac{1}{4}G(Y^1 Z^2 - Y^2 Z^1, X^p)$ for every $p \in \{1, \dots, n\}$ and all $X, Y, Z \in A^n$. But, on account of (7),

$$(30) \quad \overline{G}_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^p (Y, Z) = \frac{1}{4}G(Y^1 Z^2 - Y^2 Z^1, X^p),$$

too. Consequently, by (16), we may replace L by $L - \overline{G}$ and assume that $k = 0$ and $l = 0$ from now on.

Define $H(u, v) = 4i(v, u, 1)$ and $H'(u, v) = 4j(v, 1, u)$ for all $u, v \in A$. (26) and (27) show that H and H' satisfy (3). From (19) and (20) we deduce that $i(x, y, z) =$

$\frac{1}{4}H(y, x)z$ and $j(x, y, z) = \frac{1}{4}H'(z, x)y$ for all $x, y, z \in A$. Therefore $i(X^2, Y^1, Z^p) = \frac{1}{4}H(Y^1, X^2)Z^p$ and $j(X^2, Y^p, Z^1) = \frac{1}{4}H'(Z^1, X^2)Y^p$ for every $p \in \{1, \dots, n\}$ and all $X, Y, Z \in A^n$. But

$$(31) \quad \overline{H}_{\mathbb{R}^n}^L \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^p (Y, Z) = \frac{1}{4}(H(Y^1, X^2) - H(Y^2, X^1))Z^p,$$

$$(32) \quad \overline{H}_{\mathbb{R}^n}^R \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^p (Y, Z) = \frac{1}{4}(H'(Z^1, X^2) - H'(Z^2, X^1))Y^p$$

on account of (8) and (9). Consequently, by (16), we may replace L by $L - \overline{H}^L - \overline{H}^R$ and assume that $i = 0$ and $j = 0$ from now on (note that the conditions $k = 0$ and $l = 0$ remain valid).

Define $F(u) = 2g(u, 1, 1)$ and $F'(u) = 2h(u, 1, 1)$ for every $u \in A$. From (17) and (24) we deduce that $g(x, y, z) = \frac{1}{2}F(xy)z$ and from (18) and (25) that $h(x, y, z) = \frac{1}{2}F'(xz)y$ for all $x, y, z \in A$. Therefore $g(X^1, Y^2, Z^p) = \frac{1}{2}F(X^1Y^2)Z^p$ and $h(X^1, Y^p, Z^2) = \frac{1}{2}F'(X^1Z^2)Y^p$ for every $p \in \{1, \dots, n\}$ and all $X, Y, Z \in A^n$. But, on account of (5) and (6),

$$(33) \quad \overline{F}_{\mathbb{R}^n}^L \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^p (Y, Z) = \frac{1}{2}F(X^1Y^2)Z^p,$$

$$(34) \quad \overline{F}_{\mathbb{R}^n}^R \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^p (Y, Z) = \frac{1}{2}F'(X^1Z^2)Y^p,$$

too. Consequently, by (16), we may replace L by $L - \overline{F}^L - \overline{F}^R$ and assume that also $g = 0$ and $h = 0$ from now on.

But now $d = 0$, $e = 0$, $f = 0$ as well. Indeed, from (16) with $p = 3$, (15) and (1) with $\varphi: \mathbb{R}^n \ni x \rightarrow (x^1, x^2, x^2 + x^3, x^4, \dots, x^n) \in \mathbb{R}^n$ and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that $0 = L_{\mathbb{R}^n} \left(x^1(\partial/\partial x^1) dx^1 \wedge dx^2 \right)_X^3 (Y, Z) = -d(X^1, Y^2, Z^2) - e(X^2, Y^1, Z^2) - f(X^2, Y^2, Z^1)$, which yields the desired conclusion.

From (14), (16) with $p = 3$ and (1) with $\varphi: \mathbb{R}^n \ni x \rightarrow (x^1 + x^3, x^2, \dots, x^n) \in \mathbb{R}^n$ and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that

$$(35) \quad \begin{aligned} L_{\mathbb{R}^n} \left((x^1 + x^3) \frac{\partial}{\partial x^1} (dx^1 + dx^3) \wedge dx^2 \right)_X^1 (Y, Z) \\ = a(X^1 + X^3, Y^1 + Y^3, Z^2) + b(X^1 + X^3, Y^2, Z^1 + Z^3) \\ + c(X^2, Y^1 + Y^3, Z^1 + Z^3). \end{aligned}$$

From (14), (16) with $p = 3$ and (1) with $\varphi: \mathbb{R}^n \ni x \rightarrow (x^1, x^2, x^1 + x^3, x^4, \dots, x^n) \in \mathbb{R}^n$ and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that $L_{\mathbb{R}^n} \left(x^1(\partial/\partial x^1 - \partial/\partial x^3) dx^1 \wedge dx^2 \right)_X^3 (Y, Z) = -a(X^1, Y^1, Z^2) - b(X^1, Y^2, Z^1) - c(X^2, Y^1, Z^1)$. This, after us-

ing (16) with $p = 3$ again and then interchanging x^1 and x^3 , yields

$$(36) \quad L_{\mathbb{R}^n} \left(x^3 \frac{\partial}{\partial x^1} dx^3 \wedge dx^2 \right)_X^1 (Y, Z) \\ = a(X^3, Y^3, Z^2) + b(X^3, Y^2, Z^3) + c(X^2, Y^3, Z^3).$$

From (14), (16) with $p = 3$ and (1) with $\varphi: U \ni x \rightarrow (x^1, x^2, x^1 x^3, x^4, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^1 \neq 0\}$, and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that $L_{\mathbb{R}^n} (x^1(\partial/\partial x^1 - (x^3/x^1)\partial/\partial x^3) dx^1 \wedge dx^2)_X^3 (Y, Z) = -(X^3/X^1)(a(X^1, Y^1, Z^2) + b(X^1, Y^2, Z^1) + c(X^2, Y^1, Z^1))$. This, after using (16) with $p = 3$ again and then interchanging x^1 and x^3 , yields

$$(37) \quad L_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^3 \wedge dx^2 \right)_X^1 (Y, Z) \\ = \frac{X^1}{X^3} (a(X^3, Y^3, Z^2) + b(X^3, Y^2, Z^3) + c(X^2, Y^3, Z^3)).$$

Combining (35), (36), (37) with (14) we see that

$$(38) \quad L_{\mathbb{R}^n} \left(x^3 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^1 (Y, Z) \\ = a(X^1, Y^3, Z^2) + a(X^3, Y^1, Z^2) + b(X^1, Y^2, Z^3) \\ + b(X^3, Y^2, Z^1) + c(X^2, Y^1, Z^3) + c(X^2, Y^3, Z^1) \\ - \frac{X^1}{X^3} (a(X^3, Y^3, Z^2) + b(X^3, Y^2, Z^3) + c(X^2, Y^3, Z^3)).$$

On the other hand, (38) and (1) with $\varphi: U \ni x \rightarrow (\frac{1}{2}(x^1)^2, x^2, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^1 > 0\}$, and $t = x^3(\partial/\partial x^1) dx^1 \wedge dx^2$ implies that

$$(39) \quad L_{\mathbb{R}^n} \left(x^3 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^1 (Y, Z) \\ = \frac{1}{X^1} \left(a \left(\frac{(X^1)^2}{2}, Y^3, Z^2 \right) + a(X^3, X^1 Y^1, Z^2) + b \left(\frac{(X^1)^2}{2}, Y^2, Z^3 \right) \right. \\ \left. + b(X^3, Y^2, X^1 Z^1) + c(X^2, X^1 Y^1, Z^3) + c(X^2, Y^3, X^1 Z^1) \right) \\ - \frac{X^1}{2X^3} (a(X^3, Y^3, Z^2) + b(X^3, Y^2, Z^3) + c(X^2, Y^3, Z^3)).$$

Comparing (38) with (39) (both multiplied by $2X^1 X^3$) we get

$$(40) \quad 2wxa(x, y, z) - 2x^2a(w, y, z) = wa(x^2, y, z) - x^2a(w, y, z),$$

$$(41) \quad 2wxa(x, y, z) = 2xa(x, wy, z),$$

$$(42) \quad -2w^2c(x, y, z) = -w^2c(x, y, z).$$

From (42) we see at once that $c = 0$.

Taking $x = 1$ in (40) we deduce that a is A -linear with respect to the first variable. By (41), it is also A -linear with respect to the second.

From (14) and (1) with $\varphi: U \ni x \rightarrow (x^1, \frac{1}{2}(x^2)^2, x^3, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^2 > 0\}$ and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that

$$(43) \quad \begin{aligned} L_{\mathbb{R}^n} \left(x^1 x^2 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^1(Y, Z) \\ = a(X^1, Y^1, X^2 Z^2) + b(X^1, X^2 Y^2, Z^1). \end{aligned}$$

On the other hand, from (14), (15) and (1) with $\varphi: U \ni x \rightarrow (x^1 x^2, x^2, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n: x^2 \neq 0\}$ and $t = x^1(\partial/\partial x^1) dx^1 \wedge dx^2$ it follows that

$$(44) \quad \begin{aligned} L_{\mathbb{R}^n} \left(x^1 x^2 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^1(Y, Z) \\ = \frac{1}{X^2} (a(X^1 X^2, X^2 Y^1 + X^1 Y^2, Z^2) + b(X^1 X^2, Y^2, X^2 Z^1 + X^1 Z^2)). \end{aligned}$$

Comparing (43) with (44) (both multiplied by X^2) we get

$$(45) \quad wa(x, y, wz) = a(wx, wy, z),$$

$$(46) \quad 0 = a(wx, wy, z) + b(wx, y, wz).$$

Since a is A -linear with respect to the first and second variables, from (45) we deduce that it is also A -linear with respect to the third. Thus $a(x, y, z) = \frac{1}{2}Exyz$ for all $x, y, z \in A$, where $E = 2a(1, 1, 1)$. In addition, by (46), $b = -a$. Therefore $a(X^1, Y^1, Z^2) + b(X^1, Y^2, Z^1) = \frac{1}{2}EX^1(Y^1 Z^2 - Y^2 Z^1)$ for all $X, Y, Z \in A^n$. But, on account of (4),

$$(47) \quad \begin{aligned} \overline{E}_{\mathbb{R}^n} \left(x^1 \frac{\partial}{\partial x^1} dx^1 \wedge dx^2 \right)_X^p(Y, Z) \\ = \begin{cases} \frac{1}{2}EX^1(Y^1 Z^2 - Y^2 Z^1) & \text{if } p = 1, \\ 0 & \text{if } p \in \{2, \dots, n\}. \end{cases} \end{aligned}$$

Consequently, by (14)–(16) and the lemma we conclude that $L = \overline{E}$ (where L actually stands for $L - \overline{G} - \overline{H}^L - \overline{H}'^R - \overline{F}^L - \overline{F}'^R$ with the original L), which completes the proof that E, F, F', G, H, H' exist.

Analysis of formulas (30)–(34) and (47) makes it obvious that they are uniquely determined, and the theorem is proved. \square

Remark. The author does not know whether every linear lifting can be represented in the form (10) when $n = 2$, but it is easy to see that then this representation need not be unique. Indeed, for every $C \in A$ and every derivation D of the algebra A we may put $E = C$, $F(u) = Cu$, $F'(u) = -Cu$, $G(u, v) = -uD(v)$, $H(u, v) = uD(v)$, $H'(u, v) = -uD(v)$ for all $u, v \in A$. Then (by the lemma, (30)–(34) and (47)) the right-hand side of (10) equals zero in the case $n = 2$.

The remainder of the paper will be devoted to some corollaries of the theorem.

If L is given by (10), then from (4)–(9) it is evident that the two liftings $\overline{E} + \frac{1}{2}(F - F')^{\text{L}} + \frac{1}{2}(-F + F')^{\text{R}} + \overline{G} + \frac{1}{2}(H - H')^{\text{L}} + \frac{1}{2}(-H + H')^{\text{R}}$ and $\frac{1}{2}(F + F')^{\text{L}} + \frac{1}{2}(F + F')^{\text{R}} + \frac{1}{2}(H + H')^{\text{L}} + \frac{1}{2}(H + H')^{\text{R}}$ transform all skew symmetric tensor fields into skew symmetric and symmetric ones, respectively, and the sum of them equals L . This yields the two corollaries.

Corollary 1. *If $n \geq 3$, then for every linear lifting L of skew symmetric tensor fields of type (1, 2) to skew symmetric tensor fields of type (1, 2) on T^A there are unique $E \in A$, \mathbb{R} -linear $F: A \rightarrow A$, \mathbb{R} -bilinear $G: A \times A \rightarrow A$ satisfying (2) and \mathbb{R} -bilinear $H: A \times A \rightarrow A$ satisfying (3) such that*

$$L = \overline{E} + \overline{F}^{\text{L}} - \overline{F}^{\text{R}} + \overline{G} + \overline{H}^{\text{L}} - \overline{H}^{\text{R}}.$$

Corollary 2. *If $n \geq 3$, then for every linear lifting L of skew symmetric tensor fields of type (1, 2) to symmetric tensor fields of type (1, 2) on T^A there are unique \mathbb{R} -linear $F: A \rightarrow A$ and \mathbb{R} -bilinear $H: A \times A \rightarrow A$ satisfying (3) such that*

$$L = \overline{F}^{\text{L}} + \overline{F}^{\text{R}} + \overline{H}^{\text{L}} + \overline{H}^{\text{R}}.$$

Combining the theorem of this paper with that of [3] we easily obtain

Corollary 3. *If $n \geq 3$, then for every linear lifting L of tensor fields of type (1, 2) to tensor fields of type (1, 2) on T^A there are unique $E, E' \in A$, \mathbb{R} -linear $F, F', F'', F''': A \rightarrow A$, \mathbb{R} -bilinear $G, G': A \times A \rightarrow A$ satisfying (2) and \mathbb{R} -bilinear $H, H', H'', H''': A \times A \rightarrow A$ satisfying (3) such that*

$$L = (\overline{E} + \overline{F}^{\text{L}} + \overline{F}^{\text{R}} + \overline{G} + \overline{H}^{\text{L}} + \overline{H}^{\text{R}}) \circ \text{sk} \\ + (\overline{E'} + \overline{F}''^{\text{L}} + \overline{F}'''^{\text{R}} + \overline{G'} + \overline{H}''^{\text{L}} + \overline{H}'''^{\text{R}}) \circ \text{sy},$$

where sk and sy denote the alternation and symmetrization of tensor fields of type (1, 2) and the liftings of symmetric tensor fields are also given by formulas (4)–(9).

We end the paper with an application of our result in classification of all affine liftings of linear connections to Weil bundles (the contrast between these liftings and those of general connections to some non-Weil bundles is worth pointing out, see for instance [5]). Namely, combining our theorem with that of [4] we easily obtain

Corollary 4. *If $n \geq 3$, then for every affine lifting L of linear connections to linear connections on T^A there are unique $E \in A$, \mathbb{R} -linear $F, F' : A \rightarrow A$, \mathbb{R} -bilinear $G, G' : A \times A \rightarrow A$ satisfying (2) and \mathbb{R} -bilinear $H, H', H'', H''' : A \times A \rightarrow A$ satisfying (3) such that*

$$L_M \nabla = \nabla^A + \overline{E}_M T + \overline{F}_M^L T + \overline{F}_M^R T + \overline{G}_M T + \overline{H}_M^L T + \overline{H}_M^R T \\ + \widetilde{G}'_M(\text{tr } R) + \widetilde{H}''^L_M(\text{tr } R) + \widetilde{H}'''^R_M(\text{tr } R)$$

for every n -dimensional manifold M and every linear connection ∇ on M , where ∇^A denotes the complete lift of ∇ (see [14], [7]), T and R the torsion and curvature tensors of ∇ and \widetilde{G}' , \widetilde{H}''^L , \widetilde{H}'''^R are the unique liftings of 2-forms to tensor fields of type (1, 2) on T^A such that

$$(\widetilde{G}'_U t)_X^p(Y, Z) = G'((T^A t_{qr})(X) \cdot Y^q \cdot Z^r, X^p), \\ (\widetilde{H}''^L_U t)_X^p(Y, Z) = H''((T^A t_{qr})(X) \cdot Y^q, X^r) \cdot Z^p, \\ (\widetilde{H}'''^R_U t)_X^p(Y, Z) = H'''((T^A t_{qr})(X) \cdot Z^q, X^r) \cdot Y^p$$

for every open subset U of \mathbb{R}^n , every 2-form t on U , every $p \in \{1, \dots, n\}$, every $X \in T^A U$ and all $Y, Z \in A^n$.

References

- [1] *J. Dębecki*: Linear liftings of skew-symmetric tensor fields to Weil bundles. Czech. Math. J. 55 (130) (2005), 809–816.
- [2] *J. Dębecki*: Linear liftings of p -forms to q -forms on Weil bundles. Monatsh. Math. 148 (2006), 101–117.
- [3] *J. Dębecki*: Linear liftings of symmetric tensor fields of type (1, 2) to Weil bundles. Ann. Polon. Math. 92 (2007), 13–27.
- [4] *J. Dębecki*: Affine liftings of torsion-free connections to Weil bundles. Colloq. Math. 114 (2009), 1–8.
- [5] *M. Doupovec, W. M. Mikulski*: On the existence of prolongation of connections. Czech. Math. J. 56 (131) (2006), 1323–1334.
- [6] *D. J. Eck*: Product-preserving functors on smooth manifolds. J. Pure Appl. Algebra 42 (1986), 133–140.
- [7] *J. Gancarzewicz, W. Mikulski, Z. Pogoda*: Lifts of some tensor fields and connections to product preserving functors. Nagoya Math. J. 135 (1994), 1–41.

- [8] *G. Kainz, P. Michor*: Natural transformations in differential geometry. Czech. Math. J. *37 (112)* (1987), 584–607.
- [9] *I. Kolář*: On natural operators on vector fields. Ann. Global Anal. Geom. *6* (1988), 109–117.
- [10] *I. Kolář, P. W. Michor, J. Slovák*: Natural Operations in Differential Geometry. Springer-Verlag, Berlin, 1993.
- [11] *J. Kurek, W. M. Mikulski*: Canonical symplectic structures on the r th order tangent bundle of a symplectic manifold. Extr. Math. *21* (2006), 159–166.
- [12] *O. Luciano*: Categories of multiplicative functors and Weil’s infinitely near points. Nagoya Math. J. *109* (1988), 69–89.
- [13] *W. M. Mikulski*: The geometrical constructions lifting tensor fields of type $(0, 2)$ on manifolds to the bundles of A -velocities. Nagoya Math. J. *140* (1995), 117–137.
- [14] *A. Morimoto*: Prolongations of connections to bundles of infinitely near points. J. Diff. Geom. *11* (1976), 479–498.
- [15] *A. Weil*: Théorie des points proches sur les variétés différentielles. Colloques Internat. Centre Nat. Rech. Sci. *52* (1953), 111–117. (In French.)

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