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OPTIMAL SUBLINEAR INEQUALITIES INVOLVING GEOMETRIC AND POWER MEANS

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Abstract. There are many relations involving the geometric means $G_n(x)$ and power means $[A_n(x^\gamma)]^{1/\gamma}$ for positive n-vectors x. Some of them assume the form of inequalities involving parameters. There then is the question of sharpness, which is quite difficult in general. In this paper we are concerned with inequalities of the form $(1-\lambda)G_n^\gamma(x) + \lambda A_n^\gamma(x) \geqslant A_n(x^\gamma)$ and $(1-\lambda)G_n^\gamma(x) + \lambda A_n^\gamma(x) \leqslant A_n(x^\gamma)$ with parameters $\lambda \in \mathbb{R}$ and $\gamma \in (0,1)$. We obtain a necessary and sufficient condition for the former inequality, and a sharp condition for the latter. Several applications of our results are also demonstrated.

Keywords: geometric mean, power mean, Hermitian matrix, permanent of a complex, simplex, arithmetic-geometric inequality

MSC 2010: 26D15, 26E60

1. Introduction

Means are basic to the whole subject of inequalities and their applications (see e.g. [1]). There are many relations involving the geometric mean $G_n(x)$ and the power mean $[A_n(x^\gamma)]^{1/\gamma}$ for nonnegative n-vectors x. Some of them take on the form of inequalities involving parameters. Three such relations are obtained in [4], [5], [6] which motivate our results that follow. Let $n \geqslant 2$ be a fixed integer. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector of n nonnegative numbers, and let $x^\gamma = (x_1^\gamma, x_2^\gamma, \ldots, x_n^\gamma)$. Let $G_n(x)$ be the geometric mean

$$G_n(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$$

and $[A_n(x^\gamma)]^{1/\gamma}$ the power mean, where

$$A_n(x^\gamma) = \frac{1}{n}\sum_{i=1}^{n} x_i^\gamma, \quad \gamma \neq 0.$$

In [4] it is shown that if $\gamma \geqslant 2$ and $\lambda = n^{\gamma-1}$, then

$$(1 - \lambda)G_n^\gamma(x) + \lambda A_n^\gamma(x) \geqslant A_n(x^\gamma), \tag{1}$$

where we use $A_n(x)$ instead of $A_n(x^1)$ for the sake of convenience. The question then naturally arises as to whether such an inequality is sharp or whether the conditions $\gamma \geqslant 2$ or $\lambda = n^{\gamma-1}$ are necessary. In [5], it is shown that under the condition that $\lambda = n^{\gamma-1}$, (1) holds if and only if $\gamma \geqslant n/(n-1)$.

Note that the condition $\gamma \geqslant n/(n-1)$ implies $\gamma > 1$. Therefore it is of interest to consider the case when $0 < \gamma \leqslant 1$. The case when $\gamma = 1$ can be discarded, however, since (1) becomes an equality.

Another natural question is whether (1) holds if the inequality is reversed. Indeed, in [6] it is shown that if $1 \leqslant \gamma \leqslant n$ and $\lambda = [n/(n-1)]^{\gamma-1}$, then

$$(1 - \lambda)G_n^\gamma(x) + \lambda A_n^\gamma(x) \leqslant A_n(x^\gamma). \tag{2}$$

Again, we are left with the sublinear case when $0 < \gamma < 1$.

In this paper, we will consider the case when $0 < \gamma < 1$. We will show the following two results.

Theorem 1. *Let $0 < \gamma < 1$. Then (1) holds for all $x = (x_1, x_2, \ldots, x_n) \in [0, \infty)^n$ if and only if $\lambda \geqslant [(n-1)/n]^{1-\gamma}$, and equality in (1) holds if and only if either*

$$x_1 = x_2 = \ldots = x_n, \tag{3}$$

or

$$\lambda = \left(\frac{n-1}{n}\right)^{1-\gamma} \text{ and exists } i\colon\ 1 \leqslant i \leqslant n \text{ such that } x_1 = x_2 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_n \text{ and } x_i = 0. \tag{4}$$

Theorem 2. *Let* $0 < \gamma < 1$. *If*

$$\lambda \leqslant \frac{\gamma}{n-1}\Big[\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)}\Big]^{2-\gamma},$$

then (2) *holds for all* $x = (x_1, x_2, \ldots, x_n) \in [0,\infty)^n$; *furthermore, equality in* (2) *holds if, in addition,* $x_1 = x_2 = \ldots = x_n$.

2. An application of (1)

Before proving Theorem 1, we first illustrate its use in obtaining bounds for an integral mean. To this end, recall that the permanent of a complex n by n matrix $A = (a_{ij})$ is

$$\operatorname{per} A := \sum_{\sigma\in S_n} a_{1\sigma_1} a_{2\sigma_2} \ldots a_{n\sigma_n},$$

where the summation is over the set S_n of all permutations of $\{1, 2, \ldots, n\}$. Let B_m be a finite but nonempty subset of the simplex

$$\Omega_{n,m} = \Big\{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0,\infty)^n \colon \sum_{i=1}^{n} \alpha_i = m > 0\Big\} \tag{5}$$

and $\mu = \mu(\alpha)$ a positive function defined on B_m. Let $f\colon (0,\infty)^n \to (0,\infty)$ be defined by

$$f(x_1, x_2, \ldots, x_n) = \sum_{\alpha\in B_m} \frac{1}{n!}\mu(\alpha)\operatorname{per}\big(x_j^{\alpha_i}\big)_{n\times n}.$$

Then

$$Q = \frac{1}{(b-a)^n}\int_{[a,b]^n}\Big(\frac{f(x^\delta)}{f(1,1,\ldots,1)}\Big)^{1/\delta m}\mathrm{d}x_1 \ldots \mathrm{d}x_n,\ \delta > 0, 0 < a < b,$$

is an integral mean over the parallelepiped $[a,b]^n$.

A s s e r t i o n 1. If $\tau = \max\limits_{(\alpha_1,\alpha_2,\ldots,\alpha_n)\in B_m}\{\max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\}$ and $\delta \in (0, 1/\tau)$ as well as $\lambda \geqslant [(n-1)/n]^{1-\delta\tau}$, then

$$\Big(\frac{n}{n+1}\frac{b^{(n+1)/n} - a^{(n+1)/n}}{b-a}\Big)^n \leqslant Q \leqslant (1-\lambda)\Big(\frac{n}{n+1}\frac{b^{(n+1)/n} - a^{(n+1)/n}}{b-a}\Big)^n + \lambda\frac{a+b}{2}. \tag{6}$$

P r o o f. We first recall that [7], [8]

$$[G_n(x)]^{\alpha_1+\alpha_2+\ldots+\alpha_n} \leqslant \frac{1}{n!}\operatorname{per}\big(x_j^{\alpha_i}\big) \leqslant \prod_{j=1}^{n}\Big(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha_j}\Big)$$

for any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in [0, \infty)^n$ and $x = (x_1, x_2, \dots, x_n) \in (0, \infty)^n$. Recall further that the power mean $M_n^{[t]}(x)$ for $x \in (0, \infty)^n$ is defined by

$$M_n^{[t]}(x) = \begin{cases} (A_n(x^t))^{1/t}, & t \in \mathbb{R}, t \neq 0, \\ G_n(x), & t = 0. \end{cases}$$

Then in view of [2], [3], we see that for any $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in B_m$ and any $x = (x_1, x_2, \dots, x_n) \in (0, \infty)^n$,

$$\begin{aligned} [G_n(x)]^m &\leqslant \frac{1}{n!} \operatorname{per}\left(x_j^{\alpha_i}\right) \leqslant \prod_{j=1}^n \frac{1}{n} \sum_{i=1}^n x_i^{\alpha_j} = \prod_{j=1}^n [M_n^{[\alpha_j]}(x)]^{\alpha_j} \\ &\leqslant \prod_{j=1}^n [M_n^{[\tau]}(x)]^{\alpha_j} = [M_n^{[\tau]}(x)]^m. \end{aligned}$$

Hence,

$$\begin{aligned} [G_n(x)]^m f(1, 1, \dots, 1) &= \sum_{\alpha \in B_m} \mu(\alpha) [G_n(x)]^m \leqslant f(x) \\ &\leqslant \sum_{\alpha \in B_m} \mu(\alpha) [M_n^{[\tau]}(x)]^m = [M_n^{[\tau]}(x)]^m f(1, 1, \dots, 1), \end{aligned}$$

or

$$G_n(x) \leqslant \left[\frac{f(x)}{f(1, 1, \dots, 1)}\right]^{1/m} \leqslant M_n^{[\tau]}(x).$$

Replacing x_i by x_i^δ and taking the $(1/\delta)$-th power of all the terms in the resulting inequalities, we obtain

$$G_n(x) = \left[G_n(x^\delta)\right]^{1/\delta} \leqslant \left[\frac{f(x^\delta)}{f(1, 1, \dots, 1)}\right]^{1/\delta m} \leqslant [M_n^{[\tau]}(x^\delta)]^{1/\delta} = M_n^{[\delta\tau]}(x). \tag{7}$$

If we now apply Theorem 1, we have

$$\begin{aligned} M_n^{[\delta\tau]}(x) = \left[A_n(x^{\delta\tau})\right]^{1/\delta\tau} &\leqslant \left[(1 - \lambda) G_n^{\delta\tau}(x) + \lambda A_n^{\delta\tau}(x)\right]^{1/\delta\tau} \\ &\leqslant (1 - \lambda) G_n(x) + \lambda A_n(x), \end{aligned} \tag{8}$$

and hence

$$\begin{aligned} &\int_{[a,b]^n} \left(\frac{f(x^\delta)}{f(1, 1, \dots, 1)}\right)^{1/\delta m} \mathrm{d}x_1 \dots \mathrm{d}x_n \leqslant \int_{[a,b]^n} M_n^{[\delta\tau]}(x) \, \mathrm{d}x_1 \dots \mathrm{d}x_n \\ &\qquad \leqslant \int_{[a,b]^n} [(1 - \lambda) G_n(x) + \lambda A_n(x)] \, \mathrm{d}x_1 \dots \mathrm{d}x_n \\ &\qquad = (1 - \lambda) \int_{[a,b]^n} \left(\prod_{i=1}^n x_i^{1/n}\right) \mathrm{d}x_1 \dots \mathrm{d}x_n + \lambda \int_{[a,b]^n} \left(\frac{1}{n} \sum_{i=1}^n x_i\right) \mathrm{d}x_1 \dots \mathrm{d}x_n \\ &\qquad = (b - a)^n \left[(1 - \lambda) \left(\frac{n}{n+1} \frac{b^{(n+1)/n} - a^{(n+1)/n}}{b - a}\right)^n + \lambda \frac{a + b}{2}\right]. \end{aligned}$$

This shows that the second inequality in (6) is true. The first inequality in (6) is similarly proved. This completes the proof of Assertion 1.

3. An application of (2)

There are quite a few inequalities involving the power of eigenvalues of Hermitian matrices. We can add more by means of Theorem 2. To be more precise, let $A = (a_{ij})$ be an n by n positive definite Hermitian matrix and $\lambda_1, \lambda_2, \dots, \lambda_n$ its eigenvalues. Let $\operatorname{diag}(x)$ be the diagonal matrix with the components of $x = (x_1, x_2, \dots, x_n)$ as its diagonal elements. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Then $A = U \operatorname{diag}(\lambda) U^*$ for some unitary matrix U (where U^* is the conjugate transpose of U). Let $0 < \gamma \leqslant 1$. Then

$$\begin{aligned} A^\gamma &= U \operatorname{diag}(\lambda^\gamma) U^*, \\ A_n(\lambda^\gamma) &= \frac{1}{n} \operatorname{tr}(A^\gamma), \\ G_n(\lambda^\gamma) &= (\det A)^{\gamma/n}, \end{aligned}$$

and

$$\begin{aligned} (P_n^{[k]}(\lambda^\gamma))^{1/\gamma} &= \left[\frac{k!(n-k)!}{n!} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \prod_{j=1}^{k} \lambda_{i_j}^\gamma \right]^{1/\gamma k} \\ &= \left[\frac{k!(n-k)!}{n!} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \det\left(A^\gamma\left[i_1, i_2, \dots, i_k | i_1, i_2, \dots, i_k\right]\right) \right]^{1/\gamma k}, \end{aligned}$$

where $P_n^{[k]}(x)$ is the k-th symmetric mean of a positive vector $x = (x_1, x_2, \dots, x_n)$:

$$P_n^{[k]}(x) = \left[\frac{k!(n-k)!}{n!} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \prod_{j=1}^{k} x_{i_j} \right]^{1/k}, \; k = 1, 2, \dots, n,$$

and $M\left[i_1, i_2, \dots, i_p | j_1, j_2, \dots, j_q\right]$ is the p by q submatrix obtained from an n by n matrix M by striking out rows that are not indexed by $i_1, i_2, \dots, i_p$ and columns that are not indexed by $j_1, j_2, \dots, j_q$.

Assertion 2. Let $n \geqslant 3$, $k \in \{2, 3, \dots, n-1\}$ and let A be a positive definite Hermitian $n \times n$ matrix. Assume

$$\gamma \in (0,1), \; \theta \in \Big(0, \frac{\gamma}{n-1}\Big[\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)}\Big]^{2-\gamma}\Big], \; \omega \in \Big[\Big(\frac{n-1}{n}\Big)^{1-\gamma}, 1\Big)$$

and

$$p \in \Big(0, \frac{n-k}{k(n-1)}\Big], \; q \in \Big[\frac{n}{n-1}\Big(1-\frac{k}{n}\Big)^{1/k}, 1\Big).$$

Then

$$(9)\quad (1-\theta)(\det A)^{\gamma/n}+\theta\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\gamma}\leqslant\frac{1}{n}\operatorname{tr}(A^{\gamma})\leqslant(1-\omega)\,(\det A)^{\gamma/n}+\omega\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\gamma}$$

and

$$(10)\quad (\det A)^{(1-\theta p)/n}\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\theta p}\leqslant(P_n^{[k]}(\lambda^{\gamma}))^{1/\gamma}\leqslant(1-\omega q)\,(\det A)^{1/n}+\omega q\Big(\frac{1}{n}\operatorname{tr}A\Big),$$

where equalities hold in (9) and (10) if all eigenvalues of A are equal.

Proof. (9) is a direct consequence of Theorems 1 and 2. Next, we recall from [3], [9] that when $n\geqslant 3$ and $2\leqslant k\leqslant n-1$, then

$$[G_n(x)]^{1-p}[A_n(x)]^p\leqslant P_n^{[k]}(x)\leqslant(1-q)G_n(x)+qA_n(x)$$

holds for $p\leqslant(n-k)/k(n-1)$ and $q\geqslant n(1-k/n)^{1/k}/(n-1)$. By Theorem 2 and the arithmetic-geometric mean inequality, we may now see that

$$\begin{aligned}P_n^{[k]}(\lambda^{\gamma})&\geqslant[G_n(\lambda^{\gamma})]^{1-p}\,[A_n(\lambda^{\gamma})]^p\geqslant[G_n(\lambda^{\gamma})]^{1-p}\,[(1-\theta)G_n^{\gamma}(\lambda)+\theta A_n^{\gamma}(\lambda)]^p\\&=(\det A)^{\gamma(1-p)/n}\Big[(1-\theta)(\det A)^{\gamma/n}+\theta\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\gamma}\Big]^p\\&\geqslant(\det A)^{\gamma(1-p)/n}\Big[(\det A)^{\gamma(1-\theta)/n}\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\gamma\theta}\Big]^p\\&=\Big[(\det A)^{(1-\theta p)/n}\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\theta p}\Big]^{\gamma},\end{aligned}$$

and by Theorem 1 and the inequality for power means [2], [3],

$$\begin{aligned}P_n^{[k]}(\lambda^{\gamma})&\leqslant(1-q)G_n(\lambda^{\gamma})+qA_n(\lambda^{\gamma})\leqslant(1-q)G_n(\lambda^{\gamma})+q\,[(1-\omega)G_n^{\gamma}(\lambda)+\omega A_n^{\gamma}(\lambda)]\\&=(1-q)\,(\det A)^{\gamma/n}+q\Big[(1-\omega)\,(\det A)^{\gamma/n}+\omega\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\gamma}\Big]\\&=(1-\omega q)\,(\det A)^{\gamma/n}+\omega q\Big(\frac{1}{n}\operatorname{tr}A\Big)^{\gamma}\\&\leqslant\Big[(1-\omega q)(\det A)^{1/n}+\omega q\Big(\frac{1}{n}\operatorname{tr}A\Big)\Big]^{\gamma}.\end{aligned}$$

Furthermore, as can be checked easily, the above inequalities hold if all the eigenvalues of A are equal. This completes the proof of Assertion 2.

4. Preparatory results

We will need the following preparatory results.

Lemma 1. *Let* $0 < \gamma < 1$. *Let* $H\colon [0,\infty)^n \to \mathbb{R}$ *be defined by*

$$H(x) = A_n(x^\gamma) - (1-\lambda)G_n^\gamma(x) \tag{11}$$

where $\lambda \in \mathbb{R}$. *If* $x = (x_1, x_2, \ldots, x_n)$ *is a relative extremum of* H *over the interior of the simplex* $\Omega_{n,n}$ *defined by* (5), *then* k *components of* x, *where* $k \in \{1, 2, \ldots, n-1\}$, *are equal to each other and the other components are equal to each other as well.*

Proof. If $x = (x_1, x_2, \ldots, x_n)$ is a relative extremum of H over the interior of the simplex $\Omega_{n,n}$, then by the Lagrange multiplier method, for some $\mu \in \mathbb{R}$, the function $L(x) = H(x) + \mu\Big(\sum\limits_{i=1}^n x_i - n\Big)$ must satisfy

$$\frac{\partial L}{\partial x_j} = \frac{\gamma}{n}x_j^{\gamma-1} - (1-\lambda)\frac{\gamma}{n}\left(\prod_{i=1}^n x_i\right)^{\gamma/n} x_j^{-1} + \mu = 0,\ j = 1, 2, \ldots, n,$$

or equivalently, for some $\mu \in \mathbb{R}$,

$$\begin{aligned}
x_1^\gamma + \frac{n\mu}{\gamma}x_1 - (1-\lambda)\left(\prod_{i=1}^n x_i\right)^{\gamma/n} &= 0,\\
x_2^\gamma + \frac{n\mu}{\gamma}x_2 - (1-\lambda)\left(\prod_{i=1}^n x_i\right)^{\gamma/n} &= 0,\\
&\vdots\\
x_n^\gamma + \frac{n\mu}{\gamma}x_n - (1-\lambda)\left(\prod_{i=1}^n x_i\right)^{\gamma/n} &= 0,
\end{aligned}$$

where $0 < x_j < n$ for $j = 1, 2, \ldots, n$. But the function

$$\Psi(t) = t^\gamma + \frac{n\mu}{\gamma}t - (1-\lambda)\left(\prod_{i=1}^n x_i\right)^{\gamma/n}$$

is strictly concave on $(0, n)$, as can be seen from

$$\Psi''(t) = \gamma(\gamma-1)t^{\gamma-2} < 0, \quad t \in (0, n);$$

thus $\Psi(t)$ has at most two real roots in $(0, n)$. This shows that k components of x, where $k \in \{1, 2, \ldots, n-1\}$, are equal to each other and the other components are equal to each other as well. The proof is complete. $\square$

Lemma 2. *Let $0 < \gamma < 1$. Let*

$$(12) \qquad \varphi(t,k) = \frac{(kt^\gamma + n - k)/n - t^{\gamma k/n}}{((kt+n-k)/n)^\gamma - t^{\gamma k/n}}, \quad 0 \leqslant t < 1,\ 1 \leqslant k \leqslant n-1.$$

Then (2) *holds for all $x \in [0,\infty)^n$ if and only if*

$$(13) \qquad \lambda \leqslant \inf_{0\leqslant t<1, 1\leqslant k\leqslant n-1} \varphi(t,k).$$

Furthermore, let $\varphi(t_0,k_0) = \inf_{0\leqslant t<1,1\leqslant k\leqslant n-1} \varphi(t,k)$ and $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. Then equality holds in (2) *if and only if $x_1 = x_2 = \ldots = x_n$ or $\lambda = \varphi(t_0,k_0)$ and $x_1 = x_2 = \ldots = x_{k_0} = t_0 x_n$ and $x_{k_0+1} = x_{k_0+2} = \ldots = x_n$.*

P r o o f. First note that (2) holds if and only if

$$H(x) \geqslant \lambda A_n^\gamma(x),$$

where H is defined by (11). To show that (2) holds for $x \in [0,\infty)^n$, it suffices to show that it holds for x in the simplex $\Omega_{n,n}$ (defined by (5)). But for $x \in \Omega_{n,n}$, we have $A_n^\gamma(x) = 1$. Thus to show that (2) holds for $x \in [0,\infty)^n$, it suffices to show that $H(x) \geqslant \lambda$ for $x \in \Omega_{n,n}$. We need to consider two cases: (i) x belongs to the boundary of $\Omega_{n,n}$, and (ii) x is a relative extremum of H over $\Omega_{n,n}$. In the former case, some component of x, say x_n, is 0. Then from Jensen's inequality, $A_n(x^\gamma) \geqslant n^{\gamma-1} A_n^\gamma(x)$, so that

$$\begin{aligned} H(x) &= A_n(x^\gamma) - (1-\lambda) G_n^\gamma(x) = A_n(x^\gamma) \\ &\geqslant n^{\gamma-1} A_n^\gamma(x) = n^{\gamma-1} = \varphi(0, n-1) \\ &\geqslant \inf_{0\leqslant t<1,1\leqslant k\leqslant n-1} \varphi(t,k) \geqslant \lambda \end{aligned}$$

as desired. In the latter case, in view of Lemma 1 we may assume without loss of generality that there are two numbers u, v such that $0 < v \leqslant u$ and the first $n-k$ components of x are equal to u and the rest equal to v. Since

$$(n-k)u + kv = n,$$

we have

$$0 < v \leqslant u < \frac{n}{n-k}.$$

Let $t = v/u$. Then $0 < t \leqslant 1$, $u = n/(kt + n - k)$ and

$$\begin{aligned} H(x) &= \frac{1}{n}\sum_{i=1}^{n} x_i^{\gamma} - (1-\lambda)\Big(\prod_{i=1}^{n} x_i\Big)^{\gamma/n} \\ &= \frac{1}{n}\left[(n-k)u^{\gamma} + kv^{\gamma}\right] - (1-\lambda)\left(u^{n-k}v^{k}\right)^{\gamma/n} \\ &= \Big[\frac{1}{n}\left(kt^{\gamma} + n - k\right) - (1-\lambda)t^{\gamma k/n}\Big]u^{\gamma} \\ &= n^{\gamma-1}[kt^{\gamma} + n - k - (1-\lambda)nt^{\gamma k/n}](kt + n - k)^{-\gamma}. \end{aligned}$$

If $t = 1$, then

$$H(x) = \lambda;$$

while if $0 < t < 1$, since the arithmetic-geometric mean inequality implies

$$\Big(\frac{kt + n - k}{n}\Big)^{\gamma} > t^{\gamma k/n},$$

we see that (13) implies

$$\lambda\Big[\Big(\frac{kt + n - k}{n}\Big)^{\gamma} - t^{\gamma k/n}\Big] \leqslant \frac{kt^{\gamma} + n - k}{n} - t^{\gamma k/n}, \tag{14}$$

which in turn implies

$$H(x) \geqslant \lambda.$$

Next, we show that (13) is necessary. Indeed, if we take x to be a vector whose first k $(1 \leqslant k \leqslant n-1)$ components are equal to $t \in [0, 1)$ and the rest are equal to 1, then substituting x into (2), we obtain $\varphi(t, k) \geqslant \lambda$.

If $x_1 = x_2 = \ldots = x_n$, then equality holds in (2); otherwise, equality holds in (2) if and only if $\lambda = \inf\limits_{0\leqslant t<1, 1\leqslant k\leqslant n-1} \varphi(t, k) = \varphi(t_0, k_0)$, $x_1 = x_2 = \ldots = x_{k_0} = v$, $x_{k_0+1} = x_{k_0+2} = \ldots = x_n = u$ and $v/u = t_0$, that is, $\lambda = \varphi(t_0, k_0)$, $x_1 = x_2 = \ldots x_{k_0} = t_0 x_n$ and $x_{k_0+1} = x_{k_0+2} = \ldots = x_n$. The proof is complete.

Lemma 3. *Let $0 < \gamma < 1$. Then (1) holds for all $x \in [0, \infty)^n$ if and only if*

$$\lambda \geqslant \sup_{0\leqslant t<1, 1\leqslant k\leqslant n-1} \varphi(t, k), \tag{15}$$

where $\varphi(t, k)$ is defined by (12). Furthermore, let $\sup\limits_{0\leqslant t<1, 1\leqslant k\leqslant n-1} \varphi(t, k) = \varphi(t_0^*, k_0^*)$ and $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. Then equality holds in (1) if and only if $x_1 = x_2 = \ldots = x_n$ or $\lambda = \varphi(t_0^*, k_0^*)$ and $x_1 = x_2 = \ldots = x_{k_0^*} = t_0^* x_n$ and $x_{k_0^*+1} = x_{k_0^*+2} = \ldots = x_n$.

P r o o f. As in the proof of Lemma 2, to show that (1) holds for all $x \in [0,\infty)^n$ it suffices to show $H(x) \leqslant \lambda$ for all $x \in \Omega_{n,n}$. If x belongs to the boundary of $\Omega_{n,n}$, then some component of x, say x_n, is 0. Thus

$$\begin{aligned} H(x) &= A_n(x^\gamma) - (1-\lambda)G_n^\gamma(x) = \frac{n-1}{n}\Big(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i^\gamma\Big) \\ &\leqslant \frac{n-1}{n}\Big(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\Big)^\gamma = \Big(\frac{n-1}{n}\Big)^{1-\gamma} = \varphi(0,1) \\ &\leqslant \sup_{0\leqslant t<1, 1\leqslant k\leqslant n-1} \varphi(t,k). \end{aligned}$$

Hence, from (15) we see further that $H(x) \leqslant \lambda$. If x is a relative extremum of H over $\Omega_{n,n}$, then the same argument for proving (14) leads us to

$$\lambda\Big[\Big(\frac{kt+n-k}{n}\Big)^\gamma - t^{\gamma k/n}\Big] \geqslant \frac{kt^\gamma + n-k}{n} - t^{\gamma k/n}$$

or $H(x) \leqslant \lambda$.

Finally, if we take x to be a vector whose first k $(1 \leqslant k \leqslant n-1)$ components are equal to $t \in [0,1)$ and the rest are equal to 1, then substituting x into (1) we obtain $\varphi(t,k) \leqslant \lambda$.

If $x_1 = x_2 = \ldots = x_n$, then equality holds in (1); otherwise, equality holds in (1) if and only if $\lambda = \sup\limits_{0\leqslant t<1, 1\leqslant k\leqslant n-1} \varphi(t,k) = \varphi(t_0^*, k_0^*)$, $x_1 = x_2 = \ldots = x_{k_0^*} = v$, $x_{k_0^*+1} = x_{k_0^*+2} = \ldots = x_n = u$ and $v/u = t_0^*$, that is, $\lambda = \varphi(t_0^*, k_0^*)$, $x_1 = x_2 = \ldots x_{k_0^*} = t_0^* x_n$ and $x_{k_0^*+1} = x_{k_0^*+2} = \ldots = x_n$. The proof is complete.

5. Proofs of main results

Two real numbers α and β are said to be of the same sign, denoted by $\alpha \backsim \beta$, if $\alpha > 0 \Rightarrow \beta > 0$, $\alpha = 0 \Rightarrow \beta = 0$ and $\alpha < 0 \Rightarrow \beta < 0$. It is easily seen that if $\alpha, \beta > 0$, then $\alpha - \beta \backsim \ln\alpha - \ln\beta$.

We now turn to the proof of Theorem 1. In view of Lemma 3, it suffices to show that

$$\Big(\frac{n-1}{n}\Big)^{1-\gamma} \geqslant \sup_{0\leqslant t<1, 1\leqslant k\leqslant n-1} \varphi(t,k), \tag{16}$$

where equality holds if and only if (3) or

$$x_1 = x_2 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_n \quad \text{and} \quad x_i = 0. \tag{17}$$

Note that (16) is equivalent to

$$\Psi(t,k) := \frac{1}{t^{\gamma k/n}}\Big[\xi\Big(\frac{kt+n-k}{n}\Big)^{\gamma} + (1-\xi)t^{\gamma k/n} - \frac{kt^{\gamma}+n-k}{n}\Big] \geqslant 0,$$

where $\xi = [(n-1)/n]^{1-\gamma}$ for $t \in (0,1)$ and $k \in \{1,2,\ldots,n-1\}$. Now,

$$\Psi(t,k) = \xi\Big[\frac{kt^{1-k/n}+(n-k)t^{-k/n}}{n}\Big]^{\gamma} + (1-\xi) - \frac{kt^{\gamma(1-k/n)}+(n-k)t^{-\gamma k/n}}{n},$$

and

$$\begin{aligned}\frac{\partial \Psi(t,k)}{\partial t} &= \frac{\gamma k(n-k)}{n^2}t^{-\gamma k/n-1}\Big[\xi\Big(\frac{kt+n-k}{n}\Big)^{\gamma-1}(t-1) - (t^{\gamma}-1)\Big] \\ &\backsim \xi\Big(\frac{kt+n-k}{n}\Big)^{\gamma-1}(t-1) - (t^{\gamma}-1)\end{aligned}$$

for $t \in (0,1)$ and $k \in \{1,2,\ldots,n-1\}$. The function

$$G(t,k) = \xi\Big(\frac{kt+n-k}{n}\Big)^{\gamma-1}(t-1) - (t^{\gamma}-1),\ t \in (0,1), k \in \{1,2,\ldots,n-1\},$$

satisfies

$$G(t,k) \leqslant G(t,1) = \xi\Big(\frac{t+n-1}{n}\Big)^{\gamma-1}(t-1) - (t^{\gamma}-1),\ t \in (0,1), k \in \{1,2,\ldots,n-1\},$$

and

$$\begin{aligned}\frac{\mathrm{d}G(t,1)}{\mathrm{d}t} &= \xi\Big(\frac{t+n-1}{n}\Big)^{\gamma-2}\frac{\gamma t+n-\gamma}{n} - \gamma t^{\gamma-1} \\ &\backsim \ln\Big[\xi\Big(\frac{t+n-1}{n}\Big)^{\gamma-2}\frac{\gamma t+n-\gamma}{n}\Big] - \ln\big(\gamma t^{\gamma-1}\big) \\ &= \ln\frac{\xi}{\gamma} + (\gamma-2)\ln\frac{t+n-1}{n} + \ln\frac{\gamma t+n-\gamma}{n} + (1-\gamma)\ln t.\end{aligned}$$

Let

$$h(t) = \ln\frac{\xi}{\gamma} + (\gamma-2)\ln\frac{t+n-1}{n} + \ln\frac{\gamma t+n-\gamma}{n} + (1-\gamma)\ln t,\ t \in (0,1).$$

Since

$$\begin{aligned}h'(t) &= \frac{\gamma-2}{t+n-1} + \frac{\gamma}{\gamma t+n-\gamma} + \frac{1-\gamma}{t} \\ &\backsim (\gamma-2)t(\gamma t+n-\gamma) + \gamma t(t+n-1) + (1-\gamma)(t+n-1)(\gamma t+n-\gamma) \\ &= -(1-\gamma)\left[(1-\gamma)n+\gamma\right]t + (1-\gamma)(n-1)(n-\gamma) \\ &= (1-\gamma)\left[(1-\gamma)n+\gamma\right]\Big[-t + \frac{(n-1)(n-\gamma)}{(1-\gamma)n+\gamma}\Big] \\ &\backsim -t + \frac{(n-1)(n-\gamma)}{(1-\gamma)n+\gamma} \\ &\geqslant -t+1 \\ &> 0,\end{aligned}$$

we see that

$$-\infty = h(0) < h(t) < h(1) = \ln\frac{\xi}{\gamma} = \ln\frac{1}{\gamma}\Big(\frac{n-1}{n}\Big)^{1-\gamma}.$$

But

$$\Big(\frac{n-1}{n}\Big)^{1-\gamma} \geqslant \Big(\frac{2-1}{2}\Big)^{1-\gamma} = 2^{\gamma-1} > \gamma$$

for $\gamma \in (0,1)$, thus $\ln(\xi/\gamma) > 0$. Consequently, the function h has a unique root $t_0 \in (0,1)$. Since $h(t) < 0$ for $t \in (0,t_0)$ and $h(t) > 0$ for $t \in (t_0,1)$, we now see that $\mathrm{d}G(t,1)/\,\mathrm{d}t < 0$ for $t \in (0,t_0)$ and $\mathrm{d}G(t,1)/\,\mathrm{d}t > 0$ for $t \in (t_0,1)$. This and the fact that $1-\xi((n-1)/n)^{\gamma-1} = 0$ imply

$$\begin{aligned}\frac{\partial\Psi(t,k)}{\partial t} \backsim G(t,k) \leqslant G(t,1) &< \max\{G(0,1), G(1,1)\}\\ &= \max\Big\{1-\xi\Big(\frac{n-1}{n}\Big)^{\gamma-1}, 0\Big\} = 0.\end{aligned}$$

By virtue of

$$\Psi(1,k) = 0$$

for $k \in \{1,2,\ldots,n-1\}$, we see further that $\Psi(t,k) > 0$ for $t \in (0,1)$ and $k \in \{1,2,\ldots,n-1\}$.

Finally, we consider the conditions of equality in (1). According to the above proof we have

$$\sup_{0\leqslant t<1,1\leqslant k\leqslant n-1} \varphi(t,k) = \varphi(0,1) = \Big(\frac{n-1}{n}\Big)^{1-\gamma}.$$

By Lemma 3, equality holds in (1) if and only if $x_1 = x_2 = \ldots = x_n$ or $\lambda = ((n-1)/n)^{1-\gamma}$ and exists $i\colon\ 1 \leqslant i \leqslant n$ such that $x_1 = x_2 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_n$ and $x_i = 0$. The proof is complete.

We now turn to the proof of Theorem 2. In view of Lemma 2, it suffices to show that

$$\varphi(t,k) \geqslant \frac{\gamma}{n-1}\Big[\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)}\Big]^{2-\gamma} \tag{18}$$

for $t \in [0,1)$ and $k \in \{1,2,\ldots,n-1\}$. To this end, let us write

$$\varphi(t,k) = \frac{p_k(t)}{q_k(t)}$$

where

$$p_k(t) = \frac{kt^{(n-k)\gamma/n} + (n-k)t^{-\gamma k/n}}{n} - 1,$$

and

$$q_k(t) = \Big(\frac{kt^{(n-k)/n} + (n-k)t^{-k/n}}{n}\Big)^{\gamma} - 1$$

for $k \in \{1, 2, \ldots, n-1\}$. Since $p_k(1) = q_k(1) = 0$ and

$$p_k'(t) = \frac{\gamma k(n-k)}{n^2} t^{-\gamma k/n - 1}(t^{\gamma} - 1)$$

and

$$q_k'(t) = \frac{\gamma k(n-k)}{n^2}\Big(\frac{kt + n - k}{n}\Big)^{\gamma-1} t^{-\gamma k/n-1}(t-1),$$

we see that there is $\zeta_k \in (t, 1)$ such that

$$\begin{aligned}\varphi(t,k) &= \frac{p_k(t)}{q_k(t)} = \frac{p_k(t) - p_k(1)}{q_k(t) - q_k(1)} = \frac{p_k'(\zeta_k)}{q_k'(\zeta_k)} \\ &= \frac{\zeta_k^{\gamma} - 1}{[(k\zeta_k + n - k)/n]^{\gamma-1}(\zeta_k - 1)} \\ &\geqslant \frac{\zeta_k^{\gamma} - 1}{[((n-1)\zeta_k + 1)/n]^{\gamma-1}(\zeta_k - 1)}\end{aligned}$$

where the last inequality is obtained by substituting $k = n-1$. Let

$$u(t) = t^{\gamma} - 1$$

and

$$v(t) = \Big[\frac{(n-1)t + 1}{n}\Big]^{\gamma-1}(t-1).$$

Since $u(1) = v(1) = 0$, $u'(t) = \gamma t^{\gamma-1}$ and

$$v'(t) = \frac{1}{n}\Big[\frac{(n-1)t+1}{n}\Big]^{\gamma-2}\left[\gamma(n-1)t + 1 + (1-\gamma)(n-1)\right],$$

we see further that there is $\eta_k \in (\zeta_k, 1)$ such that

$$\varphi(t,k) \geqslant \frac{u(\xi_k) - u(1)}{v(\xi_k) - v(1)} = \frac{u'(\eta_k)}{v'(\eta_k)} = \frac{\gamma}{w(\delta_k)},$$

where

$$w(t) = t^{\gamma-2}\left\{-(1-\gamma)(n-1) + \left[1 + (1-\gamma)(n-1)\right]t\right\}$$

and

$$\delta_k = \frac{n - 1 + \eta_k^{-1}}{n} > 1.$$

Since $(1-\gamma)\,[1+(1-\gamma)(n-1)]\,t^{\gamma-3}>0$ and

$$w'(t)=(1-\gamma)\,[1+(1-\gamma)(n-1)]\,t^{\gamma-3}\Big[\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)}-t\Big],$$

the equation $w'(t)=0$ has a unique root in the interval $[1,\infty)$:

$$t^*=\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)},$$

and $w'(t)\geqslant 0$ on $[1,t^*]$ and $w'(t)\leqslant 0$ on $[t^*,\infty)$. Hence

$$w(t)\leqslant w(t^*)=(n-1)\Big[\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)}\Big]^{\gamma-2},\ t>1.$$

Summarizing,

$$\varphi(t,k)\geqslant\frac{\gamma}{w(t^*)}=\frac{\gamma}{n-1}\Big[\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)}\Big]^{2-\gamma}$$

for $t\in[0,1)$ and $k\in\{1,2,\ldots,n-1\}$. The proof is complete.

6. Three examples

Theorem 2 offers an explicit sufficient condition for λ to satisfy in order that (2) holds. However, Lemma 2 offers a necessary and sufficient condition, which unfortunately is not explicit. Provided explicit data are given, Lemma 2 may offer better results. For example, suppose we are given $n=10$ and $\gamma=1/2$. Then using commercial software, we may find that

$$\begin{aligned}\inf_{0\leqslant t<1;1\leqslant k\leqslant 9}\varphi(t,k)&=\inf_{0\leqslant t<1;1\leqslant k\leqslant 9}\frac{\frac{1}{10}(kt^{1/2}+10-k)-t^{k/20}}{(\frac{1}{10}(kt+10-k))^{1/2}-t^{k/20}}\\&=\varphi(0.0013465750656368116\ldots,9)\\&=0.3068771309760594\ldots\end{aligned}$$

so that (2) holds if and only if $\lambda\leqslant 0.3068771309760594\ldots$, and in view of Lemma 2, equality holds in (2) if and only if $x_1=x_2=\ldots=x_{10}$ or $\lambda=0.3068771309760594\ldots$ and there exists $i\in\{1,2,\ldots,10\}$ such that

$$x_1=x_2=\ldots=x_{i-1}=(0.0013465750656368116\ldots)x_i=x_{i+1}=\ldots=x_{10}.$$

On the other hand, if we apply Theorem 2, we may only conclude that (2) holds when

$$\lambda \leqslant \frac{1}{18}\Big(\frac{27}{11}\Big)^{3/2} = 0.2146407595819928\ldots.$$

As another example, consider the case when $n = 2$ in (2). Then

$$\inf_{0\leqslant t<1;1\leqslant k\leqslant n-1} \varphi(t,k) = \inf_{0\leqslant t<1} \varphi(t,1).$$

By Theorem 2 we have

$$\inf_{0\leqslant t<1} \varphi(t,1) \geqslant \gamma.$$

On the other hand,

$$\inf_{0\leqslant t<1} \varphi(t,1) \leqslant \lim_{t\to 1} \varphi(t,1) = \gamma,$$

thus

$$\inf_{0\leqslant t<1;1\leqslant k\leqslant n-1} \varphi(t,k) = \inf_{0\leqslant t<1} \varphi(t,1) = \gamma.$$

In other words, if $n = 2$ and $0 < \gamma < 1$, then (2) holds for all $x \in [0,\infty)^n$ if and only if $\lambda \leqslant \gamma$.

As another application of our results, we will show the following result.

Theorem 3. *For any* $x = (x_1, x_2, \ldots, x_n) \in (0,\infty)^n$ *we have*

$$\frac{A_n(x\ln x) - A_n(x)\ln A_n(x)}{n-1-\ln(n-1)} \leqslant A_n(x) - G_n(x) \leqslant \frac{A_n(x\ln x) - A_n(x)\ln A_n(x)}{\ln n - \ln(n-1)}, \tag{19}$$

where $x\ln x$ *is the short hand notation for* $(x_1\ln x_1, x_2\ln x_2, \ldots, x_n\ln x_n)$; *furthermore, equalities hold if in addition* $x_1 = x_2 = \ldots = x_n$.

To see the proof, let $\gamma \in (0,1)$. If we let

$$\lambda = \frac{\gamma}{n-1}\Big[\frac{(2-\gamma)(n-1)}{1+(1-\gamma)(n-1)}\Big]^{2-\gamma},$$

then by Theorem 2, (2) holds, which is equivalent to

$$\frac{1-\lambda}{1-\gamma}\,(G_n^\gamma(x) - A_n^\gamma(x)) \leqslant \frac{A_n(x^\gamma) - A_n^\gamma(x)}{1-\gamma}.$$

By taking limits on both sides as $\gamma \to 1$, we see that

$$\Big(\lim_{\gamma\to 1}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma}\Big)(G_n(x) - A_n(x)) \leqslant A_n(x)\ln A_n(x) - A_n(x\ln x). \tag{20}$$

Since

$$\ln\lambda = \ln\gamma + (2-\gamma)\{\ln((2-\gamma)(n-1)) - \ln(1+(1-\gamma)(n-1))\} - \ln(n-1),$$

we see that

$$\begin{aligned}\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma} &= \frac{1}{\gamma} - \{\ln((2-\gamma)(n-1)) - \ln(1+(1-\gamma)(n-1))\} \\ &\quad + (2-\gamma)\Big(-\frac{1}{2-\gamma} + \frac{n-1}{1+(1-\gamma)(n-1)}\Big).\end{aligned}$$

Since $\lambda \to 1$ as $\gamma \to 1$, we have

$$\lim_{\gamma\to 1}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma} = \lim_{\gamma\to 1}\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma} = n - 1 - \ln(n-1). \tag{21}$$

By (20) and (21), we see that the first inequality in (19) holds.

Similarly, if we let

$$\lambda = \Big(\frac{n-1}{n}\Big)^{1-\gamma},$$

then by Theorem 1 we obtain

$$\Big(\lim_{\gamma\to 1}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma}\Big)(G_n(x) - A_n(x)) \geqslant A_n(x)\ln A_n(x) - A_n(x\ln x).$$

By means of arguments similar to those discussed above, we may show that in this case,

$$\lim_{\gamma\to 1}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma} = \ln n - \ln(n-1).$$

Hence the latter inequality in (19) holds.

Obviously, when $x_1 = x_2 = \ldots = x_n$, equalities hold in (19). The proof is complete.

In [10], [11] several applications on power means are obtained.

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