Yankui Song

On relatively almost Lindelöf subsets

Mathematica Bohemica, Vol. 134 (2009), No. 2, 183–190

Persistent URL: http://dml.cz/dmlcz/140653

134 (2009) MATHEMATICA BOHEMICA No. 2, 183–190

ON RELATIVELY ALMOST LINDELÖF SUBSETS

Yankui Song, Nanjing

(Received March 11, 2008)

Abstract. A subspace Y of a space X is almost Lindelöf (strongly almost Lindelöf) in X if for every open cover $\mathscr{U}$ of X (of Y by open subsets of X), there exists a countable subset $\mathscr{V}$ of $\mathscr{U}$ such that $Y \subseteq \bigcup\{\overline{V}\colon V \in \mathscr{V}\}$. In this paper we investigate the relationships between relatively almost Lindelöf subset and relatively strongly almost Lindelöf subset by giving some examples, and also study various properties of relatively almost Lindelöf subsets and relatively strongly almost Lindelöf subsets.

Keywords: Lindelöf space, strongly Lindelöf subset, almost Lindelöf subset, strongly almost Lindelöf subset

MSC 2010: 54D15, 54D20

1. Introduction

By a space we mean a topological space. Recall from [1], [2], [4] that a subspace Y of a space X is Lindelöf in X if for every open cover $\mathscr{U}$ of X there exists a countable subfamily covering Y. A space X is *almost Lindelöf* (see [5], [6]) if for every open cover $\mathscr{U}$ of X there exists a countable subfamily $\mathscr{V}$ such that $X = \bigcup\{\overline{V}\colon V \in \mathscr{V}\}$. Motivated by the classes of these spaces, the following classes of spaces are given:

Definition 1.1. A subspace Y of a space X is *strongly Lindelöf* in X if for every open cover $\mathscr{U}$ of Y by open subsets of X there exists a countable subset $\mathscr{V}$ of $\mathscr{U}$ such that $Y \subseteq \bigcup\{V\colon V \in \mathscr{V}\}$.

Definition 1.2. A subspace Y of a space X is *almost Lindelöf* in X if for every open cover $\mathscr{U}$ of X there exists a countable subset $\mathscr{V}$ of $\mathscr{U}$ such that $Y \subseteq \bigcup\{\overline{V}\colon V \in \mathscr{V}\}$.

The author acknowledges supports from the NSF of China (Grant No 10571081) and the National Science Foundation of Jiangsu Higher Education Institutions of China (Grant No 07KJB-110055).

Definition 1.3. A subspace Y of a space X is *strongly almost Lindelöf* in X if for every open cover $\mathscr{U}$ of Y by open subsets of X there exists a countable subset $\mathscr{V}$ of $\mathscr{U}$ such that $Y \subseteq \bigcup\{\overline{V}\colon V \in \mathscr{V}\}$.

Let X be a space and Y a subspace of X. From the above definitions it is clear that if Y is Lindelöf in X, then Y is almost Lindelöf in X; if Y is strongly Lindelöf in X, then Y is Lindelöf in X; if Y is strongly Lindelöf in X, then Y is strongly almost Lindelöf in X and if Y is strongly almost Lindelöf in X, then Y is almost Lindelöf in X, but the converse implications do not hold.

The purpose of this note is to investigate the relationships between the spaces given above by giving some examples and also to study various properties of relatively almost Lindelöf subsets and relatively strongly almost Lindelöf subsets.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. For a space X and a subspace Y of X, the *extent* $e(Y, X)$ of Y in X [2] is defined as the smallest cardinal number κ such that the cardinality of every discrete subspace Y which is closed in X is not greater than κ. Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal and ω_2 the second uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Let $\langle a, b\rangle$ denote the ordered pair having a as the first coordinate and b as the second coordinate. If $(X, \leqslant)$ is a partially ordered set and $a < b$ in X, then (a, b) denotes the interval $\{x \in X\colon a < x < b\}$. Other terms and symbols that we do not define will be used as in [3].

2. Some examples of relatively almost Lindelöf subsets

In this section we give some examples of relatively almost Lindelöf subsets and relatively strongly almost Lindelöf subsets. First, we give two positive results which are proved easily:

Theorem 2.1. *Let X be a regular space and Y a subspace of X. If Y is almost Lindelöf in X, then Y is Lindelöf in X.*

Theorem 2.2. *Let X be a regular space and Y a subspace of X. If Y is almost Lindelöf in X, then Y is Lindelöf in X.*

Now, we give an example showing that Theorems 2.1 and 2.2 are not true for Hausdorff spaces.

Example 2.3. There exist a Hausdorff almost Lindelöf space X and two subspaces Y_1 and Y_2 of X such that Y_1 is almost Lindelöf in X but Y_1 is not Lindelöf in X and Y_2 is strongly almost Lindelöf in X but Y is not Lindelöf in X.

P r o o f. Let $D=\{d_\alpha\colon \alpha<\omega_1\}$ be a discrete space of cardinality ω_1. Let $X=([0,1]\times D)\cup\{a\}$ where $a\notin[0,1]\times D$. Define a basic open set for a topology on X as follows: $[0,1]\times D$ has the usual product topology and is an open subspace of X; the basic open sets containing a take the form $\{a\}\cup\bigcup\{[0,1)\times\{d_\alpha\}\colon \alpha>\beta\}$ where $\beta<\omega_1$. The topology generated by these basic open sets is Hausdorff but it is not regular, since a cannot be separated from the closed subset $\{\langle 1,d_\alpha\rangle\colon \alpha<\omega_1\}$ by disjoint open subsets of X. Each subset $[0,1]\times\{d_\alpha\}$ is compact for each $\alpha<\omega_1$. Clearly, X is not Lindelöf, since $\{\langle 1,d_\alpha\rangle\colon \alpha<\omega_1\}$ is a discrete closed subset of X of cardinality ω_1. Let $Y_1=\{\langle 1,d_\alpha\rangle\colon \alpha<\omega_1\}$ and $Y_2=\{\langle 1,d_\alpha\rangle\colon \alpha<\omega_1\}\cup\{a\}$.

First, we show that X is almost Lindelöf. To this end, let $\mathscr{U}$ be an open cover of X. Then there is a $U_a\in\mathscr{U}$ such that $a\in U_a$. Thus there exists an $\alpha_0<\omega_1$ such that

$$U_a'=\{a\}\cup\bigcup\{[0,1)\times\{d_\alpha\}\colon \alpha>\alpha_0\}\subseteq U_a.$$

For each $\alpha\leqslant\alpha_0$, since $[0,1]\times\{d_\alpha\}$ is compact, there exists a finite subfamily $\mathscr{V}_\alpha$ of $\mathscr{U}$ such that

$$[0,1]\times\{d_\alpha\}\subseteq\bigcup\mathscr{V}_\alpha.$$

If we put $\mathscr{V}=\bigcup\limits_{\alpha\leqslant\alpha_0}\mathscr{V}_\alpha\cup\{U_a\}$, then $\mathscr{V}$ is a countable subfamily of $\mathscr{U}$. By the definition of topology of X it is not difficult to see that $X=\bigcup\{\overline{V}\colon V\in\mathscr{V}\}$, which shows that X is almost Lindelöf. Thus, Y_1 is almost Lindelöf in X, since every subset of an almost Lindelöf space X is almost Lindelöf in X.

Next, we show that Y_1 is not Lindelöf in X. Let $U_\alpha=[0,1]\times\{d_\alpha\}$ for each $\alpha<\omega_1$ and

$$U_a=\{a\}\cup([0,1)\times D).$$

Let us consider the open cover $\mathscr{U}=\{U_\alpha\colon \alpha<\omega_1\}\cup\{U_a\}$ of X. Let $\mathscr{V}$ be any countable subfamily of $\mathscr{U}$. Let

$$\alpha_0=\sup\{\alpha\colon U_\alpha\in\mathscr{V}\}.$$

Then $\alpha_0<\omega_1$, since $\mathscr{V}$ is countable. If we pick $\alpha'>\alpha_0$, then $\langle 1,d_\alpha'\rangle\notin\bigcup\mathscr{V}$, since $U_{\alpha'}$ is the only element of $\mathscr{U}$ containing $\langle 1,d_\alpha'\rangle$ and $U_{\alpha'}\notin\mathscr{V}$, which shows that Y_1 is not Lindelöf in X.

Similarly to the proof that Y_1 is not Lindelöf in X we may show that Y_2 is not Lindelöf in X.

Finally, we show that Y_2 is strongly almost Lindelöf in X. Let $\mathscr{U}$ be an open cover of Y_2 by open subsets of X. Then there is a $U_a\in\mathscr{U}$ such that $a\in U_a$. Thus there exists an $\alpha_0<\omega_1$ such that

$$U_a'=\{a\}\cup\bigcup\{[0,1)\times\{d_\alpha\}\colon \alpha>\alpha_0\}\subseteq U_a.$$

For each $\alpha \leqslant \alpha_0$ we pick $V_\alpha \in \mathscr{U}$ such that

$$\langle 1, d_\alpha\rangle \in V_\alpha.$$

If we put $\mathscr{V}'=\{V_\alpha\colon\ \alpha \leqslant \alpha_0\}$, then

$$\{\langle 1, d_\alpha\rangle\colon\ \alpha \leqslant \alpha_0\} \subseteq \bigcup \mathscr{V}'.$$

Let $\mathscr{V}=\mathscr{V}' \cup\{U_a\}$. Then $\mathscr{V}$ is a countable subfamily of $\mathscr{U}$. By the definition of topology of X it is not difficult to see that

$$Y_2 \subseteq \bigcup\{\overline{V}\colon\ V \in \mathscr{V}\},$$

which shows that Y_2 is strongly almost Lindelöf in X.

Remark 2.4. In Example 2.3, since Y_2 is not Lindelöf in X, Y_2 is not strongly Lindelöf in X. Thus, Example 2.3 shows that there exist a Hausdorff space X and a subspace Y of X such that Y is strongly almost Lindelöf in X but Y is not strongly Lindelöf in X.

Remark 2.5. By Theorem 2.2, for a subset Y of a regular space X, if Y is strongly almost Lindelöf in X, then Y is strongly Lindelöf in X, hence Y is Lindelöf in X. But the statement need not be true for the class of Hausdorff spaces. Example 2.3 shows that there exist a Hausdorff space X and a subspace Y of X such that Y is strongly almost Lindelöf in X but Y is not Lindelöf in X. The converse need not be true for the class of Tychonoff spaces as the following example shows.

Example 2.6. There exist a regular space X and a subspace Y of X such that Y is Lindelöf in X but X is not Lindelöf and Y is not strongly almost Lindelöf in X.

Proof. Let $X_1 = \omega_1$ with the usual topology and $X_2 = \{x_\alpha\colon\ \alpha < \omega_1 + 1\}$ with the following topology: If $\alpha < \omega_1$, then $\{x_\alpha\}$ is open. A set containing x_{ω_1} is open if and only if its complement is countable. With this topology, X_2 is regular and Lindelöf. Let $X = X_1 \times X_2$ and $Y = \{0\} \times \{x_\alpha\colon\ \alpha < \omega_1\}$. Since $X_1 \times \{0\}$ is homeomorphic to ω_1, it is not Lindelöf. Thus X is not Lindelöf, since X_1 is a closed subset of X.

To show that Y is Lindelöf in X, let $\mathscr{U}$ be an open cover of X. Since $\{0\} \times X_2$ is homeomorphic to X_2, $\{0\} \times X_2$ is a Lindelöf subset, hence there exists a countable subfamily $\mathscr{V}$ of $\mathscr{U}$ such that $\{0\} \times X_2 \subseteq \bigcup \mathscr{V}$, thus $Y \subseteq \bigcup \mathscr{V}$, which shows that Y is Lindelöf in X.

Next, we show that Y is not strongly almost Lindelöf in X. Put $U_\alpha = X_1 \times \{x_\alpha\}$ for each $\alpha < \omega_1$. Let us consider the open cover

$$\mathscr{U} = \{U_\alpha\colon\ \alpha < \omega_1\}$$

of Y by open sets of X. Let $\mathscr{V}$ be any countable subfamily of $\mathscr{U}$. If we put

$$\alpha_0 = \sup\{\alpha\colon\ U_\alpha \in \mathscr{V}\},$$

then $\alpha_0 < \omega_1$, since $\mathscr{V}$ is countable. Hence,

$$\bigcup \mathscr{V} = \bigcup\{\overline{V}\colon\ V \in \mathscr{V}\} \subseteq X_1 \times \{x_\alpha\colon\ \alpha < \alpha_0 + 1\}.$$

If we pick $\alpha' > \alpha_0$, then $\langle 0, x_{\alpha'}\rangle \in U_{\alpha'}$ and $U_{\alpha'} \notin \mathscr{V}$, since $U_{\alpha'}$ is the only element of $\mathscr{V}$ containing $\langle 0, x_{\alpha'}\rangle$, hence $\langle 0, x_{\alpha'}\rangle \notin \bigcup\{\overline{V}\colon\ V \in \mathscr{V}\}$, which completes the proof.

Remark 2.7. In Example 2.6, since Y is Lindelöf in X, Y is almost Lindelöf in X; since Y is not strongly almost Lindelöf in X, hence Y is not strongly Lindelöf in X. Thus, by Example 2.6, we can see that

(1) there exist a Tychonoff space X and a subspace Y of X such that Y is almost Lindelöf in X but Y is not strongly almost Lindelöf in X;

(2) there exist a Tychonoff space X and a subspace Y of X such that Y is Lindelöf in X but Y is not strongly Lindelöf in X.

Arhangel'skii [2] showed that if Y is Lindelöf in X, then the extent $e(Y,X)$ of Y in X is countable. Clearly, if Y is strongly Lindelöf in X, then the extent $e(Y,X)$ of Y in X is countable. From Theorems 2.1 and 2.2 it follows that if Y is almost Lindelöf (strongly almost Lindelöf) in a regular space X, then the extent $e(Y,X)$ of Y in X is countable. Now, we give an example showing that the statement is not true for the class of Hausdorff spaces.

Example 2.8. For every infinite cardinal κ, there exist a Hausdorff space X and a subspace Y of X such that Y is strongly almost Lindelöf in X and $e(Y,X) \geqslant \kappa$.

Proof. For every infinite cardinal κ, let D be a discrete space of cardinality κ. Let $X = ([0,1]\times D)\cup\{a\}$ where $a \notin [0,1]\times D$. Define a basic open set for a topology on X as follows: $[0,1]\times D$ has the usual product topology and is an open subspace of X; the basic open sets containing a take the form $\{a\}\cup\bigcup\{[0,1)\times(D\setminus F\}$ where F is a countable subset of D. The topology generated by these basic open sets is Hausdorff but it is not regular. Each subset $[0,1]\times\{\alpha\}$ is compact for each $\alpha < \omega_1$. Clearly, X is not Lindelöf if $\kappa \geqslant \omega$, since $\{\langle 1,\alpha\rangle\colon\ \alpha < \omega_1\}$ is a discrete closed subset of X of cardinality ω_1. Let

$$Y = \{\langle 1, d\rangle\colon\ d \in D\} \cup \{a\}.$$

Then $e(Y,X) \geqslant \kappa$, since Y is a closed discrete subspace of X with $|Y| = \kappa$. As in the proof of Example 2.3, we can prove that Y is strongly almost Lindelöf in X.

3. Various topological properties of relatively almost Lindelöf subsets and strongly relatively almost Lindelöf subsets

From the definition of relatively almost Lindelöf subset it is not difficult to see that every subset of an almost Lindelöf space X is almost Lindelöf in X. If $A \subseteq B \subseteq X$ and A is almost Lindelöf in B, then A is almost Lindelöf in X. However, the converse is not true as the following example shows.

Example 3.1. There exist a Tychonoff space X and two subsets A and B of X with $A \subseteq B$ such that A is almost Lindelöf in X but A is not almost Lindelöf in B.

Proof. Let $X = \{x_\alpha\colon \alpha < \omega_2\} \cup \{a\}$ with the following topology: If $\alpha < \omega_2$, then $\{x_\alpha\}$ is open. A set containing a is open if and only if its complement is countable. With this topology, X is regular and Lindelöf. Let $A = \{x_\alpha\colon \alpha < \omega_1\}$ and $B = \{x_\alpha\colon \alpha < \omega_2\}$. Then A is almost Lindelöf in X, since X is Lindelöf.

Next, we note that A is not almost Lindelöf in B, since B is a discrete space and A is a subspace of B with $|A| = \omega_1$, which completes the proof.

Remark 3.2. It is not difficult to see that A is not strongly almost Lindelöf in X in Example 3.1. Thus Example 3.1 shows that a subset of a Lindelöf space X need not be strongly almost Lindelöf in X.

From the construction of Example 3.1 it is not difficult to see that B is an open subset of X. In the following, we give an example showing that this is also not true for a closed set.

Example 3.3. There exist a Hausdorff space X and two subsets A and B of X with $A \subseteq B$ such that A is almost Lindelöf in X and B is closed in X but A is not almost Lindelöf in B.

Proof. Let X be the same space X as in Example 2.3. Let $A = Y_1$ and $B = Y_2$. Then A is almost Lindelöf in X, since X is almost Lindelöf. But A is not almost Lindelöf in B, since B is a discrete space and A is a discrete subspace of B with $|A| = \omega_1$.

Now, we give a positive result which can be easily proved.

Proposition 3.4. *If $A \subseteq B \subseteq X$, where B is a clopen subset of X, then A is almost Lindelöf in B iff A is almost Lindelöf in X.*

For relatively strongly almost Lindelöf subsets, it is not difficult to prove the following proposition.

Proposition 3.5. *If $A \subseteq B \subseteq X$, then A is strongly almost Lindelöf in B iff A is strongly almost Lindelöf in X.*

Theorem 3.6. *Let $f\colon X \to Y$ be a continuous function. If A is almost Lindelöf in X, then $f(A)$ is almost Lindelöf in Y.*

Proof. Let $\{U_\alpha\colon \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(U_\alpha)\colon \alpha \in \Lambda\}$ is an open cover of X. Since A is almost Lindelöf in X, there exists a countable subset $\{\alpha_i\colon i \in \omega\}$ of Λ such that $A \subseteq \bigcup\limits_{i\in\omega} \overline{f^{-1}(U_{\alpha_i})}$ and thus

$$f(A) \subset \Bigl(\bigcup_{i\in\omega} \overline{f^{-1}(U_{\alpha_i})}\Bigr) \subseteq \bigcup_{i\in\omega} f(\overline{f^{-1}(U_{\alpha_i})}) \subseteq \bigcup_{i\in\omega} \overline{f(f^{-1}(U_{\alpha_i}))} \subseteq \bigcup_{i\in\omega} \overline{U_{\alpha_i}}.$$

Hence, $f(A)$ is almost Lindelöf in Y, which completes the proof.

Similarly to the previous result, one can prove the following:

Theorem 3.7. *Let $f\colon X \to Y$ be a continuous function. If A is strongly almost Lindelöf in X, then $f(A)$ is strongly almost Lindelöf in Y.*

Proposition 3.8. *If A is almost Lindelöf in X and Y is a compact space, then $A \times Y$ is almost Lindelöf in $X \times Y$.*

Proof. Let $\mathscr{U}$ be an open cover of $X \times Y$. Without loss of generality, we can assume that $\mathscr{U}$ consists of basic open sets of $X \times Y$. Since $\{x\} \times Y$ is a compact subset of $X \times Y$ for each $x \in X$, there exists a finite subfamily $\{U_{x,i} \times V_{x,i}\colon i = 1, 2, \ldots n_x\}$ of $\mathscr{U}$ such that

$$\{x\} \times Y \subseteq \bigcup \{U_{x,i} \times V_{x,i}\colon 1 \leqslant i \leqslant n_x\}.$$

Let $W_x = \bigcap \{U_{x_i}\colon 1 \leqslant i \leqslant n_x\}$. Then

$$\{x\} \times Y \subseteq \bigcup \{W_x \times V_{x,i}\colon 1 \leqslant i \leqslant n_x\}.$$

Let $\mathscr{W} = \{W_x\colon x \in X\}$. Then $\mathscr{W}$ is an open cover of X. Hence, there is a countable subfamily $\{W_{x_j}\colon j \in \omega\}$ of $\mathscr{W}$ such that

$$A \subseteq \bigcup_{j\in\omega} \overline{W_{x_j}},$$

since A is almost Lindelöf in X. Let $\mathscr{V} = \{U_{x_j,i} \times V_{x_j,i}\colon 1 \leqslant i \leqslant n_{x_j}, j \in \omega\}$. Then $\mathscr{V}$ is a countable subfamily of $\mathscr{U}$. To show that $A \times Y \subseteq \bigcup \{\overline{U_{x_j,i} \times V_{x_j,i}}\colon 1 \leqslant i \leqslant n_{x_j}, j \in \omega\}$, let $\langle s, t\rangle \in A \times Y$ be fixed. Let $U_s \times V_t$ be any open neighborhood of

$\langle s,t\rangle$ in $X\times Y$ where U_s and V_t are open neighborhoods of s and t in X and Y, respectively. Since $A\subseteq \bigcup_{j\in\omega}\overline{W_{x_j}}$, there exists a $j\in\omega$ such that $s\in\overline{W_{x_j}}$, hence $U_s\cap W_{x_j}\neq\emptyset$. Thus,

$$(U_s\times V_t)\cap\left(\bigcup\{W_{x_j}\times V_{x_j,i}\colon\ 1\leqslant i\leqslant n_{x_j}\}\right)\neq\emptyset.$$

Therefore,

$$(U_s\times V_t)\cap\left(\bigcup\{U_{x_j,i}\times V_{x_j,i}\colon\ 1\leqslant i\leqslant n_{x_j}\}\right)\neq\emptyset.$$

We have

$$\langle s,t\rangle\in\overline{\bigcup\{U_{x_j,i}\times V_{x_j,i}\colon\ 1\leqslant i\leqslant n_{x_j}\}}=\bigcup\{\overline{U_{x_j,i}\times V_{x_j,i}}\colon\ 1\leqslant i\leqslant n_{x_j}\}.$$

This implies $\langle s,t\rangle\in\bigcup\{\overline{U_{x_j,i}\times V_{x_j,i}}\colon\ 1\leqslant i\leqslant n_{x_j}, j\in\omega\}$. Hence, $A\times Y\subseteq\bigcup\{\overline{U_{x_j,i}\times V_{x_j,i}}\colon\ 1\leqslant i\leqslant n_{x_j}, j\in\omega\}$, which shows that $A\times Y$ is strongly almost Lindelöf in $X\times Y$.

Similarly, we can prove the following:

Proposition 3.9. *If A is strongly almost Lindelöf in X and Y is a compact space, then $A\times Y$ is strongly almost Lindelöf in $X\times Y$.*

Acknowledgements. The author would like to thank the referees for their kind help and valuable comments.

References

[1] *A. V. Arhangel'skii, M. M. Hamdi, Genedi*: The beginnings of the theory of relative topological properties. General Topology. Spaces and Functions, MGU, Moskva (1989), 3–48. (In Russian.)

[2] *A. V. Arhangel'skii*: A generic theorem in the theory of cardinal invariants of topological spaces. Comment. Math. Univ. Carolinae *36* (1995), 303–325.

[3] *R. Engelking*: General Topology. Revised and completed edition, Heldermann, 1989.

[4] *Lj. D. Kočinac*: Some relative topological properties. Mat. Vesn. *44* (1992), 33–44.

[5] *F. Cammaroto, G. Santoro*: Some counterexamples and properties on generalizations of Lindelöf spaces. Int. J. Math. Math. Sci. *19* (1996), 737–746.

[6] *S. Willard, U. N. B. Dissanayake*: The almost Lindelöf degree. Can. Math. Bull. *27* (1984), 452–455.

Author's address: *Yankui Song*, Department of Mathematics, Nanjing Normal University, 210097, P.R. China, e-mail: `songyankui@njnu.edu.cn`.