Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Y. S. Pawar; S. S. Khopade

α-ideals and annihilator ideals in 0-distributive lattices

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 49 (2010), No. 1, 63--74

Persistent URL: `http://dml.cz/dmlcz/140738`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **49**, 1 (2010) 63–74

α-ideals and Annihilator Ideals in 0-distributive Lattices

Y. S. PAWAR [1], S. S. KHOPADE [2]

Department of Mathematics, Shivaji University,
Kolhapur, India
[1] *e-mail: pawar_y_s@yahoo.com*
[2] *e-mail: ssk27_01@rediffmail.com*

(Received October 7, 2009)

Abstract

In a 0-distributive lattice sufficient conditions for an α-ideal to be an annihilator ideal and prime ideal to be an α-ideal are given. Also it is proved that the images and the inverse images of α-ideals are α-ideals under annihilator preserving homomorphisms.

Key words: 0-distributive lattice, α-ideal, annihilator ideal, quasi-complemented lattice.

2000 Mathematics Subject Classification: 06D99

1 Introduction

Varlet [7] introduced 0-distributive lattices as a generalization of distributive lattices and pseudo-complemented lattices. The theory of 0-distributive lattices was further studied by Balasubramani, Venkatanarsimhan [1,2], C. Jayaram [5], and Pawar and Mane [6]. W. H. Cornish [3] introduced and studied the properties of α-ideals in distributive lattices. Generalization of the concept of α-ideals in 0-distributive lattices is carried out by C. Jayaram [5]. Additional properties of α-ideals in 0-distributive lattices were obtained by Pawar and Mane [6] which generalize the results of Cornish [3].

In this paper a joint study of annihilator ideals and α-ideals in 0-distributive lattices is continued to supplement the results of Jayaram [5] and Pawar, Mane [6]. We prove that the image and the inverse image of an α-ideal are α-ideals under annihilator preserving homomorphism of a 0-distributive lattice. Further it is proved that in a 0-distributive lattice the set of all prime ideals is an antichain if every prime ideal is an annihilator ideal. This result facilitates us to characterize finite Boolean algebras. Several properties of α-ideals in a 0-distributive quasi-complemented lattice are studied. Necessary and sufficient conditions for the set of all α-ideals of a 0-distributive lattice to be semi-complemented are furnished.

2 Preliminaries

For basic concepts in lattice theory, we refer to Grätzer [4]. Throughout this paper by a lattice $L = \langle L, \wedge, \vee \rangle$ we mean a bounded lattice with the least element 0 and the greatest element 1. L is said to be 0-distributive if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for all a, b, c in L. For any non-empty subset A of L define $A^* = \{x \in L \mid x \wedge a = 0 \text{ for each } a \in A\}$. A^* is called the annihilator of A. When $A = \{x\}$, $A^* = \{x\}^* = (x]^*$, where $(x]$ is the principal ideal generated by $x \in L$, i.e. $(x] = \{y \in L \mid y \leq x\}$. Note that L is 0-distributive if and only if for any non-empty subset A of L, $A^*$is an ideal of L. An ideal I of L is called an annihilator ideal if $I = A^*$ for some non-empty subset A of L, or equivalently if $I = I^{**}$. An element $d \in L$ is called dense if $(d]^* = (0]$. An ideal I of L is called a dense ideal if $I^* = (0]$. We denote the collection of all dense elements by D. An ideal I of L is called an α-ideal if $(x]^{**} \subseteq I$ for every $x \in I$. We denote the set of all α-ideals of L by $I_\alpha(L)$. L is pseudo-complemented if for every $a \in L$ there exists $a^* \in L$ such that $(a]^* = (a^*]$. L is called quasi-complemented if for any $a \in L$ there is $b \in L$ such that $a \wedge b = 0$ and $(a \vee b]^* = \{0\}$. An ideal $J \neq (0]$ of L is a semi-complement of an ideal I of L if $I \cap J = (0]$. A family $\mathcal{K}$ of ideals of L is said to be semi complemented if every element of $\mathcal{K}$ has a semi-complement in it. L is said to be disjunctive if for all $a < b$ there is $c \in L$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$. By a homomorphism we mean a 0-1 lattice homomorphism. For an ideal I of L, we define

$$I^e = \left\{x \in L \,\middle|\, (a]^* \subseteq (x]^* \text{ for some } a \in I\right\}.$$

Note that I^e coincides with $I' = \left\{x \in L \,\middle|\, x \in (a]^{**} \text{ for some } a \in I\right\}$ defined by Jayram [5].

In sequel, we need the following results:

Result 1 *Every annihilator ideal in L is an α-ideal.*

Result 2 [7] *A lattice L is quasi-complemented if for any $x \in L$ there exists $y \in L$ such that $(x]^{**} = (y]^*$.*

Result 3 [5] *For any ideal I in L the set*

$$I^e = \{x \in L \big| (a]^* \subseteq (x]^* \textit{ for some } a \in I\}$$

is the smallest α-ideal containing I and an ideal I in L is an α-ideal if and only if $I = I^e$.

Result 4 *L is 0-distributive if and only if*

$$(a \vee b]^* = (a]^* \cap (b]^* \textit{ for all } a, b \in L.$$

Result 5 [6] *For any ideal I in a 0-distributive lattice L, the following are equivalent:*

1. *I is an α-ideal.*
2. *$I = \cup_{x \in L}(x]^{**}$.*
3. *For any x, y in L, $(x]^* = (y]^*$ and $x \in I \Rightarrow y \in I$.*

Result 6 [7] *L is a 0-distributive lattice if and only if the set $I(L)$ of all ideals of L forms a pseudo-complemented lattice.*

Result 7 [4] *Let $\langle L, \wedge, \vee \rangle$ be a pseudo-complemented lattice and*

$$S(L) = \left\{ a \in L \;\middle|\; a = a^{**} \right\}.$$

Then $\langle S(L), \sqcap, \sqcup \rangle$ forms a Boolean algebra, where for $a, b \in S(L)$ we have

$$a \sqcap b = a \wedge b \quad \text{and} \quad a \sqcup b = \left(a^* \wedge b^*\right)^*.$$

Result 8 *Any 0-distributive and disjunctive lattice is distributive.*

Result 9 [4] *In a distributive lattice L, if $a < b$, then there exists a prime ideal P containing a but not b.*

Result 10 [5] *Let I be an α-ideal of a 0-distributive lattice L and S be a meet sub-semilattice of L such that $I \cap S = \emptyset$. Then there exists a prime α-ideal P of L such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Result 11 [4] (Nachbin's Theorem) *Let L be a bounded distributive lattice. L is a Boolean lattice if and only if $\wp$, the set of all prime ideals of L, is an antichain.*

Result 12 [4] *If $f\colon L_1 \to L_2$ is a 0-1 lattice homomorphism, then*

1. *For any ideal I of L_1, $f(I)$ is an ideal of L_2.*
2. *For any ideal J of L_2, $f^{-1}(J)$ is an ideal of L_1.*
3. $\operatorname{Ker} f = \left\{ x \in L_1 \;\middle|\; f(x) = 0' \right\}$ *is an ideal of L_1 where $0'$ is the least element of L_2.*

3 α-ideals and annihilator ideals

Throughout this section L denotes a bounded $0-$distributive lattice and D be the set of all dense elements of L.

We begin with the following result.

Theorem 3.1 *Let S be any non-empty subset of L which is closed under $\wedge$ operation. Define,*

$$I = \left\{ x \in L \;\middle|\; x \wedge y = 0 \text{ for some } y \in S \right\}.$$

Then I is an α-ideal of L.

Proof Obviously $0 \in I$. If $x_1 \leq x_2$ in L and $x_2 \in I$ then $x_1 \in I$.

Now, let $x_1, x_2 \in I$. Then $x_1 \wedge s_1 = 0$ and $x_2 \wedge s_2 = 0$ for some $s_1, s_2 \in S$. Hence $x_1 \wedge (s_1 \wedge s_2) = 0$ and $x_2 \wedge (s_1 \wedge s_2) = 0$ imply $(x_1 \vee x_2) \wedge (s_1 \wedge s_2) = 0$ (since L is 0-distributive). As $s_1 \wedge s_2 \in S$, we get $x_1 \vee x_2 \in I$. Thus I is an ideal of L. Let $x \in I$ and $y \in (x]^{**}$. Clearly $x \in I$ implies $x \wedge s = 0$ for some $s \in S$. But then $s \in (x]^*$ and hence, $y \wedge s = 0$. This shows that $y \in I$ and consequently $(x]^{**} \subseteq I$. Hence I is an α-ideal of L. □

From Theorem 3.1, the result of Pawar and Mane ([6], Theorem 3) follows as a corollary.

Corollary 3.2 *For any filter F in L,*

$$O(F) = \{x \in L \mid x \wedge y = 0 \text{ for some } y \in F\}$$

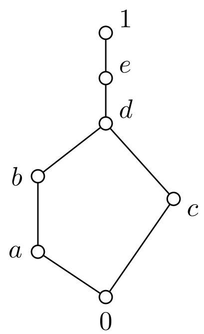
is an α-ideal of L.

An ideal I of L is called a 0-ideal if $I = O(F)$ for some filter F [6]. Hence we get

Corollary 3.3 *Every 0-ideal of L is an α-ideal of L.*

Every minimal prime ideal in L is an α-ideal ([6], Theorem 1) but not every prime ideal in L is necessarily an α-ideal. For this consider the following example:

Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ whose Hasse diagram is as in Fig. 1. The ideal $(e]$ is a prime ideal but not an α-ideal. For $d \in (e]$, $(d]^{**} = L \nsubseteq (e]$.



L

Fig. 1

Sufficient condition for a prime ideal in L to be an α-ideal is given in the following theorem.

Theorem 3.4 *If a prime ideal P of L is non-dense, then P is an α-ideal.*

Proof By assumption, $P^* \neq (0]$. Hence there exists $x \in P^*$ such that $x \neq 0$. But then $(x]^* \supseteq P^{**}$ gives $(x]^* \supseteq P$ as $P \subseteq P^{**}$. Further if $t \in (x]^*$, then $t \wedge x = 0 \in P$. But as P is a prime ideal, $t \in P$ (since $P \cap P^* = (0] \Longrightarrow x \notin P$). But this shows that $(x]^* \subseteq P$. Combining both the inclusions, we get $P = (x]^*$. Hence P is an annihilator ideal. By Result 1, P is an α-ideal. □

A sufficient condition for a 0-distributive lattice to be quasi-complemented is given in the following theorem.

Theorem 3.5 *If no proper α-ideal of L is dense, then L is quasi-complemented.*

Proof Let $x \in L$ and put $I = (x]^* \vee (x]^{**}$. Then I^e is an α-ideal in L (by Result 3). Further, $(x]^* \subseteq I \subseteq I^e$, we get $(I^e)^* \subseteq (x]^{**}$ and $(x]^{**} \subseteq I \subseteq I^e$ imply $(I^e)^* \subseteq (x]^*$. Hence $(I^e)^* \subseteq (x]^* \cap (x]^{**} = (0]$. Therefore I^e is a dense α-ideal in L. By the assumption $I^e = L$. As $D \neq \emptyset$, there exists $d \in D$ such that $d \in I^e$. There exists $t \in I$ such that $(t]^* \subseteq (d]^* = (0]$ and hence $(t]^* = (0]$. As $t \in I = (x]^* \vee (x]^{**}$, we have $t \leq a \vee b$ for some $a \in (x]^*$ and $b \in (x]^{**}$. Hence $a \wedge b = 0$. Further $(a \vee b]^* \subseteq (t]^* = (0]$ gives $(a]^* \cap (b]^* = (0]$ (see Result 4) and hence $(a]^* \subseteq (b]^{**}$. As $b \in (x]^{**}$, we get $(b]^{**} \subseteq \{(x]^{**}\}^{**} = (x]^{**}$. Thus we have $(a]^* \subseteq (x]^{**}$. At the same time $a \in (x]^*$ gives $(a]^* \supseteq (x]^{**}$. Combining both the inclusions we get $(a]^* = (x]^{**}$. Thus for any $x \in L$, there exists $a \in L$ such that $(a]^* = (x]^{**}$. Hence L is quasi-complemented (by Result 2). □

Consider the 0-distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is as in Fig. 2. All α-ideals of L are principal ideals. Note that not every principal ideal in a 0-distributive lattice is an α-ideal, e.g. $(a]$.

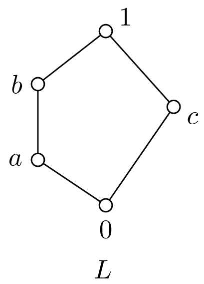


Fig. 2

Every finite 0-distributive lattice is both quasi-complemented and pseudo-complemented. This follows by the following theorem.

Theorem 3.6 *Let every α-ideal in L be a principal ideal. Then L is both quasi-complemented and pseudo-complemented.*

Proof Let $x \in L$. Then $(x]^*$ is an α-ideal in L (by Result 1). By assumption, $(x]^* = (a]$ for some $a \in L$. Then $x^* = a$. Therefore L is pseudo-complemented. Further $(x]^{**} = (a]^*$ shows that L is quasi-complemented (by Result 2). □

Theorem 3.7 *If L is a quasi-complemented lattice and if a prime ideal P of L contains no dense element, then P is an α-ideal.*

Proof Let $(a]^* = (b]^*$ and $a \in P$ for some $a, b \in L$. As L is quasi-complemented, there exists $x \in L$ such that $(a]^{**} = (x]^*$ (by Result 2). Hence $a \wedge x = 0$. Further

$$\begin{aligned}(a \vee x]^* &= (a]^* \cap (x]^* \quad \text{(by Result 4)}\\ &= (a]^* \cap (a]^{**} = (0].\end{aligned}$$

This implies $a \vee x \in D$. As $P \cap D = \emptyset$, $a \vee x \notin P$. We have $x \notin P$ due to $a \in P$. Clearly $x \in (a]^*$ implies $x \in (b]^*$ and hence $b \wedge x = 0$. As $b \wedge x \in P$ and $x \notin P$ we get $b \in P$. Thus given $(a]^* = (b]^*$ and $a \in P$ we conclude $b \in P$. Finally, P is an α-ideal (by Result 5). □

Corollary 3.8 *If L is a quasi-complemented lattice such that no proper α-ideal of L is a dense ideal, then every prime dense ideal of L contains a dense element.*

Proof Let P be a prime dense ideal of L. Assume $P \cap D = \emptyset$. Then by Theorem 3.7, P is an α-ideal. As P is a proper α-ideal, by assumption, P is non-dense, which is a contradiction. Hence $P \cap D \neq \emptyset$ and the result follows. □

In L, every annihilator ideal is an α-ideal but not conversely. A sufficient condition for an α-ideal to be an annihilator ideal is given in the following theorem.

Theorem 3.9 *If every dense ideal in L contains a dense element, then every α-ideal in L is an annihilator ideal.*

Proof Let I be an α-ideal of L. Clearly $I \subseteq I \vee I^*$ gives $(I \vee I^*)^* \subseteq I^*$ and $I^* \subseteq I \vee I^*$ yields $(I \vee I^*)^* \subseteq I^{**}$. Hence $(I \vee I^*)^* \subseteq I^* \cap I^{**} = (0]$ showing that $(I \vee I^*)$ is a dense ideal of L. By hypothesis, $(I \vee I^*) \cap D \neq \phi$. Let $d \in (I \vee I^*) \cap D$. As $d \in I \vee I^*$ we have $d \leq a \vee b$ for some $a \in I$ and $b \in I^*$. Hence by Result 4, $(a]^* \cap (b]^* \subseteq (d]^* = (0]$ gives $(b]^* \subseteq (a]^{**}$. Let $x \in I^{**}$. Then $b \wedge x = 0$ as $b \in I^*$. Thus $x \in (b]^* \subseteq (a]^{**}$. As $a \in I$ and I is an α-ideal, we get $(a]^{**} \subseteq I$. But then $x \in I$ shows that $I^{**} \subseteq I$. But as $I \subseteq I^{**}$ always holds, we get $I = I^{**}$ and the result follows. □

Though an α-ideal of L needs not be an annihilator ideal, there are some 0-distributive lattices in which every α-ideal is an annihilator ideal.

In a 0-distributive lattice L whose Hasse diagram is as in Fig. 2, the α-ideals are $(0], (b], (c]$ and $(1] = L$. Each of them is an annihilator ideal.

The set $I_\alpha(L)$ of all α-ideals of L forms a complete distributive lattice with $(0]$ as the least element and L as the greatest element and the set theoretic intersection as the infimum. The supremum of $I, J \in I_\alpha(L)$ is given by $I \oplus J = (I \vee J)^e$ (see [4]). As L is a 0-distributive lattice, the set $I(L)$ of all ideals of L forms a pseudo-complemented lattice (see Result 6). Hence the set $\{I \in I(L) \mid I = I^{**}\}$ forms a Boolean algebra (by Result 7). But $I = I^{**}$ if and only if I is an annihilator ideal of L. Thus the set $I_o(L)$ of all annihilator ideals of L forms a Boolean algebra. For a 0-distributive lattice in which every α-ideal is an annihilator ideal we have the set $I_\alpha(L)$ of all α-ideals of L forms a complete Boolean algebra.

We know that $L^* = (0]$. Hence L is always a dense ideal of L. But L needs not be the unique dense ideal of L. For this consider the following example.

Consider the 0-distributive lattice $L = \{0, a, b, c, d, e, 1\}$ whose Hasse diagram is in Fig. 3. Here $(a]^* = (b]^* = (c]^* = (d]^* = (e]^* = (0]$. Thus $(a]$, $(b]$, $(c]$, $(d]$, $(e]$ are all proper dense ideals of L.

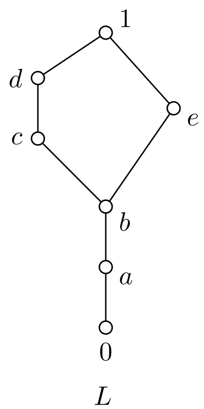


Fig. 3

Theorem 3.10 *If the lattice $I_\alpha(L)$ is semi complemented, then L is the only dense α-ideal in L.*

Proof Obviously, $L \in I_\alpha(L)$ as $L^* = (0]$ and $(0]^* = L$. Suppose that there exists a proper α-ideal I of L such that $I^* = (0]$. By assumption there exists $J \in I_\alpha(L)$, $J \neq (0]$ such that $I \cap J = (0]$. But then $J \subseteq I^* = (0]$ implies $J = (0]$, which is a contradiction. Hence L is the only dense, α-ideal of L. $\square$

Remark 1 Not every ideal has to be an α-ideal in a 0-distributive lattice. Indeed, consider the 0-distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in Fig. 2. Let $I = (a] = \{0, a\}$. Then I is an ideal of L. Now for $a \in I$, $(a]^{**} = \{0, a, b\} \nsubseteq I$. Hence $I = (a]$ is not an α-ideal. We know that L is disjunctive if and only if $(a]^* = (b]^*$ gives $a = b$. Hence if L is a disjunctive lattice, then every ideal in L is an α-ideal.

Remark 2 In a bounded distributive lattice, not every ideal has to be an α-ideal. For this consider the following example: The set $L = \{1, 2, 4, 5, 10, 20\}$ with respect to divisibility is a distributive lattice. The ideal $(2] = \{1, 2\}$ is not an α-ideal of L. The Hasse diagram of L is given in Fig. 4.

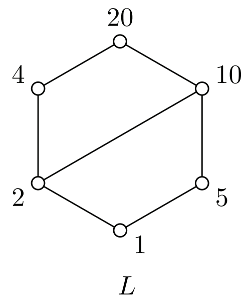


Fig. 4

Under the condition of disjunctivity in L we have

Theorem 3.11 *Let L be a bounded distributive lattice. L is disjunctive if and only if every ideal in L is an α-ideal.*

Proof ($\Rightarrow$) In a 0-distributive, disjunctive lattice, every ideal is an α-ideal and any distributive lattice with 0 is a 0-distributive lattice. Hence the implication follows.

($\Leftarrow$) Let every ideal in a distributive lattice L be an α-ideal. We prove that L is disjunctive. Let $x, y \in L$ be such that $(x]^* = (y]^*$ and $x \neq y$. As L is distributive, there is a prime ideal P containing exactly one of them. Assume that $y \in P$. As $(x]^* = (y]^*$ and $y \in P$ imply $x \in P$, since P is an α-ideal, which is a contradiction. Hence $(x]^* = (y]^*$ gives $x = y$ and shows that L is disjunctive. □

From the proof of Theorem 3.11, we immediately get

Corollary 3.12 *Let L be a bounded distributive lattice. Then the following statements are equivalent:*

1. *L is disjunctive.*
2. *Every ideal of L is an α-ideal.*
3. *Every prime ideal of L is an α-ideal.*

Theorem 3.13 *If any proper α-ideal of L is non dense, then any dense ideal of L contains a dense element.*

Proof Let I be a dense ideal of L, i.e. $I^* = (0]$. We have $I \subseteq I^e$ by Result 3.

Hence $(I^e)^* \subseteq I^* = (0]$ imply $(I^e)^* = (0]$. Let $(I^e) \cap D = \emptyset$. Since I^e is an α-ideal (by Result 3), there is a prime α-ideal P such that $I^e \subseteq P$ and $P \cap D = \emptyset$ (by Result 10). As P is a proper α-ideal, by assumption, $P^* \neq (0]$. But $I^e \subseteq P$ yields $P^* \subseteq (I^e)^* = (0]$. Thus $P^* = (0]$, which is a contradiction. Therefore $(I^e) \cap D \neq \emptyset$. Let $d \in (I^e) \cap D$. Then $d \in (I^e)$ gives the existence of $a \in I$ such that $(a]^* \subseteq (d]^* = (0]$. Hence $a \in I \cap D$ and consequently $I \cap D \neq \emptyset$. □

Combining the above results we get

Theorem 3.14 *The following statements are equivalent in L:*

1. $I^* \neq (0]$ *for any proper α-ideal I of L.*
2. $I \cap D \neq \emptyset$ *for any dense ideal I of L.*
3. *Every α-ideal is an annihilator ideal.*
4. $I_\alpha(L)$ *is semi-complemented.*
5. $I_\alpha(L)$ *has a unique dense element.*

Further any of the above conditions imply that L is quasi-complemented.

In the following theorem we give a sufficient condition for the collection of all prime ideals of L to be an antichain.

Theorem 3.15 *Let L be a 0-distributive lattice in which every prime ideal is an annihilator ideal. Then $\wp$, the collection of all prime ideals of L, is an antichain.*

Proof Let $P, Q \in \wp$ be such that $P \subset Q$. Choose $q \in Q \setminus P$ and $x \in Q^*$. Then $x \wedge q = 0 \in P$. As P is a prime ideal and $q \notin P$, we get $x \in P$. Thus $Q^* \subseteq P$. $P \subset Q$ implies $Q^* \subseteq P^*$. Thus $Q^* \subseteq P \cap P^*$ and so $Q^{**} = L$. By assumption, $Q^{**} = Q$, thus $Q = L$, which is a contradiction, as Q is a proper ideal. Hence no two of prime ideals are comparable. $\square$

Corollary 3.16 *A bounded distributive lattice L is a Boolean lattice if every prime ideal in L is an annihilator ideal.*

Proof By Theorem 3.15, the collection $\wp$ of all prime ideals of L is an antichain and hence by Nachbin's Theorem (Result 11), L is a Boolean lattice. $\square$

If I is an ideal in a finite Boolean lattice L, then $I = (x]$ for some $x \in L$. Hence $I^* = (x']$, where x' denotes the complement of x in L. Therefore $I^{**} = (x''] = (x] = I$. Thus every ideal in a finite Boolean lattice is an annihilator ideal. Hence we have

Theorem 3.17 *For a finite distributive lattice L the following statements are equivalent:*

1. *Every ideal in L is an annihilator ideal.*
2. *Every prime ideal is an annihilator ideal.*
3. $\wp$ *is an antichain.*
4. *L is a Boolean lattice.*

4 Annihilator preserving homomorphisms and α-ideals

Throughout this section L and L' denote bounded 0-distributive lattices with the least elements 0 and $0'$ respectively and $f\colon L \to L'$ denotes a 0-1 lattice homomorphism. f is called an annihilator preserving if $f(A^*) = \{f(A)\}^*$ for any $(0] \subset A \subset L$. We say that f^{-1} preserves annihilators if $f^{-1}(B^*) = \{f^{-1}(B)\}^*$ for any $(0'] \subset B \subset L'$.

Theorem 4.1 *Let $f\colon L \to L'$ be a homomorphism. Then we have:*

1. *If f is an annihilator preserving epimorphism, then for every annihilator ideal A of L, $f(A)$ is an annihilator ideal of L'.*
2. *If f^{-1} preserves annihilators, then for every annihilator ideal B of L', $f^{-1}(B)$ is an annihilator ideal of L.*

Proof (1) Let A be an annihilator ideal of L, i.e. $A^{**} = A$. By Result 12, $f(A)$ is an ideal of L'. As f is annihilator preserving, $\{f(A)\}^{**} = f(A^{**}) = f(A)$. This shows that $f(A)$ is an annihilator ideal of L'.

(2) Let B be an annihilator ideal of L'. Hence $B^{**} = B$. By Result 12, $f^{-1}(B)$ is an ideal of L. Since f^{-1} preserves annihilators, we get

$$\{f^{-1}(B)\}^{**} = f^{-1}(B^{**}) = f^{-1}(B).$$

This proves $f^{-1}(B)$ is an annihilator ideal of L. □

Corollary 4.2 *If $f\colon L \to L'$ is a homomorphism such that f^{-1} preserves the annihilators, then* $\operatorname{Ker} f$ *is an annihilator ideal and hence an α-ideal in L.*

Proof $\operatorname{Ker} f = \{x \in L \mid f(x) = 0'\}$ where $0'$ is the least element in L'. Hence $\operatorname{Ker} f = f^{-1}((0'])$. As $(0']$ is an annihilator ideal in L', by Theorem 4.1, $\operatorname{Ker} f$ is an annihilator ideal in L and hence an α-ideal in L (by Result 1). □

Theorem 4.3 *Let $f\colon L \to L'$ be an epimorphism. If* $\operatorname{Ker} f = \{0\}$*, then f is annihilator preserving and f^{-1} preserves annihilators.*

Proof (1) We prove that f is an annihilator preserving map.

Let $(0] \subset A \subset L$. Then we have $f(A^*) \subseteq (f(A))^*$. Let $x \in (f(A))^* \subseteq L'$. As f is onto, there exists $y \in L$ such that $f(y) = x \in (f(A))^*$

$$\begin{aligned}
&\Longrightarrow\quad f(y) \wedge f(a) = 0' \text{ for all } a \in A\\
&\Longrightarrow\quad f(y \wedge a) = 0'\\
&\Longrightarrow\quad y \wedge a \in \operatorname{Ker} f = \{0\}\\
&\Longrightarrow\quad y \wedge a = 0 \text{ for all } a \in A\\
&\Longrightarrow\quad y \in A^*\\
&\Longrightarrow\quad f(y) \in f(A^*) \text{ i.e. } x \in f(A^*).
\end{aligned}$$

Hence $(f(A))^* \subseteq f(A^*)$. Combining both the inclusions we get $f(A^*) = (f(A))^*$. This proves that f is annihilator preserving.

(2) We prove that f^{-1} preserves annihilators.

Let $(0] \subset A \subset L'$ and $x \in \{f^{-1}(A)\}^*$. Then $x \wedge a = 0$ for all $a \in f^{-1}(A)$.

$\Longrightarrow x \wedge a = 0$ for all $f(a) \in A$
$\Longrightarrow f(x) \wedge f(a) = 0'$ for all $f(a) \in A$
$\Longrightarrow f(x) \in A^*$
$\Longrightarrow x \in f^{-1}(A^*)$

Hence $\{f^{-1}(A)\}^* \subseteq f^{-1}(A^*)$.

Suppose $x \in f^{-1}(A^*)$ and $a \in f^{-1}(A)$. Then $f(x) \in A^*$ and $f(a) \in A$. Hence $f(x) \wedge f(a) = 0'$ gives $f(x \wedge a) = 0'$. Thus $x \wedge a \in \operatorname{Ker} f = \{0\}$. Therefore $x \wedge a = 0$, for all $a \in f^{-1}(A)$. Thus $x \in \{f^{-1}(A)\}^*$. This shows that $\{f^{-1}(A)\}^* \subseteq f^{-1}(A^*)$. Combining the two inclusions we get $f^{-1}(A^*) = \{f^{-1}(A)\}^*$. □

Theorem 4.4 *Let $f\colon L \to L'$ be an annihilator preserving epimorphism. If $\operatorname{Ker} f = \{0\}$, then we have:*

$$A^* = B^* \text{ if and only if } \{f(A)\}^* = \{f(B)\}^*$$

for any non-empty subsets A, B of L.

Proof Assume that $A^* = B^*$. Then clearly $f(A^*) = f(B^*)$. Since f is an annihilator preserving we get $\{f(A)\}^* = \{f(B)\}^*$.

Conversely, suppose $\{f(A)\}^* = \{f(B)\}^*$. Let $x \in A^*$. Then $x \wedge a = 0$ for all $a \in A$.

$\Longrightarrow f(x \wedge a) = 0'$ for all $a \in A$
$\Longrightarrow f(x) \wedge f(a) = 0'$ for all $a \in A$
$\Longrightarrow f(x) \in \{f(A)\}^*$
$\Longrightarrow f(x) \in \{f(B)\}^*$ by assumption
$\Longrightarrow f(x) \wedge f(b) = 0'$ for all $b \in B$
$\Longrightarrow f(x \wedge b) = 0'$ for all $b \in B$
$\Longrightarrow x \wedge b \in \operatorname{Ker} f = \{0\}$ for all $b \in B$
$\Longrightarrow x \wedge b = 0$ for all $b \in B$
$\Longrightarrow x \in B^*$

Hence $A^* \subseteq B^*$. Similarly we can prove $B^* \subseteq A^*$. Thererforе $A^* = B^*$. □

A necessary and sufficient condition for the inverse image of an α-ideal to be an α-ideal is given in the following theorem.

Theorem 4.5 *Let $f\colon L \to L'$ be an epimorphism. For every α-ideal J' of L', $f^{-1}(J')$ is an α-ideal in L if and only if for each $x' \in L'$, $f^{-1}\left((x']^*\right)$ is an α-ideal in L.*

Proof ($\Rightarrow$) Choose any $x' \in L'$. As the annihilator ideal $(x']^*$ is an α-ideal (by Result 1), the proof of 'only if part' follows by assumption.

($\Leftarrow$) Let J' be an α-ideal of L'. Then $f^{-1}(J')$ is an ideal of L (by Result 12). Let $x, y \in L$ be such that $(x]^* = (y]^*$ and $x \in f^{-1}(J')$. We claim that $(f(x)]^* = (f(y)]^*$.

Indeed, $f(t) \in (f(x)]^*$ implies $f(t) \wedge f(x) = 0$, thus $f(x) \in (f(t)]^*$ and $x \in f^{-1}\left[(f(t)]^*\right]$. By assumption $f^{-1}\left[(f(t)]^*\right]$ is an α-ideal of L. As $(x]^* = (y]^*$ and $x \in f^{-1}\left[(f(t)]^*\right]$, we get $y \in f^{-1}\left[(f(t)]^*\right]$ (by Result 5). But then $f(t) \wedge f(y) = 0$ gives $f(y) \in (f(t)]^*$ thus $y \in f^{-1}\left[(f(t)]^*\right]$ and consequently $f(t) \in (f(y)]^*$. Thus we get $(f(x)]^* \subseteq (f(y)]^*$. Similarly, we can prove that $(f(y)]^* \subseteq (f(x)]^*$. Hence $(f(x)]^* = (f(y)]^*$. Now $x \in f^{-1}(J')$ yields $f(x) \in J'$. As J' is an α-ideal, $(f(x)]^* = (f(y)]^*$ and $f(x) \in J'$ give $f(y) \in J'$ (by Result 5). But then $y \in f^{-1}(J')$. Thus $(x]^* = (y]^*$ and $x \in f^{-1}(J')$ imply $y \in f^{-1}(J')$. Hence by Result 5, $f^{-1}(J')$ is an α-ideal of L. □

We prove that the images and the inverse images of α-ideals under annihilator preserving homomorphism of 0-distributive lattices are again α-ideals.

Theorem 4.6 *Let $f\colon L \to L'$ be an annihilator preserving epimorphism.*

1. *If I is an α-ideal of L, then $f(I)$ is an α-ideal of L'.*
2. *If J' is an α-ideal of L', then $f^{-1}(J')$ is an α-ideal of L.*

Proof (1) Let I be an α-ideal of L. By Result 12, $f(I)$ is an ideal of L'. Let $f(a) \in f(I)$, i.e. $a \in I$. As I is an α-ideal, $(a]^{**} \subseteq I$. Hence $f((a]^{**}) \subseteq f(I)$. Since f is annihilator preserving we have $f((a]^{**}) = (f(a)]^{**}$. Thus $(f(a)]^{**} \subseteq f(I)$. This shows that $f(I)$ is an α-ideal of L'.

(2) Let J' be an α-ideal of L'. Let $x, y \in L$ with $(x]^* = (y]^*$ and $x \in f^{-1}(J')$. $(x]^* = (y]^*$ implies $f((x]^*) = f\left((y]^*\right)$. By assumption, $(f(x)]^* = (f(y)]^*$. Further $x \in f^{-1}(J')$ gives $f(x) \in J'$. Now $(f(x)]^* = (f(y)]^*$, $f(x) \in J'$ and J' is an α-ideal of L', consequently $f(y) \in J'$ (by Result 5) and $y \in f^{-1}(J')$. Thus $(x]^* = (y]^*$ and $x \in f^{-1}(J')$ imply $y \in f^{-1}(J')$. Finally, by Result 5, $f^{-1}(J')$ is an α-ideal of L. □

Acknowledgement. Authors are thankful to UGC, New Delhi for financial support through scheme F. No. 33-109/2007 (SR) and the referee for his valuable suggestions.

References

[1] Balasubramani, P., Venkatanarsimhan: *Characterizations of the 0-distributive lattice.* Indian J. Pure Appl. Math. **32**, 3 (2001), 315–324.

[2] Balasubramani, P.: *Stone topology of the set of the set of prime filters of a 0-distributive lattice.* Indian J. Pure Appl. Math. **35**, 2 (2004), 149–158.

[3] Cornish, W. H.: *Annulets and $\propto$-ideals in a distributive lattice.* J. Aust. Math. Soc. **15**, 1 (1975), 70–77.

[4] Grätzer, G.: Lattice Theory – First concepts and distributive lattices. *Freeman and Company*, San Francisco, 1971.

[5] Jayaram, C.: *Prime α-ideals in a 0-distributive lattice.* Indian J. Pure Appl. Math. **17**, 3 (1986), 331–337.

[6] Pawar, Y. S., Mane, D. N.: *α-ideals in 0-distributive semilattices and 0-distributive lattices.* Indian J. Pure Appl. Math. **24**, 7-8 (1993), 435–443.

[7] Varlet, J.: *A generalization of the notion of pseudo-complementedness.* Bull. Soc. Roy. Liege **37** (1968), 149–158.