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Mathematica Bohemica, Vol. 133 (2008), No. 3, 299–311

Persistent URL: `http://dml.cz/dmlcz/140619`

MODAL OPERATORS ON BOUNDED RESIDUATED l-MONOIDS

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(Received March 27, 2007)

Abstract. Bounded residuated lattice ordered monoids (Rl-monoids) form a class of algebras which contains the class of Heyting algebras, i.e. algebras of the propositional intuitionistic logic, as well as the classes of algebras of important propositional fuzzy logics such as pseudo MV-algebras (or, equivalently, GMV-algebras) and pseudo BL-algebras (and so, particularly, MV-algebras and BL-algebras). Modal operators on Heyting algebras were studied by Macnab (1981), on MV-algebras were studied by Harlenderová and Rachůnek (2006) and on bounded commutative Rl-monoids in our paper which will apear in Math. Slovaca. Now we generalize modal operators to bounded Rl-monoids which need not be commutative and investigate their properties also for further derived algebras.

Keywords: residuated l-monoid, residuated lattice, pseudo BL-algebra, pseudo MV-algebra

MSC 2010: 06D35, 06F05

1. Introduction

Residuated lattice ordered monoids (Rl-monoids) form a large class of algebras containing the class of all lattice ordered groups (l-groups) as well as classes of algebras of several propositional logics. Recall that commutative Rl-monoids were introduced in [26] as a common generalization of Abelian l-groups and Heyting algebras (i.e. algebras of the intuitionistic propositional logic). At the same time, the algebras of fuzzy logics, such as MV-algebras [4] (see also [5]) (which are also categorically equivalent to Wajsberg algebras [9]), i.e., algebras of the Łukasiewicz infinite valued logic, and BL-algebras [13], i.e., algebras of Hájek's basic fuzzy logic, can be recognized as special cases of bounded commutative Rl-monoids.

The first author was supported by the Council of Czech Government, MSM 6198959214.

More generally, Rl-monoids without the requirement of the commutativity of the semigroup binary operation were introduced in [16] and further developed in [17]. Analogously as in the commutative cases, the algebras of non-commutative generalizations of fuzzy logics can be also viewed as special cases of bounded Rl-monoids. Namely, GMV-algebras [21] or pseudo MV-algebras [11] (which are also categorically equivalent to pseudo Wajsberg algebras [3]), i.e., algebras of the non-commutative Łukasiewicz logic [19], and pseudo BL-algebras [6], i.e., algebras of Hájek's non-commutative basic fuzzy logic [14], form proper classes of the class of bounded Rl-monoids.

Modal operators (special cases of closure operators) on Heyting algebras were introduced and studied by Macnab in [20]. Analogously, modal operators on MV-algebras were introduced in [15]. A generalization of those operators to all bounded commutative Rl-monoids were studied by the authors in [23].

In this paper we define and study modal operators on bounded Rl-monoids which need not be commutative. Many of results are obtained for the variety of good normal Rl-monoids that contains, among others, both the variety of good pseudo BL-algebras and that of Heyting algebras.

2. Bounded Rl-monoids

A *bounded* Rl-*monoid* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid (need not be commutative).

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leqslant z$ iff $x \leqslant y \rightarrow z$ iff $y \leqslant x \rightsquigarrow z$ for any $x, y \in M$.

(iv) $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

Recall that the lattice $(M; \vee, \wedge)$ is distributive and that bounded Rl-monoids form a variety of algebras of the indicated type. Moreover, the bounded Rl-monoids can be recognized as bounded integral generalized BL-algebras in the sense of [1] and [2] and hence it is possible to prove (see [7]) that the operation "$\odot$" distributes over the lattice operations "$\vee$" and "$\wedge$".

In what follows, by an Rl-monoid we will mean a *bounded* Rl-monoid.

If the operation "$\odot$" in an Rl-monoid M is commutative then M is called a *commutative* Rl-*monoid.*

For any Rl-monoid M we define two unary operations (negations) "$^-$" and "$^\sim$" on M such that $x^- := x \rightarrow 0$ and $x^\sim := x \rightsquigarrow 0$ for every $x \in M$.

Recall that the algebras of the mentioned propositional logics are characterized in the class of Rl-monoids as follows:

An Rl-monoid M is

a) a pseudo BL-algebra ([18]) if and only if M satisfies the identities of pre-linearity $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;

b) a GMV-algebra (pseudo MV-algebra) ([22]) if and only if M fulfils the identities $x^{-\sim} = x = x^{\sim-}$;

c) a Heyting algebra ([26]) if and only if the operations "$\odot$" and "$\wedge$" coincide on M.

Basic properties of bounded Rl-monoids have been given in many articles (e.g. [25]), here we present some of them which will be used in this paper.

Lemma 1. *In any bounded* Rl-*monoid M we have for any $x, y \in M$:*

(1) $x \leqslant y \Longleftrightarrow x \to y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$.

(2) $x \leqslant y \Longrightarrow z \to x \leqslant z \to y$, $z \rightsquigarrow x \leqslant z \rightsquigarrow y$.

(3) $x \leqslant y \Longrightarrow y \to z \leqslant x \to z$, $y \rightsquigarrow z \leqslant x \rightsquigarrow z$.

(4) $1^{-\sim} = 1 = 1^{\sim-}$, $0^{-\sim} = 0 = 0^{\sim-}$.

(5) $x \leqslant x^{-\sim}$, $x \leqslant x^{\sim-}$.

(6) $x^{-\sim-} = x^{-}$, $x^{\sim-\sim} = x^{\sim}$.

(7) $x^{-} \odot x = 0 = x \odot x^{\sim}$.

(8) $x \to (y \to z) = (x \odot y) \to z$, $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$.

(9) $(x \odot y)^{-} = x \to y^{-}$, $(x \odot y)^{\sim} = y \rightsquigarrow x^{\sim}$.

(10) $(x \vee y)^{-} = x^{-} \wedge y^{-}$, $(x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}$.

An Rl-monoid M is called *good* if M satisfies the identity $x^{-\sim} = x^{\sim-}$. (The notion of a good Rl-monoid is a generalization of that of a good pseudo BL-algebra introduced in [12].)

Obviously, every GMV-algebra or every commutative Rl-monoid is good. Moreover (see also [8]), if M_1 and M_2 are non-trivial GMV-algebras then their ordinal sum is a good Rl-monoid which need not be either a GMV-algebra or a pseudo BL-algebra.

Let M be a good Rl-monoid. According to [8], we define a binary operation "$\oplus$" on M as follows:

$$\forall x, y \in M; x \oplus y := (y^{-} \odot x^{-})^{\sim}.$$

By [25], every good Rl-monoid fulfils the identity $(x^{-} \odot y^{-})^{\sim} = (x^{\sim} \odot y^{\sim})^{-}$.

Lemma 2. *If M is a good* Rl-*monoid and $x, y \in M$ then*

$$(x \oplus y)^{-\sim} = x^{-\sim} \oplus y^{-\sim} = x^{-\sim} \oplus y = x \oplus y^{-\sim} = x \oplus y.$$

P r o o f. $(x \oplus y)^{-\sim} = (y^- \odot x^-)^{\sim-\sim} = (y^- \odot x^-)^\sim = x \oplus y$,

$$
\begin{aligned}
x^{-\sim} \oplus y^{-\sim} &= (y^{-\sim-} \odot x^{-\sim-})^\sim = (y^- \odot x^-)^\sim = x \oplus y, \\
&= (y^{-\sim-} \odot x^-)^\sim = x \oplus y^{-\sim} \;= (y^- \odot x^{-\sim-})^\sim = x^{-\sim} \oplus y.
\end{aligned}
$$

□

Proposition 3 ([8]). *Let M be a good* Rl-*monoid. Then for all $x, y, z \in M$:*

(i) $x \oplus y = (y^\sim \odot x^\sim)^-$;

(ii) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(iii) $x,\ y \leqslant x \oplus y$;

(iv) $x \oplus 0 = x^{-\sim} = 0 \oplus x$;

(v) $x \oplus 1 = 1 = 1 \oplus x$;

(vi) $x \oplus y = x^- \rightsquigarrow y^{-\sim} = y^\sim \to x^{-\sim}$.

3. Modal operators—definition and properties

Definition. Let M be an Rl-monoid. A mapping $f\colon\ M \longrightarrow M$ is called a *modal operator* on M if, for any $x,\ y \in M$,

1. $x \leqslant f(x)$;
2. $f(f(x)) = f(x)$;
3. $f(x \odot y) = f(x) \odot f(y)$.

Moreover, if an Rl-monoid M is good and for any $x,\ y \in M$,

4. $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus y)$,

then a modal operator f is called *strong*.

Proposition 4. a) *If f is a modal operator on an* Rl-*monoid M and $x,\ y \in M$ then*

(i) $x \leqslant y \Longrightarrow f(x) \leqslant f(y)$;

(ii) $f(x \to y) \leqslant f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))$,
$f(x \rightsquigarrow y) \leqslant f(x) \rightsquigarrow f(y) = f(f(x) \rightsquigarrow f(y)) = x \rightsquigarrow f(y) = f(x \rightsquigarrow f(y))$;

(iii) $f(x) \leqslant (x \to f(0)) \rightsquigarrow f(0)$, $f(x) \leqslant (x \rightsquigarrow f(0)) \to f(0)$;

(iv) $x^- \odot f(x) \leqslant f(0)$, $f(x) \odot x^\sim \leqslant f(0)$;

(v) $f(x \vee y) \leqslant f(x \vee f(y)) = f(f(x) \vee f(y))$.

b) *If M is a good* Rl-*monoid and f is a strong modal operator on M then*

(vi) $x \oplus f(0) \geqslant f(x^{-\sim}) \geqslant f(x)$, $f(0) \oplus x \geqslant f(x^{-\sim}) \geqslant f(x)$.

P r o o f. The proof of (i), (ii), (iv) and (v) runs as in [23].

(iii) By (i) and (ii), we have

$$(f(x) \to f(0)) \odot f(x) = f(x) \wedge f(0) = f(0)$$
$$\Longrightarrow f(x) \leqslant (f(x) \to f(0)) \rightsquigarrow f(0) = (x \to f(0)) \rightsquigarrow f(0).$$

The latter inequality can be proved similarly.
(vi) By Proposition 3(vi), Lemma 1(2)(5) and properties (ii), (i) above,

$$x \oplus f(0) = x^{-} \rightsquigarrow f(0)^{-\sim} \geqslant x^{-} \rightsquigarrow f(0) = f(x^{-} \rightsquigarrow f(0)) \geqslant f(x^{-} \rightsquigarrow 0)$$
$$= f(x^{-\sim}) \geqslant f(x).$$

The proof of the remaining inequalities is analogous. □

Proposition 5. *Let M be a good* Rl-*monoid. If f is a strong modal operator on M and $x, y \in M$ then*
(i) $f(x \oplus y) = f(f(x) \oplus f(y))$;
(ii) $x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x$.

P r o o f. (i) By the definition of a strong modal operator it si obvious.
(ii) We have $f(x \oplus f(0)) = f(x \oplus 0) = f(x^{-\sim}) \Longrightarrow f(x^{-\sim}) = f(x \oplus f(0)) \geqslant x \oplus f(0) \geqslant f(x^{-\sim})$. □

Therem 6. *Let M be an* Rl-*monoid and $f\colon M \longrightarrow M$ be a mapping. Then f is a modal operator on M if and only if for any x, $y \in M$ it is satisfied:*
(a) $x \to f(y) = f(x) \to f(y)$;
(b) $x \rightsquigarrow f(y) = f(x) \rightsquigarrow f(y)$;
(c) $f(x) \odot f(y) \geqslant f(x \odot y)$.

P r o o f. Let a mapping f fulfil conditions (a)–(c).
Properties 1 and 2 from the definition of a modal operator follow in the same way as in the commutative case (see [23]).
We prove the property 3 from definition:
$x \odot y \leqslant f(x \odot y) \Longrightarrow x \leqslant y \to f(x \odot y) = f(y) \to f(x \odot y) \Longrightarrow x \odot f(y) \leqslant f(x \odot y) \Longrightarrow f(y) \leqslant x \rightsquigarrow f(x \odot y) = f(x) \rightsquigarrow f(x \odot y) \Longrightarrow f(x) \odot f(y) \leqslant f(x \odot y) \Longrightarrow f(x) \odot f(y) = f(x \odot y)$. □

R e m a r k 7. M. Galatos and C. Tsinakis introduced in [10] the notion of a *nucleus* of a residuated lattice L as a closure operator γ on L satisfying $\gamma(a)\gamma(b) \leqslant \gamma(ab)$. From this point of view, a modal operator f on an Rl-monoid M is a nucleus of M satisfying $f(x) \odot f(y) \geqslant f(x \odot y)$.

As a consequence we obtain:

Proposition 8. *Let M be an* Rl-*monoid and $f\colon M \longrightarrow M$ be a mapping. Then f is a nucleus of M if and only if f satisfies* (a) *and* (b) *of Theorem* 6.

P r o o f. Considering the previous explication it remains to prove the isotony of f.

We first note $y \to f(y) = f(y) \to f(y) = 1$. Further, $x \leqslant y \Longrightarrow 1 = y \to f(y) \leqslant x \to f(y) = f(x) \to f(y) \Longrightarrow f(x) \leqslant f(y)$. $\square$

R e m a r k 9. In [23, Corollary 8], which is the analogy of Proposition 8 for the commutative case, we required the isotony of f besides. It is evident now that this requirement is superfluous.

An Rl-monoid is called *normal* if it satisfies the identities

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}$$
$$(x \odot y)^{\sim-} = x^{\sim-} \odot y^{\sim-}.$$

For example, every good pseudo BL-algebra or every Heyting algebra is normal [25].

Let M be a good Rl-monoid and $a \in M$. We denote by $\varphi_a\colon M \longrightarrow M$ the mapping such that $\varphi_a(x) = a \oplus x$ for every $x \in M$. By Lemma 2, we have φ_a coincides with $\varphi_{a^{-\sim}}$.

Denote by

$$I(M) = \{a \in M\colon\ a \odot a = a\}$$

the set of all multiplicative idempotents in an Rl-monoid M. It is obvious that $0, 1 \in I(M)$. By [17, Lemma 2.8.3], $a \odot x = a \wedge x$ holds for any $a \in I(M)$, $x \in M$. Further we can prove, if M is a normal Rl-monoid and $a \in I(M)$ then also $a^{-\sim} \in I(M)$. Recall that by [8], $(M;\ \oplus)$ is a semigroup. Further, if $a^- \in I(M)$ then $a \oplus a = (a^- \odot a^-)^\sim = a^{-\sim}$.

Theorem 10. *If M is a good and normal* Rl-*monoid and $a \in M$ then φ_a is a strong modal operator on M if and only if a^-, $a^{-\sim} \in I(M)$.*

P r o o f. a) Let $a, x, y \in M,\ a^-, a^{-\sim} \in I(M)$.

1. $\varphi_a(x) = a \oplus x = (x^- \odot a^-)^\sim \geqslant x^{-\sim} \geqslant x$.
2. $\varphi_a(\varphi_a(x)) = a \oplus (a \oplus x) = (a \oplus a) \oplus x = a^{-\sim} \oplus x = a \oplus x = \varphi_a(x)$.
3. We first prove that $a \oplus x = (a \vee x)^{-\sim}$.

$a \oplus x = (x^- \odot a^-)^\sim = (x^- \wedge a^-)^\sim = (x \vee a)^{-\sim}$, by Lemma 1(10).

Now, we prove condition 3 from the definition of a modal operator.

By Lemma 2, the normality of M and the distributivity "$\odot$" over "$\vee$", we have

$$\begin{aligned}\varphi_a(x)\odot\varphi_a(y) &= (a\oplus x)\odot(a\oplus y) = (a^{-\sim}\oplus x)\odot(a^{-\sim}\oplus y)\\ &= (a^{-\sim}\vee x)^{-\sim}\odot(a^{-\sim}\vee y)^{-\sim} = ((a^{-\sim}\vee x)\odot(a^{-\sim}\vee y))^{-\sim}\\ &= ((a^{-\sim}\odot a^{-\sim})\vee(x\odot a^{-\sim})\vee(a^{-\sim}\odot y)\vee(x\odot y))^{-\sim}\\ &= (a^{-\sim}\vee(x\odot y))^{-\sim} = a^{-\sim}\oplus(x\odot y) = a\oplus(x\odot y) = \varphi_a(x\odot y).\end{aligned}$$

4. By [8], $(M;\oplus)$ is a semigroup.

For this reason and by the above and Lemma 2,

$$\begin{aligned}\varphi_a(x\oplus\varphi_a(y)) &= a\oplus(x\oplus(a\oplus y)) = ((a\oplus a)\oplus x)\oplus y = (a^{-\sim}\oplus x)\oplus y\\ &= (a\oplus x)\oplus y = a\oplus(x\oplus y) = \varphi_a(x\oplus y).\end{aligned}$$

b) Let φ_a be a strong modal operator on M. Then $\varphi_a(x\odot y)=\varphi_a(x)\odot\varphi_a(y)$, hence $a\oplus(x\odot y)=(a\oplus x)\odot(a\oplus y)$ for any x, $y\in M$. For $x=y=0$, we obtain $a\oplus 0=(a\oplus 0)\odot(a\oplus 0)$, thus $a^{-\sim}=a^{-\sim}\odot a^{-\sim}$, from this $a^{-\sim}\in I(M)$.

Further, $\varphi_a(x\oplus y)=\varphi_a(x\oplus\varphi_a(y))$, hence $a\oplus(x\oplus y)=a\oplus(x\oplus(a\oplus y))$ for any x, $y\in M$. For $x=y=0$, we have $a^{-\sim}=a\oplus 0=a\oplus(0\oplus 0)=a\oplus(0\oplus(a\oplus 0))=(a\oplus 0)\oplus a^{-\sim}=a^{-\sim}\oplus a^{-\sim}$, thus $a^{-\sim}=(a^{-\sim-}\odot a^{-\sim-})^{\sim}$. From this it follows that $a^{-}=(a^{-}\odot a^{-})^{\sim-}=a^{-\sim-}\odot a^{-\sim-}=a^{-}\odot a^{-}$ and so $a^{-}\in I(M)$. $\square$

Theorem 11. *Let M be a good normal* Rl*-monoid and f be a modal operator on M such that $f(x)=f(x^{-\sim})$ for all $x\in M$. Then f is strong if and only if $f=\varphi_{f(0)}$ and $f(0)^{-}\in I(M)$.*

Proof. Let f be a modal operator on M satisfying $f(x)=f(x^{-\sim})$ for every $x\in M$.

If f is strong then by Proposition 5, $f(x)=f(x^{-\sim})=x\oplus f(0)$ for any $x\in M$. Hence $f=\varphi_{f(0)}$ and moreover, by Theorem 10, $f(0)^{-}$, $f(0)^{-\sim}\in I(M)$.

If f is any modal operator then $f(0)^{-\sim}=f(0\odot 0)^{-\sim}=(f(0)\odot f(0))^{-\sim}=f(0)^{-\sim}\odot f(0)^{-\sim}$, thus $f(0)^{-\sim}\in I(M)$.

Therefore conversely, if $f=\varphi_{f(0)}$ and $f(0)^{-}\in I(M)$, then by Theorem 10, we conclude that f is strong. $\square$

Corollary 12. *If M is a* GMV*-algebra (pseudo* MV*-algebra, equivalently) and f is a modal operator on M, then f is strong if and only if $f=\varphi_{f(0)}$.*

Proof. It is sufficient to show that $f(0)^{-}\in I(M)$ for any modal operator f.

It is known (see [11] and [22]) that the set $I(M)$ coincides with the set $B(M)$ of all elements having complements in the lattice $(M;\vee,\wedge,0,1)$ in any GMV-algebra M, and if $x\in B(M)$ and x' is its complement, then $x'=x^{-}=x^{\sim}\in B(M)=I(M)$.

Since $f(0)\in I(M)$ in our case, we obtain $f(0)^{-}\in I(M)$ as well. $\square$

Let M be an Rl-monoid and $a \in I(M)$. Let us consider mappings $\psi_a^1\colon M \longrightarrow M$ and $\psi_a^2\colon M \longrightarrow M$ such that $\psi_a^1(x) := a \to x$ and $\psi_a^2(x) := a \rightsquigarrow x$ for every $x \in M$.

Proposition 13. *If M is an* Rl*-monoid and $a \in I(M)$ then for any x, $y \in M$*

$$x \to \psi_a^1(y) = \psi_a^1(x) \to \psi_a^1(y),$$
$$x \rightsquigarrow \psi_a^2(y) = \psi_a^2(x) \rightsquigarrow \psi_a^2(y).$$

P r o o f. By the definition of ψ_a^1 and Lemma 1(8), $x \to \psi_a^1(y) = x \to (a \to y) = (x \odot a) \to y = (a \odot x) \to y$ and $\psi_a^1(x) \to \psi_a^1(y) = (a \to x) \to (a \to y) = ((a \to x) \odot a) \to y = (a \wedge x) \to y = (a \odot x) \to y$.

The other identity would be proved analogously. $\square$

Corollary 14. *Let M be an* Rl*-monoid and $a \in I(M)$. Then ψ_a^1 is a modal operator on M if and only if for any x, $y \in M$*

$$x \rightsquigarrow \psi_a^1(y) = \psi_a^1(x) \rightsquigarrow \psi_a^1(y),$$
$$\psi_a^1(x) \odot \psi_a^1(y) \geqslant \psi_a^1(x \odot y).$$

Let M be an Rl-monoid and f be a modal operator on M. Then $\mathrm{Fix}(f) = \{x \in M\colon f(x) = x\}$ will denote the set of all fixed elements of the operator f. By the definition of a modal operator it is obvious that $\mathrm{Fix}(f) = \mathrm{Im}(f)$.

Theorem 15. *If f is a modal operator on an* Rl*-monoid M then* $\mathrm{Fix}(f)$ *is closed under the operations $\wedge, \odot, \to$ and $\rightsquigarrow$, and* $\mathrm{Fix}(f) = (\mathrm{Fix}(f); \odot, \vee_F, \wedge, \to, \rightsquigarrow, f(0), 1)$, *where $x \vee_F y = f(x \vee y)$ for any $x, y \in \mathrm{Fix}(f)$, is an* Rl*-monoid.*

P r o o f. (i) Since f is a closure operator on the lattice $(M; \vee, \wedge)$, it holds $x \wedge y \in \mathrm{Fix}(f)$ for any x, $y \in \mathrm{Fix}(f)$, and so $(\mathrm{Fix}(f); \vee_F, \wedge)$ is a lattice.

(ii) $(\mathrm{Fix}(f); \vee_F, \wedge, f(0), 1)$ is a bounded lattice.

(iii) Let x, $y \in \mathrm{Fix}(f)$. Then $f(x \odot y) = f(x) \odot f(y) = x \odot y$, thus $x \odot y \in \mathrm{Fix}(f)$.

(iv) If y, $z \in \mathrm{Fix}(f)$ then by Proposition 5 we have $y \to z = f(y) \to f(z) = f(f(y) \to f(z)) = f(y \to z)$, hence $y \to z \in \mathrm{Fix}(f)$. For any y, $z \in \mathrm{Fix}(f)$, $y \rightsquigarrow z \in \mathrm{Fix}(f)$ analogously.

Therefore, if x, y, $z \in \mathrm{Fix}(f)$ then $x \odot y$, $y \to z$, $x \rightsquigarrow z \in \mathrm{Fix}(f)$ and for this reason $x \odot y \leqslant z$ holds in $\mathrm{Fix}(f)$ if and only if $x \leqslant y \to z$ and it is equivalent to $y \leqslant x \rightsquigarrow z$.

(v) By foregoing, $\mathrm{Fix}(f)$ also satisfies the identities $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$. $\square$

Let M be an Rl-monoid. For $a \in I(M)$, let

$$I(a) := [0, a] = \{x \in M \colon\ 0 \leqslant x \leqslant a\}.$$

Theorem 16. *Let M be an* Rl-*monoid and $a \in I(M)$. For any x, $y \in I(a)$ we set $x \odot_a y = x \odot y$, $x \to_a y := (x \to y) \wedge a$ and $x \rightsquigarrow_a y := (x \rightsquigarrow y) \wedge a$. Then $I(a) = (I(a); \odot_a, \vee, \wedge, \to_a, \rightsquigarrow_a, 0, a)$ is an* Rl-*monoid.*

Proof. (i) If x, $y \in I(a)$ then $x \odot y \in I(a)$ and $x \odot a = a \odot x = x \wedge a = x$, hence $(I(a); \odot_a, a)$ is a monoid.

(ii) Obviously, $(I(a); \vee, \wedge, 0, a)$ is a bounded lattice.

(iii) Let x, $y \in I(a)$. It holds that $x \to y$ is the greatest element $z \in M$ such that $z \odot x \leqslant y$. Therefore $(x \to y) \wedge a$ is the greatest element in $I(a)$ with this property. Analogously, for y, $z \in I(a)$, $z \rightsquigarrow y$ is the greatest element $x \in M$ such that $z \odot_a x \leqslant y$. Hence, $(z \rightsquigarrow y) \wedge a$ is the greatest element in $I(a)$ with this property. That means, $z \odot_a x \leqslant y$ if and only if $z \leqslant (x \to y) \wedge a = x \to_a y$, and if and only if $x \leqslant (z \rightsquigarrow y) \wedge a = z \rightsquigarrow_a y$ for any x, $z \in I(a)$.

(iv) For any x, $y \in I(a)$ we have $(x \to_a y) \odot_a x = ((x \to y) \wedge a) \odot x = (x \to y) \odot a \odot x = (x \to y) \odot x \odot a = (x \wedge y) \wedge a = x \wedge y$. We obtain $y \odot_a (y \rightsquigarrow_a x) = x \wedge y$ analogously. □

Let M be an Rl-monoid, $a \in I(M)$ and $x \in I(a)$. We denote by x^{-_a} and $x^{\sim_a}$ the negations of an element x in $I(a)$.

Proposition 17. a) *If M is an* Rl-*monoid, $a \in I(M)$ and $x \in I(a)$, then*

$$x^{-_a} = x^- \wedge a, \quad x^{\sim_a} = x^\sim \wedge a.$$

b) *Moreover, if M is good and satisfying the identities*

$$(*) \qquad (v \wedge w)^- = v^- \vee w^-,\ (v \wedge w)^\sim = v^\sim \vee w^\sim,$$

then the Rl-*monoid $I(a)$ is good, too. If we denote by $x \oplus_a y$ the sum of elements x, $y \in I(a)$ in the* Rl-*monoid $I(a)$ then it holds*

$$x \oplus_a y = (x \oplus y) \wedge a.$$

Proof. a) $x^{-_a} = x \to_a 0 = (x \to 0) \wedge a = x^- \wedge a$, $x^{\sim_a} = x \rightsquigarrow_a 0 = (x \rightsquigarrow 0) \wedge a = x^\sim \wedge a$.

b) Let M be good and satisfy $(*)$. Then

$$\begin{aligned} x^{-_a \sim_a} &= (x^{-_a})^{\sim} \wedge a = (x^{-} \wedge a)^{\sim} \wedge a = (x^{-\sim} \vee a^{\sim}) \wedge a = (x^{-\sim} \wedge a) \vee (a^{\sim} \wedge a) \\ &= (x^{-\sim} \wedge a) \vee 0 = x^{-\sim} \wedge a = x^{\sim -} \wedge a = (x^{\sim -} \wedge a) \vee (a^{-} \wedge a) \\ &= (x^{\sim -} \vee a^{-}) \wedge a = (x^{\sim} \wedge a)^{-} \wedge a = (x^{\sim_a})^{-} \wedge a = x^{\sim_a -_a}, \end{aligned}$$

hence $I(a)$ is also a good Rl-monoid and therefore we can define $x \oplus_a y$ for any $x, y \in I(a)$.

Then it holds, using Lemma 1(9),

$$\begin{aligned} x \oplus_a y &= (y^{-_a} \odot x^{-_a})^{\sim_a} = (y^{-_a} \odot x^{-_a})^{\sim} \wedge a = (y^{-} \odot a \odot x^{-} \odot a)^{\sim} \wedge a \\ &= (y^{-} \odot x^{-} \odot a)^{\sim} \wedge a = (a \rightsquigarrow (y^{-} \odot x^{-})^{\sim}) \odot a = a \odot (a \rightsquigarrow (y^{-} \odot x^{-})^{\sim}) \\ &= a \wedge (y^{-} \odot x^{-})^{\sim} = (x \oplus y) \wedge a. \end{aligned}$$

□

R e m a r k 18. For example, every pseudo BL-algebra satisfies the identities $(*)$ (see [25]).

Let M be an Rl-monoid, $a \in I(M)$ and let f be a modal operator on M. Let us consider a mapping $f^a \colon I(a) \longrightarrow I(a)$ such that $f^a(x) := f(x) \wedge a (= f(x) \odot a)$, for every $x \in I(a)$.

Theorem 19. a) *Let M be an* Rl*-monoid, $a \in I(M)$ and f be a modal operator on M. Then f^a is a modal operator on the* Rl*-monoid $I(a)$.*

b) *If M is good and it satisfies the identities $(*)$, and f is strong, then f^a is also a strong modal operator on $I(a)$.*

P r o o f. a) Consider $x, y \in I(a)$.

1. $x \leqslant a$ and $x \leqslant f(x)$, hence $x \leqslant a \wedge f(x) = f^a(x)$.
2. $f^a(f^a(x)) = f^a(f^a(x)) = f(f(x) \wedge a) \wedge a = f(f(x) \odot a) \wedge a = (f(f(x)) \odot f(a)) \wedge a = f(x) \wedge f(a) \wedge a = f(x) \wedge a = f^a(x)$.
3. $f^a(x \odot y) f^a(x \odot y) = f(x \odot y) \wedge a = f(x) \odot f(y) \odot a \odot a = (f(x) \wedge a) \odot (f(y) \wedge a) = f^a(x) \odot f^a(y)$.

b) Let f be strong. Then

$$\begin{aligned} f^a(x \oplus_a f^a(y)) &= f^a(x \oplus_a f^a(y)) = f^a((x \oplus (f(y) \wedge a)) \wedge a) \\ &= f((x \oplus (f(y) \wedge a)) \wedge a) \wedge a \\ &= f(x \oplus (f(y) \wedge a)) \wedge f(a) \wedge a = f(x \oplus f(f(y) \wedge a)) \wedge a \\ &= f(x \oplus ((f(f(y)) \wedge f(a))) \wedge a = f(x \oplus (f(y) \wedge f(a))) \wedge a \\ &= f(x \oplus f(y \wedge a)) \wedge a = f(x \oplus f(y)) \wedge a = f(x \oplus y) \wedge a \\ &= f^a(x \oplus y). \end{aligned}$$

□

Proposition 20. *If M is an* Rl*-monoid then $I(M)$ is a subalgebra of its reduct* $(M; \odot, \vee, \wedge, 0, 1)$.

P r o o f. Suppose M is an Rl-monoid and x, $y \in I(M)$. Then

$$(x \odot y) \odot (x \odot y) = x \odot (y \odot x) \odot y = x \odot (x \odot y) \odot y = (x \odot x) \odot (y \odot y) = x \odot y,$$

thus $x \odot y = x \wedge y \in I(M)$. Further,

$$(x \vee y) \odot (x \vee y) = (x \odot x) \vee (y \odot x) \vee (x \odot y) \vee (y \odot y) = x \vee y \vee (x \odot y) = x \vee y,$$

hence also $x \vee y \in I(M)$.

Obviously, $0, 1 \in I(M)$. $\square$

Let f be a modal operator on an Rl-monoid M and $\hat{f} = f|I(M)$. Assume $x \in I(M)$. Then $f(x) = f(x \odot x) = f(x) \odot f(x)$, so $f(x) \in I(M)$. Therefore, we can consider $\hat{f}$ as the mapping of $I(M)$ into $I(M)$.

Theorem 21. *Let M be an* Rl*-monoid and let f be a modal operator on M. Then $\hat{f}\colon\ I(M) \longrightarrow I(M)$ fulfills conditions* 1, 2, 3 *from the definition of a modal operator.*

P r o o f. Theorem is the immediate consequence of previous considerations. $\square$

Theorem 22. *Let M be a good and normal* Rl*-monoid and let $x^- \in I(M)$ for any $x \in I(M)$. Then $I(M)$ is closed under the operation "$\oplus$". Furthermore, if f is a strong modal operator on M then $\hat{f}$ satisfies the condition 4 from the definition of a strong modal operator, too.*

P r o o f. Let x, $y \in I(M)$. By the proof of Theorem 10, part 3, it holds $x \oplus y = (x \vee y)^{-\sim}$. Further,

$$\begin{aligned}(x \oplus y) \odot (x \oplus y) &= (x \vee y)^{-\sim} \odot (x \vee y)^{-\sim} = ((x \vee y) \odot (x \vee y))^{-\sim} \\ &= ((x \odot x) \vee (x \odot y) \vee (y \odot x) \vee (y \odot y))^{-\sim} = (x \vee y)^{-\sim} = x \oplus y.\end{aligned}$$

Therefore, $x \oplus y \in I(M)$.

Now it is obvious that $\hat{f}$ satisfies also the condition 4 from the definition of a strong modal operator. $\square$

Let us remind that an Rl-monoid is called *representable* if and only if it is isomorphic with a subdirect product of linearly ordered Rl-monoids. It is obvious that every (bounded) linearly ordered Rl-monoid is a pseudo BL-algebra. Therefore, representable Rl-monoids are pseudo BL-algebras as well; and by [18], they form a proper subclass of the class of all pseudo BL-algebras. (Let us recall that, by [21], the class of representable commutative Rl-monoids and the class of all BL-algebras coincide.)

Theorem 23. *Let M be a representable pseudo* BL*-algebra. Then $I(M)$ is a subalgebra in M which is a Heyting algebra. If f is a modal operator on M then $\hat{f}$ is a modal operator on $I(M)$. Moreover, if M is good and $x^- \in I(M)$ for every $x \in I(M)$ and f is a strong modal operator on M, then $\hat{f}$ is a strong modal operator on $I(M)$.*

Proof. Let a representable pseudo BL-algebra M be isomorphic with a subdirect product of pseudo BL-chains M_α, $\alpha \in A$. Let $a = (a_\alpha; \alpha \in A) \in M$. Then $a \in I(M)$ if and only if $a_\alpha \in I(M_\alpha)$ for every $\alpha \in A$. Suppose $x = (x_\alpha; \alpha \in A)$, $y = (y_\alpha; \alpha \in A) \in I(M)$. Then $x_\alpha \to y_\alpha = 1$ for $y_\alpha \geqslant x_\alpha$ and $x_\alpha \to y_\alpha = y_\alpha$ for $x_\alpha > y_\alpha$. Hence $(x_\alpha \to y_\alpha; \alpha \in A) \in I(M)$ and $(x_\alpha \to y_\alpha; \alpha \in A) = x \to y$. Similarly, $(x_\alpha \rightsquigarrow y_\alpha; \alpha \in A) \in I(M)$ and $(x_\alpha \rightsquigarrow y_\alpha; \alpha \in A) = x \rightsquigarrow y$. By [17], $I(M)$ is a Heyting algebra.

Therefore, if f is a modal operator on M then $\hat{f}$ is a modal operator on $I(M)$. Further by [25], every good pseudo BL-algebra is a normal Rl-monoid. For that reason, if M is good and $x^- \in I(M)$ for every $x \in I(M)$, and if a modal operator on M is strong, then $\hat{f}$ is a strong modal operator on the Heyting algebra $I(M)$. □

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