135 (2010) MATHEMATICA BOHEMICA No. 1, 63–68

# ON THE LONELY RUNNER CONJECTURE

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(Received January 20, 2009)

*Abstract.* Suppose $k+1$ runners having nonzero distinct constant speeds run laps on a unit-length circular track. The Lonely Runner Conjecture states that there is a time at which a given runner is at distance at least $1/(k+1)$ from all the others. The conjecture has been already settled up to seven ($k \leqslant 6$) runners while it is open for eight or more runners. In this paper the conjecture has been verified for four or more runners having some particular speeds using elementary tools.

*Keywords*: congruences, arithmetic progression, bi-arithmetic progression

*MSC 2010*: 11B25, 11B75

## 1. Introduction

In 1967, Wills [14] stated a conjecture, now known as the Lonely Runner Conjecture. According to Goddyn in [4] it reads as follows:

Suppose $k+1$ runners having nonzero distinct constant speeds run laps on a unit-length circular track. The Lonely Runner Conjecture states that there is a time at which one runner is at distance at least $1/(k+1)$ from all the others.

The same conjecture was also stated independently by Cusick [6] in 1974. For $k \leqslant 3$ the conjecture was settled by Betke and Wills in [3] who were dealing with some Diophantine approximation problem and also independently by Cusick in [6] who was considering $n$-dimensional geometry view-obstruction problem. The case $k = 4$ was first proved by Cusick and Pomerence in [7] with a proof that requires a computer work. Later, Bienia et al. in [4] gave a simpler proof for $k = 4$. The case $k = 5$ was proved by Bohman, Holzman and Kleitman in [5]. A simpler proof

The study was done partially in the Mathematics research team LaMUSE of the Saint-Etienne University, France, under postdoctoral fellowship sponsored by Saint-Etienne-Metropole.

for this case was given by Renault in [13]. Recently, Barajas and Serra in ([1], [2]) proved the conjecture for $k = 6$. Some more work on this conjecture can be found in [12]. For $k \geqslant 7$ the conjecture is still open. We verify the conjecture for four or more runners having some particular speeds using elementary techniques.

## 2. Definitions and useful known results

**Definition 2.1.** Suppose $M = \{m_1, m_2, \dots, m_k\}$ where $m_i$'s are positive integers and let $\|x\|$ denote the distance of the real number $x$ from the nearest integer. Denote

$$\kappa(M) = \sup_{x\in(0,1)} \min_i \|xm_i\|.$$

The Lonely Runner Conjecture in the form of Wills and Cusick reads as follows: Suppose $M$ is a finite set of positive integers with $|M| = k$. Then

$$\kappa(M) \geqslant \frac{1}{k+1}.$$

Haralambis in [10] gave a remark which gives three equivalent definitions for $\kappa(M)$.

Remark 2.1 (Haralambis, [10]). Let $M = \{m_1, m_2, \dots, m_k\}$ and

$$\kappa_1(M) = \sup_{x\in(0,1)} \min_i \|xm_i\|,$$
$$\kappa_2(M) = \sup_{(c,m)=1} \frac{1}{m} \min_i |cm_i|_m,$$
$$\kappa_3(M) = \max_{\substack{m=m_j+m_l\\ 1\leqslant x\leqslant m/2}} \frac{1}{m} \min |xm_i|_m,$$

where $|y|_m$ denotes the absolute value of the absolutely least remainder of $y \pmod m$. Then $\kappa_1(M) = \kappa_2(M) = \kappa_3(M)$, and we denote this common value by $\kappa(M)$.

It is straightforward from the definition that

(∗) if $d$ is a positive integer such that $dM_1 = M_2$, then $\kappa(M_1) = \kappa(M_2)$;

(∗∗) if $M_1 \subset M_2$, then $\kappa(M_1) \geqslant \kappa(M_2)$.

**Definition 2.2.** A set of the form $I \cup J$ is called a bi-arithmetic progression of length $k$ with difference $d$ if both $I$ and $J$ are arithmetic progressions of difference $d$, $|I| + |J| = k$, and $I + I$, $I + J$, $J + J$ are pairwise disjoint.

Now we mention (without proof) some already known results which are useful in our discussion.

**Theorem 2.1** (Freiman, [8]). *Suppose* $|M| = k$, $k \geqslant 4$ *and* $|M+M| = 2k-1+b < 3k-3$. *Then* $M$ *is a subset of an arithmetic progression of length at most* $k+b$.

**Theorem 2.2** (Jin, [11]). *There exists a positive real number* $\varepsilon$ *and a natural number* $K$ *such that for any finite set of natural numbers* $M$ *with* $|M| = k > K$ *and* $|M+M| = 3k-3+b$ *for* $0 \leqslant b \leqslant \varepsilon k$, $M$ *is either a subset of an arithmetic progression of length at most* $2k-1+2b$ *or a subset of a bi-arithmetic progression of length at most* $k+b$.

## 3. Main results

**Theorem 3.1.** *If* $M = \{a, a+d, a+2d, \ldots, a+(k-1)d\}$ *and* $k \geqslant 1$, *then*

$$\kappa(M) \begin{cases} \geqslant \dfrac{2a+(k-1)(d-1)}{2\{2a+(k-1)d\}} & \text{if } d \text{ is odd;} \\ = \frac{1}{2} & \text{if } d \text{ is even.} \end{cases}$$

Proof. If $d$ is even then $M$ contains only odd integers. Choosing $x = 1/2$ and applying $\kappa_1(M)$, we get the result in this case. Now suppose that $d$ is odd. Let $m = 2a+(k-1)d$. Since $\gcd(d,m) = 1$, we have $dx \equiv 1 \pmod{m}$ for some integer $x$. Let $dx = 1 + mq$. Then $x$ and $mq$, hence $x$ and $(k-1)q$ are of opposite parity, and so

$$ax \equiv \frac{m-(k-1)d}{2}x = \frac{m[x-(k-1)q]-(k-1)}{2} \equiv \frac{m-(k-1)}{2} \pmod{m}.$$

Therefore, for $0 \leqslant l \leqslant k-1$,

$$(a+ld)x \equiv \frac{m}{2} + \Big(l - \frac{k-1}{2}\Big) \pmod{m}.$$

Thus applying $\kappa_3(M)$, we have

$$\kappa(M) \geqslant \frac{2a+(k-1)(d-1)}{2\{2a+(k-1)d\}}.$$

□

**Theorem 3.2.** *Suppose $|M| = k$, $k \geqslant 4$ and $|M+M| = 2k-1+b < 3k-3$. Then $\kappa(M) \geqslant 1/(k+1)$, provided also that if $M$ is a subset of an arithmetic progression with difference 1 then the first term of the arithmetic progression is greater than 1.*

Proof. It is clear from Freiman's theorem that $M$ is a subset of an arithmetic progression of length at most $k+b$. Now suppose that the first term of the arithmetic progression is $a$ and the difference is $d$. Then we have $M \subseteq \{a, a+d, \ldots, a+(k+b-1)d\}$. Without loss of generality, take $\gcd(a,d) = 1$. Then using $(*)$ and Theorem 3.1, we have

$$\kappa(M) \geqslant \kappa(\{a, a+d, \ldots, a+(k+b-1)d\}) \geqslant \frac{2a+(k+b-1)(d-1)}{2\{2a+(k+b-1)d\}}.$$

We now show that

$$\frac{2a+(k+b-1)(d-1)}{2\{2a+(k+b-1)d\}} \geqslant \frac{1}{k+1}.$$

This is true if and only if $(k+1)\{2a+(k+b-1)(d-1)\} \geqslant 2\{2a+(k+b-1)d\}$, if and only if $2a(k-1) \geqslant (k+b-1)\{2d-(k+1)(d-1)\} = (k+b-1)\{k+1-(k-1)d\}$. Notice that this is always true for $d \geqslant 2$. Therefore, now suppose that $d = 1$. Then the above inequality is equivalent to $2a(k-1) \geqslant 2(k+b-1)$. Since $2k-1+b < 3k-3$, hence, $k+b-1 < 2k-3$. Thus the inequality is true if and only if $a(k-1) \geqslant 2k-3$, if and only if $a \geqslant (2k-3)/(k-1)$ $(< 2)$. This completes the proof. □

**Theorem 3.3.** *Suppose there exists a positive real number $\varepsilon$ and a natural number $K$ such that $M$ is a finite set of positive integers with $|M| = k > K$ and $|M+M| = 3k-3+b$ for $0 \leqslant b \leqslant \varepsilon k$. Then $\kappa(M) \geqslant 1/(k+1)$ provided $M$ is not a subset of a bi-arithmetic progression, and if $M$ is a subset of an arithmetic progression with difference 1, the first term of the arithmetic progression must be greater than or equal to $2\{(\varepsilon+1)k-1\}/(k-1)$.*

Proof. It is clear from Jin's theorem that $M$ is a subset of an arithmetic progression of length at most $2k-1+2b$. Now suppose that the first term of the arithmetic progression is $a$ and the difference is $d$. Then we have $M \subseteq \{a, a+d, \ldots, a+2(k+b-1)d\}$. Without loss of generality, take $\gcd(a,d) = 1$. Then using $(*)$ and Theorem 3.1, we have

$$\kappa(M) \geqslant \kappa(\{a, a+d, \ldots, a+2(k+b-1)d\}) \geqslant \frac{2a+2(k+b-1)(d-1)}{2\{2a+2(k+b-1)d\}}.$$

We now show that

$$\frac{2a+2(k+b-1)(d-1)}{2\{2a+2(k+b-1)d\}} \geqslant \frac{1}{k+1}.$$

This is true if and only if $(k+1)\{a+(k+b-1)(d-1)\} \geqslant 2\{a+(k+b-1)d\}$, if and only if $a(k-1) \geqslant (k+b-1)\{2d-(k+1)(d-1)\} = (k+b-1)\{k+1-(k-1)d\}$. Notice that this is always true for $d \geqslant 2$ and $k \geqslant 3$. Therefore, now suppose that $d = 1$. Then the above inequality is equivalent to $a(k-1) \geqslant 2(k+b-1)$. Since $b \leqslant \varepsilon k$, hence, $k+b-1 \leqslant (\varepsilon+1)k-1$. Thus the inequality is true if and only if $a(k-1) \geqslant 2\{(\varepsilon+1)k-1\}$, if and only if $a \geqslant 2\{(\varepsilon+1)k-1\}/(k-1)$. This completes the proof. □

Observation. Theorem 3.3 gives more choices for the speeds of the runners satisfying the Lonely Runner Conjecture than Theorem 3.2 because in Theorem 3.3 the set $M$ has larger doubling property than the set $M$ in Theorem 3.2.

The following example shows that the statements of Theorem 3.2 and Theorem 3.3 are not completely equivalent, that is, this example satisfies Theorem 3.3 but not Theorem 3.2.

Example 3.1. For $k > 15$, let $M = [0, k-3] \cup \{k+10, 2k+20\}$. Then $|M| = k$ and $|M+M| = 3k+9$. The shortest arithmetic progression containing $M$ has length $2k+21$.

Acknowledgement. I am very much thankful to the referee for his/her careful reading of the paper and giving useful suggestions how to present the paper in a better form.

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