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ON EXTREMAL SIZES OF LOCALLY k-TREE GRAPHS

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Abstract. A graph G is a *locally k-tree graph* if for any vertex v the subgraph induced by the neighbours of v is a k-tree, $k \geqslant 0$, where 0-tree is an edgeless graph, 1-tree is a tree. We characterize the minimum-size locally k-trees with n vertices. The minimum-size connected locally k-trees are simply $(k+1)$-trees. For $k \geqslant 1$, we construct locally k-trees which are maximal with respect to the spanning subgraph relation. Consequently, the number of edges in an n-vertex locally k-tree graph is between $\Omega(n)$ and $O(n^2)$, where both bounds are asymptotically tight. In contrast, the number of edges in an n-vertex k-tree is always linear in n.

Keywords: extremal problems, local property, locally tree, k-tree

MSC 2010: 05C35

1. Introduction

The graphs $G = (V, E)$ considered in this paper are finite and simple, i.e., undirected, loopless and without multiple edges. Let $\mathcal{P}$ be a family of graphs. A graph G is said to satisfy *local property* $\mathcal{P}$ if for all $v \in V(G)$ we have $G[N(v)] \in \mathcal{P}$, where $N(v)$ denotes the neighbourhood of v and $G[S]$ stands for the subgraph induced by $S \subseteq V(G)$.

The graphs with local property $\mathcal{P}$ for $|\mathcal{P}| = 1$ have been studied by many authors. The study has been inspired by the Trahtenbrot-Zykov problem [23] whether, given a graph H, there exists a graph G which is locally constant, namely locally H. Summaries of the results of this type can be found in the survey papers by Hell [10] and Sedláček [19]. The major question is then the existence of any (or just finite) local realization G of H, see [4] for the nonexistence and [2], [3] for the existence of algorithms. The set of forbidden subgraphs of a graph with a local hereditary property $\mathcal{P}$ has been described by Borowiecki and Mihók [1]. Interesting extremal problems arise in case $\mathcal{P}$ is infinite. Erdős and Simonovits [6] found the maximum number of edges in a locally acyclic graph. Ryjáček and Zelinka [17] constructed locally disconnected graphs with a large number of edges while Fronček [7] found an upper bound for the number of edges of a locally linear graph (i.e., for all $v \in V(G)$, $G[N(v)]$ is a regular graph of degree 1), and a locally path graph [8] (i.e., for all $v \in V(G)$, $G[N(v)]$ is a path). Zelinka [22] studied locally tree graphs, i.e., the graphs in which the subgraph induced by $N(v)$ is a tree for all $v \in V(G)$. He proved that the minimum number of edges in a connected locally tree graph with n vertices is $2n-3$ and posed the problem of determining the maximum. The problem was addressed by Kowalska [12]. She proved that $|E(G)| \leqslant \frac{1}{2}(n^2) - \frac{1}{2}(5n) + 7$ holds for any locally tree graph G with n vertices.

Sedláček [18] introduced an N_2-local property. The edge-induced subgraph on the set of all edges of a graph G that are adjacent to a given vertex x is denoted by $N_2(x, G)$. A graph G has an N_2-*local property* $\mathcal{P}$ if the subgraph $N_2(x, G)$ has property $\mathcal{P}$ for every vertex $x \in V(G)$. The maximum size among planar N_2-locally disconnected graphs of given order was found in [16]. Also the concept of edge-local properties was studied, e.g., in [21], [13]. A graph G is said to satisfy an *edge-local property* $\mathcal{P}$ if for any edge $e = xy$ the subgraph induced by all vertices adjacent to at least one vertex x, y but different from them has property $\mathcal{P}$. An upper bound for the number of edges of edge-locally acyclic graphs was proven by Fronček [16].

In this paper we deal with infinitely many local infinite properties, namely, these are locally k-tree graphs, $k = 0, 1, \ldots$. We need to define k-trees and a specific ordering of their vertices for $k > 0$ only (because 0-trees are just edgeless graphs). To start with, let $\{v_1, \ldots, v_n\}$ be the vertex set of a graph G and let G_i denote the subgraph of G induced by vertices $\{v_1, \ldots, v_i\}$, $i \geqslant 1$. For each $v_j \in V(G_i)$, $N_i(v_j)$ and $d_i(v_j)$ denote the neighbourhood and the degree of v_j in G_i, respectively. Any subgraph of G which is isomorphic to K_k (a complete graph with k vertices) is called a k-*clique*. Assuming $1 \leqslant k \leqslant n$, the ordering $(v_1, \ldots, v_n)$ of $V(G)$ is the k-*perfect elimination ordering* (k-PEO for brevity) if vertices $\{v_1, \ldots, v_k\}$ induce a k-clique and for each $i > k$ the set $N_i(v_i)$ also induces a k-clique. Finally, the graph G is a k-*tree* with $k \geqslant 1$ if it has a k-perfect elimination ordering while, for $k \geqslant 0$, it is a *locally* k-*tree graph* if for any vertex v the subgraph induced by $N(v)$ is a k-tree.

In order to generate a PEO of a graph, Rose at al. [15] developed a method called Lexicographic Breadth First Search that has been also used in the recognition of k-trees [11].

Theorem 1 ([15]). *The complexity of a k-tree recognition is $O(kn)$, where $n = |V(G)|$.*

Theorem 2. *The complexity of a locally k-tree graph recognition is $O(km)$, where $m = |E(G)|$.*

Proof. Let G be a locally k-tree graph. It is enough to test whether or not $G[N(v)]$ is a k-tree for any $v \in V(G)$. Hence $\sum_{v\in V(G)} O(kd(v)) = O(km)$, and the theorem follows. □

In Section 2 we give basic properties of k-trees. We next prove that each $(k+1)$-tree is a locally k-tree graph. We conversely prove that each connected locally k-tree graph contains a $(k+1)$-tree as a spanning subgraph. Hence a locally k-tree graph has at least $k+1$ vertices. We characterize minimum-size locally k-tree graphs on n vertices. Namely, these are $(k+1)$-forests with exactly $\lfloor n/(k+1)\rfloor$ components. The smallest size among locally k-tree graphs on n vertices is determined and is linear in n. In fact, the smallest size is asymptotic to $(k+1)n$ if graphs are connected, and asymptotic to $nk/2$ otherwise.

In Section 3 we give some properties and constructions of locally k-tree graphs. Section 4 is devoted to the construction of maximal locally k-tree graphs.

For brevity, we will omit definitions of standard notions of graph theory we use here. For these and other related concepts we refer the reader to [5].

2. The minimum size of locally k-tree graphs

To determine the minimum size of locally k-tree graphs we need some well-known properties of k-trees.

Lemma 1. *If G is a k-tree, then $|E(G)| = \binom{k}{2} + (|V(G)| - k)k$ and, for any subgraph H of G with order at least k, we have $|E(H)| \leqslant \binom{k}{2} + (|V(H)| - k)k$.*

Theorem 3. *Let G be a k-tree and let $\{v_1, \ldots, v_k\} \subseteq V(G)$ be a set of vertices which induces K_k in G. Then the ordering $(v_1, \ldots, v_k)$ can be extended to a k-PEO of G.*

We extend Theorem 3 in the following way.

Theorem 4. *Let G be a k-tree and let H be a subgraph of G which also is a k-tree. Then each k-PEO of H can be extended to a k-PEO of G.*

Proof. Suppose that this is not the case. Let H be a subgraph of G with the maximum number of vertices which is a k-tree and such that there is a k-PEO of H which cannot be extended to a k-PEO of G. Let $(v_1, \ldots, v_n)$ be a k-PEO of G and let $(v_{i_1}, \ldots, v_{i_t})$ be an ordered subset of $\{v_1, \ldots, v_n\}$ which contains all vertices of H. Obviously $t < n$.

Let us first prove that $(v_{i_1}, \ldots, v_{i_t})$ is a k-PEO of H. Since H is a k-tree with t vertices, we have $|E(H)| = \binom{k}{2} + k(t-k)$. On the other hand, $|E(H)| \leqslant |E(G[\{v_{i_1}, \ldots, v_{i_k}\}])| + \sum\limits_{j=k+1}^{t} d_{i_j}(v_{i_j})$. Since $|E(G[\{v_{i_1}, \ldots, v_{i_k}\}])| \leqslant \binom{k}{2}$ and $\sum\limits_{j=k+1}^{t} d_{i_j}(v_{i_j}) = k(t-k)$, we conclude that the vertices $\{v_{i_1}, \ldots, v_{i_k}\}$ induce K_k and for $j = k+1, \ldots, t$ the vertices $N(v_{i_j})$ also induce K_k. Hence the ordering $(v_{i_1}, \ldots, v_{i_t})$ is a k-PEO of H.

If there exists a vertex v_j, $j > i_t$, such that $N_j(v_j) \subseteq V(H)$ then the ordering $(v_{i_1}, \ldots, v_{i_t}, v_j)$ is a k-PEO and so $G[V(H) \cup \{v_j\}]$ is a k-tree with more vertices than H, a contradiction. If such a vertex does not exist then $v_{i_1} \neq v_1$ (note that $G \neq H$). Then the vertices $v_{i_1}, \ldots, v_{i_k}$ have a common neighbour v_p, $p < i_1$. Hence $d_H(v_p) = k$ and $G[V(H) \cup \{v_p\}]$ is a k-tree, a contradiction. □

Lemma 2. *If a graph G is a $(k+1)$-tree, then G is locally k-tree.*

Proof. Let $(v_1, \ldots, v_n)$ be a $(k+1)$-PEO of a graph G and let v_j be a vertex of G. Let $(v_{i_1}, \ldots, v_{i_t})$ be an ordered subset of $\{v_1, \ldots, v_n\}$ which contains all vertices of $N(v_j)$. Since $(v_1, \ldots, v_n)$ is a $(k+1)$-PEO, it follows that the vertices $\{v_{i_1}, \ldots, v_{i_{k+1}}\}$ induce a clique and $i_{k+1} < j \leqslant i_{k+2}$. Moreover, $N_{i_p}(v_{i_p}) \subseteq N(v_j)$ for $p \geqslant k+2$. This implies that $(v_{i_1}, \ldots, v_{i_t})$ is a k-PEO of the graph induced by $N(v_j)$. □

Theorem 5. *If a connected graph G is locally k-tree, then for any subgraph T of G which is a $(k+1)$-tree there is a spanning subgraph H of G which is a $(k+1)$-tree and contains T.*

Proof. Let $H \subseteq G$ be a $(k+1)$-tree which has the maximum order and contains T. Suppose that $V(H) \neq V(G)$. Since G is connected, there exists a vertex of H adjacent to a vertex of $G - H$. Let x be the first such vertex in a $(k+1)$-PEO of H. From Theorem 4 it follows that any k-PEO of the subgraph induced by $N(x) \cap V(H)$ can be extended to a k-PEO of the subgraph induced by $N(x)$. Let $(v_1, \ldots, v_t)$ be a k-PEO of $G[N(x) \cap V(H)]$ and let y be the first vertex of the k-PEO of $G[N(x)]$ which is not in H. Then $(v_1, \ldots, v_t, y)$ is also a k-PEO. Moreover, since $xy \in E(G)$, the subgraph induced by $N(y) \cap V(H)$ is a $(k+1)$-clique. Thus $G[V(H) \cup \{y\}]$ is a $(k+1)$-tree containing T, which contradicts the maximality of H. □

Corollary 1. *If a connected graph G is locally k-tree, then G contains a spanning $(k+1)$-tree.*

Proof. It is easy to see that for any vertex $x \in V(G)$ there is a k-clique in $G[N[x]]$. Let K be a $(k+1)$-clique in G. Then by Theorem 5, K can be extended to a spanning subgraph of G which is a $(k+1)$-tree. $\square$

Corollary 2. *If G is a connected locally k-tree graph of order n and smallest size, then $|E(G)| = n(k+1) - \binom{k+2}{2}$.*

If G is not connected and is a locally 0-tree graph of order n with the minimum number of edges, then G is isomorphic to $\overline{K}_n$. More generally, a locally k-tree graph of order n with the minimum number of edges is a $(k+1)$-forest, whence $n \geqslant k+1$.

Theorem 6. *Minimum-size locally k-tree graphs on n vertices are precisely the $(k+1)$-forests with $\lfloor n/(k+1) \rfloor$ components (each of which is a $(k+1)$-tree).*

Proof. Let G_n be a minimum-size locally k-tree graph on n vertices and let $n = (k+1)p+r$ where $p = \lfloor n/(k+1) \rfloor \geqslant 1$ and $r = n \bmod (k+1)$ (whence $0 \leqslant r \leqslant k$). If $r = 0$ then clearly $G_n = pK_{k+1}$. Similarly, for any r, p is the number of components of G_n. Hence, for $r > 0$, given any G_{n-1}, adding a new vertex, say v, together with $k+1$ edges which join v to a $(k+1)$-clique of G_{n-1} gives a G_n. Moreover, each G_n with $r > 0$ can thus be obtained, which completes the proof. $\square$

Corollary 3. *Let $n = (k+1)p + r$ where $r = n \bmod (k+1)$. A minimum-size locally k-tree graph G_n on n vertices has $nk/2 + (k+2)r/2$ edges, which is asymptotically $nk/2$ as $n \to \infty$.*

Proof. From the proof of Theorem 6 it follows that $|E(G_n)| = p\binom{k+1}{2} + (k+1)r$ where $p = (n-r)/(k+1)$. Therefore $|E(G_n)| = nk/2 + (k+2)r/2$ with $r = n - (k+1)\lfloor n/(k+1) \rfloor \leqslant k$, whence the result follows. $\square$

3. Properties and constructions of locally k-tree graphs

The *union* of two vertex-disjoint graphs G and H is a graph $G \cup H$ such that $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. If $A \subseteq E(\overline{G})$, then by $G + A$ we denote a graph with $V(G+A) = V(G)$ and $E(G+A) = E(G) \cup A$.

Construction 1: k-join

Let H and H' be two vertex-disjoint copies of K_{k+1} and let $V(H) = \{v_1, \ldots, v_{k+1}\}$, $V(H') = \{v'_1, \ldots, v'_{k+1}\}$. Then a *$k$-join* of H and H', denoted by $H \oplus H'$, is defined as follows:

$$H \oplus H' = (H \cup H') + E',$$

where $E' = \{v_i v'_j \colon\ i = 1, 2, \ldots, k+1,\ j = i, i+1, \ldots, k+1\}$.

New edges, which form the set E', are called *the joining edges* of the k-join.

Figure 1. 1-join: $H \oplus H'$ and $H' \oplus H$.

The graphs $H \oplus H'$ and $H' \oplus H$ are isomorphic, but as labeled graphs they are different. The importance of this fact will be seen in the next construction, where we use special graphs to obtain non-isomorphic locally k-tree graphs.

Construction 2: k-join substitution

A k-join substitution of a graph G is a graph G' that we obtain by replacing each vertex of G by a $(k+1)$-clique K^v, and by adding edges between the cliques K^v and K^w for each edge vw of G such that the subgraph of G' induced on the vertices of the two cliques K^v, K^w is a k-join of K^v and K^w.

Remark 1. If G' is a result of a k-join substitution of G, then G' has exactly $(k+1)|V(G)|$ vertices and exactly $\binom{k+2}{2}|E(G)|$ edges.

Remark 2. If graphs H_1 and H_2 are obtained from G by the k-join substitution, they need not be isomorphic. The result of the k-join substitution depends on the labels (the order) of vertices of $(k+1)$-cliques and the order in which the k-join is performed on "adjacent" $(k+1)$-cliques.

By $K_{k+1} \overset{\oplus}{\to} G$ we will denote the set of graphs which can be obtained from G by a k-join substitution.

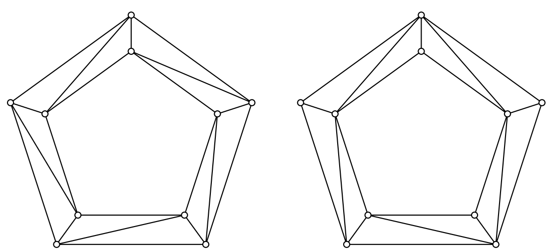


Figure 2. Two non-isomorphic graphs in $K_2 \overset{\oplus}{\to} C_5$.

Theorem 7. *If G is a K_3-free graph, then each graph in $K_{k+1} \overset{\oplus}{\to} G$ is locally k-tree.*

Proof. Let H be a graph obtained from G by a k-join substitution and let $v' \in V(H)$. Let K^v be a $(k+1)$-clique which contains the vertex v' and which replaces the vertex v of G. Since G is K_3-free, the set $N_G(v) = \{u^1, \ldots, u^t\}$ is independent. Let K^{u^i} be a $(k+1)$-clique which replaces the vertex u^i. We will show that the graph F induced by vertices $V(K^v) \cup V(K^{u^1}) \cup \ldots \cup V(K^{u^t})$ is a $(k+1)$-tree. Let $\{u_1^i, \ldots, u_{k+1}^i\}$ be the vertex set of K^{u^i}, $i = 1, \ldots, t$, which is ordered so that $|N(u_j^i) \cap V(K^v)| = k + 2 - j$, $j = 1, \ldots, k+1$. Then $(v_1, \ldots, v_{k+1}, u_1^1, \ldots, u_{k+1}^1, u_1^2, \ldots, u_{k+1}^2, \ldots, u_1^t, \ldots, u_{k+1}^t)$ is a $(k+1)$-PEO, where $(v_1, \ldots, v_{k+1})$ is an appropriate PEO of K^v. Hence F is a $(k+1)$-tree and since $N_H(v') = N_F(v')$ we obtain that the graph induced by $N_H(v')$ is a k-tree. □

If $H \in K_{k+1} \overset{\oplus}{\to} G$, then $|V(H)| = (k+1)|V(G)|$. To obtain a locally k-tree with n vertices (for arbitrary n) we can use the next lemma.

Lemma 3. *Let G be a locally k-tree graph and H a $(k+1)$-clique of G. If we join a new vertex v with all vertices of H, then the resulting graph is also locally k-tree.*

A subgraph $H \subseteq G$ is called a *K_t-factor* of G if H is a spanning subgraph of G, and H is the union of vertex-disjoint t-cliques. By $\mathcal{L}_F(k)$ we denote the set of locally k-tree graphs which contain a K_{k+1}-factor.

Lemma 4. *Let G and F be disjoint k-trees. Let G' and F' be k-cliques in G and F, respectively. Let H be a graph obtained from the graphs G and F by bijectively identifying vertices of G' with those of F' and leaving the remaining vertices unchanged. Then H is a k-tree.*

Proof. Let $V(G') = \{v_1, \ldots, v_k\}$ and $V(F') = \{w_1, \ldots, w_k\}$, where the notation is chosen such that v_i and w_i are identified into a vertex, say v_i, of H. Assume that the remaining vertices are denoted so that $(v_1, v_2, \ldots, v_{|V(G)|})$ and $(w_1, w_2, \ldots, w_{|V(F)|})$ are k-PEO's of G and F, respectively (by Theorem 3 such k-PEO's exist). Then the ordering $(v_1, v_2, \ldots, v_{|V(G)|}, w_{k+1}, \ldots, w_{|V(F)|})$ is a k-PEO of H. Hence H is a k-tree. □

Theorem 8. *Let F be a locally k-tree graph and let F' be an induced subgraph of F such that*

(1) *F' is the union of $(k+1)$-cliques, and*

(2) *there exists a locally k-tree graph G such that F' is a spanning subgraph of G, and such that for each edge $uv \in E(G) \setminus E(F')$, the distance $d_F(u, v)$ of u and v in F is at least 3.*

Then the graph obtained from F by adding the edges $E(G) \setminus E(F')$ is also a locally k-tree graph.

P r o o f. Let F'' be the graph obtained from F by adding the edges $E(G) \setminus E(F')$. Let $S \subseteq V(F'')$ be the set of vertices which are incident to edges which were added to F''. Since all edges which we added connect vertices which are in the distance at least 3, it follows that for any vertex $x \in V(F'') \setminus S$ the graphs induced by the neighbours of x in F'' and in F are the same, i.e., it is a k-tree. For any vertex $y \in S$ the graph induced by its neighbours is a gluing of two k-trees (i.e., by identification of vertices of k-cliques which was described in Lemma 4), so that the resulting graph is a k-tree. □

4. Construction of maximal locally k-tree graphs

A locally k-tree graph is *maximal*, if it is not a spanning subgraph of another locally k-tree graph. In this section, we describe a construction of maximal locally k-tree graphs for $k \geqslant 1$.

Let $G(a,b;k)$ denote a graph obtained from a complete bipartite graph $K_{a,b}$ by the k-join substitution performed on $(k+1)$-cliques which replace the vertices of an independent set of order a and $(k+1)$-cliques replacing the vertices of an independent set of order b.

Recall that $\mathcal{L}_F(k)$ is the set of locally k-tree graphs which contain a K_{k+1}-factor.

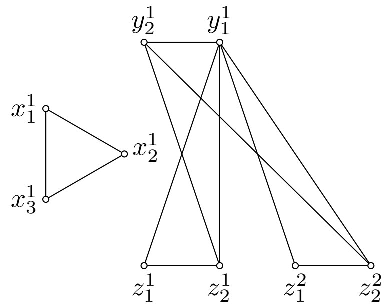
Proposition 1. *Let a, b, k be positive integers. Then $G(a,b;k) \in \mathcal{L}_F(k)$.*

P r o o f. By Theorem 7 each graph obtained from a K_3-free graph by the k-join substitution is locally k-tree. Then $G(a,b;k)$ is a locally k-tree graph. It is easy to see that $G(a,b;k)$ contains a K_{k+1}-factor, and hence, $G(a,b;k) \in \mathcal{L}_F(k)$. □

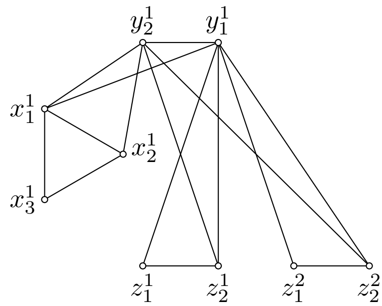
The graph $G(a,b;k)$ has exactly $(a+b)(k+1)$ vertices and exactly $ab\binom{k+2}{2}$ edges. Then the number of edges of $G(a,b;k)$ for a given number of vertices is maximized if $a=b$. In that case, $G(a,a;k)$ is a graph on $n = 2a(k+1)$ vertices and $a^2\binom{k+2}{2} = \frac{1}{8}n^2(k+2)/(k+1)$ edges. That is, we obtain the following bound.

Corollary 4. *If n is divisible by $k+1$, then a locally k-tree graph on n vertices with maximum number of edges has at least $\frac{1}{8}n^2(k+2)/(k+1)$ edges.*

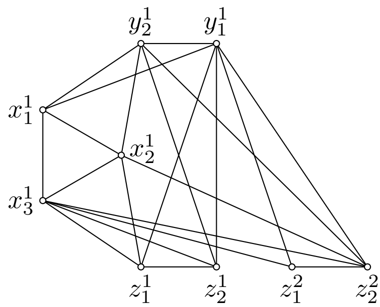
It turns out that we can do better. We describe a construction of a locally k-tree graph that works for all sufficiently large values of n, and using this graph we obtain an improved bound.



Step 1. The graph G_1



Step 2. The graph G_2



Step 2. The graph $G_3 = G(1, 1, 2; 1)$

Figure 3. Construction of maximal locally k-tree graphs.

Construction 3

For positive integers k, p, r and t, let $G(p, r, t; k)$ denote the graph obtained from graphs G_0, G_1, G_2 and G_3 defined as follows:

(i) Let G_0 denote the union of $r+t$ cliques $Y_1, \ldots, Y_r, Z_1, \ldots, Z_t$ of order $(k+1)$, and p cliques $X_1, \ldots, X_p$ of order $(k+2)$ where the vertices of each X_i are labeled $\{x_1^i, x_2^i, \ldots, x_{k+2}^i\}$, the vertices of each Y_i are labeled $\{y_1^i, y_2^i, \ldots, y_{k+1}^i\}$, and the vertices of each Z_i are labeled $\{z_1^i, z_2^i, \ldots, z_{k+1}^i\}$. Moreover, let X_i^Y denote the clique on the vertices $\{x_1^i, \ldots, x_{k+1}^i\}$, and let X_i^Z denote the clique on the vertices $\{x_2^i, \ldots, x_{k+2}^i\}$.

(ii) Let G_1 denote the graph obtained from G_0 by adding edges between each clique Y_i and each clique Z_j such that the subgraph of G_1 induced on the cliques Y_i, Z_j is a k-join of Y_i and Z_j. (Note that the subgraph of G_0 induced on the cliques $Y_1, \ldots, Y_r, Z_1, \ldots, Z_t$ is a k-join substitution of $K_{r,t}$).

(iii) Let G_2 denote the graph obtained from G_1 by adding edges between each clique Y_i and each clique X_j^Y such that the subgraph of G_2 induced on the cliques Y_i, X_j^Y is a k-join of Y_i and X_j^Y.

(iv) Let G_3 denote the graph obtained from G_2 by adding edges between each clique Z_i and each clique X_j^Z such that the subgraph of G_3 induced on cliques Z_i, X_j^Z is a k-join of Z_i and X_j^Z.

(v) Then $G(p,r,t;k)$ is the graph G_3.

Proposition 2. *The graph $G(p,r,t;k)$ has $p(k+2)+(r+t)(k+1)$ vertices and $(pr+pt+rt)\binom{k+2}{2}+p\binom{k+2}{2}+(r+t)\binom{k+1}{2}$ edges.*

For $k \geqslant 1$ and $n \geqslant 3k+4$ let us denote by $\mathcal{G}(k,n)$ the set of locally k-tree graphs of order n which can be obtained by Construction 3, i.e., $\mathcal{G}(k,n) = \{G(p,r,t;k)\colon p,r,t \geqslant 1 \text{ and } p(k+2)+(r+t)(k+1)=n\}$.

Lemma 5. *For any integer $n \geqslant (k+2)^2$ the set $\mathcal{G}(k,n)$ is nonempty.*

P r o o f. By Proposition 2, the graph $G(p,r,t;k)$ has $n = p(k+2)+(r+t)\times(k+1) = (k+1)(r+t+p)+p$ vertices. Let $a = \lfloor n/(k+1)\rfloor$ and let $b = n \bmod (k+1)$, and $0 \leqslant b \leqslant k$. We observe that $n \geqslant (k+2)^2$ implies that if $b=0$, then $a \geqslant k+4$, and if $b \geqslant 1$, then $a \geqslant k+2$. Hence, if $b=0$, we let $p=k+1$, $r=1$ and $t=a-k-3$, and if $b \geqslant 1$, we let $p=b$, $r=1$ and $t=a-b-1$. Clearly, since $k \geqslant 0$, in both cases we have $p,r,t \geqslant 1$ and $n = p(k+2)+(r+t)(k+1)$, which proves the lemma. □

Recall that a locally k-tree graph G is called maximal if G is not a spanning subgraph of any other locally k-tree graph.

We prove that the graphs in $\mathcal{G}(k,n)$ are maximal locally k-tree graphs.

Lemma 6. *Let G be a locally k-tree graph and let $E' \subseteq E(\overline{G})$ be a set of edges such that $G+E'$ is also a locally k-tree graph. Then for any edge $uv \in E'$ we have $d_G(u,v) \geqslant 3$.*

P r o o f. Let $u,v \in V(G)$ and $d_G(u,v)=2$. Then there is a vertex x such that $u,v \in N(x)$. Let $d(x)=d$. By our assumption $H = G[N(x)]$ is a k-tree, hence $|E(H+uv)| = (d-k)k+\binom{k}{2}+1$. Thus by Lemma 1, $H+uv$ is not a k-tree and $H+uv$ is not a subgraph of any k-tree. □

Lemma 7. *Let G be a locally k-tree graph. Let $E' \subseteq E(\overline{G})$ be a set of edges such that $G+E'$ also is a locally k-tree graph. Then there is a vertex v such that v is incident to $k+1$ edges of E', say $vv_1, vv_2, \ldots, vv_{k+1}$, and the vertices $v_1,\ldots,v_{k+1}$ induce a $(k+1)$-clique in G.*

P r o o f. Let $G' = G+E'$ and let w be a vertex of G which is incident to at least one edge of E'. Let $(w_1,\ldots,w_p)$ be a k-PEO of $G'[N(w)]$ such that $w,w_1,\ldots,w_k$ induce a $(k+1)$-clique in G (by Theorem 3 such a k-PEO exists). Let v be the first vertex of $(w_1, \ldots, w_p)$ which is joined with w by an edge belonging to E'. Let $\{v_1, v_2, \ldots, v_k\} \subseteq N(v)$ be the subset of $(w_1, \ldots, w_p)$ which precede v and induce a k-clique in G'. From Lemma 6 it follows that the edges of this clique are in $E(G)$. To complete the proof it suffices to show that the edges $v_1v, \ldots, v_kv$ are in E'. Suppose that this is not true and there is an edge vv_i which is not in E'. Then $d_G(v, w) = 2$ in G and the edge $vw \in E'$, a contradiction with Lemma 6. □

Theorem 9. *If $G \in \mathcal{G}(k, n)$, then G is a maximal locally k-tree graph.*

P r o o f. First we will show that G is a locally k-tree graph. The graph G_0 is a union of locally k-trees, hence it is locally k-tree. The edges which we added to G_0 in (ii) satisfy the assertions of Theorem 8, hence G_1 is locally k-tree. Similarly, the edges which we added to G_1 in (iii) and to G_2 in (iv) satisfy the assertions of Theorem 8, hence G_3 is locally k-tree. Thus G is locally k-tree. The graph G does not contain any vertex v for which there are vertices $v_1, \ldots, v_{k+1}$ which induce a $(k+1)$-clique and are in distance at least 3 with v. Then by Lemma 7, G is a maximal locally k-tree graph. □

5. The maximum size of locally k-tree graphs

In this section we characterize the graphs of $\mathcal{G}(k, n)$ with the maximum number of edges (for fixed k and n). As a consequence, we obtain a lower bound on the maximum number of edges in a locally k-tree graph on n vertices.

Lemma 8. *Let k, n, p, r, t be positive integers such that $n \geqslant (k+2)^2$ and $n = p(k+2) + (r+t)(k+1)$. If the graph $G(p, r, t; k)$ has the maximum number of edges for fixed k, n, p, then $|r - t| \leqslant 1$.*

P r o o f. Suppose that $G = G(p, r, t; k)$ is the graph with the maximum number of edges for fixed k, n, p and $r \geqslant t + 2$. Then we construct a new graph $G' = G(p', r', t'; k)$ using Construction 3 with parameters $p' = p$, $r' = r - 1$, $t' = t + 1$. Then

$$|E(G')| = |E(G)| + (r - t - 1)\binom{k+2}{2} > |E(G)|.$$

Hence the graph G' has more edges than G, a contradiction. □

Lemma 9. *Let k, n be positive integers such that $n \geqslant (k+2)(3k+2)$. If the graph $G(p,r,t;k)$ has the maximum number of edges for fixed k, n, then p satisfies one of the following conditions*

$$\frac{1}{k+2}\Big(\frac{k}{3k+2}n - \frac{k+1}{2(3k+2)}(3k^2+8k+c)\Big) \leqslant p \leqslant \frac{1}{k+2}\Big(\frac{k}{3k+2}n + \frac{k+1}{2(3k+2)}(3k^2+8k+8+c)\Big)$$

where $c=0$ if k is even, $c=-1$ if k is odd and $(n-p(k+2))/(k+1)$ is odd, and $c=1$ if k is odd and $(n-p(k+2))/(k+1)$ is even.

Proof. Let $G = G(p,r,t;k)$ have the maximum number of edges. Lemma 8 implies that $|r-t| \leqslant 1$. Then the graph G has

$$|E(G)| = p\frac{n-p(k+2)}{k+1}\binom{k+2}{2} + p\binom{k+2}{2} + \frac{n-p(k+2)}{k+1}\binom{k+1}{2} + \Big\lfloor\Big(\frac{n-p(k+2)}{2(k+1)}\Big)^2\Big\rfloor\binom{k+2}{2}$$

edges.

Case 1. k is even.

Suppose that $p < 1/(k+2)\big(k/(3k+2)n - (k+1)/(2(3k+2))(3k^2+8k)\big)$. Then the graph $G' = G(p',r',t';k)$ with parameters $p' = p+k+1$, $r' = r - \frac{1}{2}(k+2)$, $t' = t - \frac{1}{2}(k+2)$ has n vertices and more edges:

$$|E(G')| = |E(G)| + \frac{1}{4}(k+2)^2(3k+2)\Big[-p + \frac{1}{k+2}\Big(\frac{k}{3k+2}n - \frac{k+1}{2(3k+2)}(3k^2+8k)\Big)\Big].$$

Suppose that $p > 1/(k+2)\big(k/(3k+2)n + (k+1)/(2(3k+2))(3k^2+8k+8)\big)$. Then the graph $G' = G(p',r',t';k)$ with parameters $p' = p-(k+1)$, $r' = r + \frac{1}{2}(k+2)$, $t' = t + \frac{1}{2}(k+2)$ has n vertices and more edges:

$$|E(G')| = |E(G)| + \frac{1}{4}(k+2)^2(3k+2)\Big[p - \frac{1}{k+2}\Big(\frac{k}{3k+2}n + \frac{k+1}{2(3k+2)}(3k^2+8k+8)\Big)\Big].$$

Case 2. k is odd.

The proof falls naturally into two subcases.

Case 2.1. k is odd and $(n-p(k+2))/(k+1)$ is odd.

Since $(n-p(k+2))/(k+1)$ is odd, we have $|r-t|=1$ and assume that $r=t+1$. Then

$$|E(G)| = p\,\frac{n-p(k+2)}{k+1}\binom{k+2}{2} + p\binom{k+2}{2} + \frac{n-p(k+2)}{k+1}\binom{k+1}{2} + \Big(\Big(\frac{n-p(k+2)}{2(k+1)}\Big)^2 - \frac{1}{4}\Big)\binom{k+2}{2}.$$

Suppose that $p < 1/(k+2)\big(k/(3k+2n) - \frac{1}{2}(k+1)/(3k+2)(3k^2+8k-1)\big)$. Then the graph $G' = G(p', r', t'; k)$ with parameters $p' = p+k+1$, $r' = r - \frac{1}{2}(k+3)$, $t' = t - \frac{1}{2}(k+1)$ has more edges:

$$|E(G')| = |E(G)| + \frac{1}{4}(k+2)^2(3k+2)\Big[-p + \frac{1}{k+2}\Big(\frac{k}{3k+2}\,n - \frac{k+1}{2(3k+2)}(3k^2+8k-1)\Big)\Big].$$

If $p > 1/(k+2)\big(k/(3k+2)n + \frac{1}{2}(k+1)/(3k+2)(3k^2+8k+7)\big)$, then again there exists a graph with n vertices and more edges, i.e., $G' = G(p', r', t'; k)$ with parameters $p' = p-(k+1)$, $r' = r + \frac{1}{2}(k+1)$, $t' = t + \frac{1}{2}(k+3)$:

$$|E(G')| = |E(G)| + \frac{1}{4}(k+2)^2(3k+2)\Big[p - \frac{1}{k+2}\Big(\frac{k}{3k+2}n + \frac{k+1}{2(3k+2)}(3k^2+8k+7)\Big)\Big].$$

Case 2.2. k is odd and $(n-p(k+2))/(k+1)$ is even.

Since $(n-p(k+2))/(k+1)$ is even, we have that $|r-t|=0$. Then

$$|E(G)| = p\,\frac{n-p(k+2)}{k+1}\binom{k+2}{2} + p\binom{k+2}{2} + \frac{n-p(k+2)}{k+1}\binom{k+1}{2} + \Big(\frac{n-p(k+2)}{2(k+1)}\Big)^2\binom{k+2}{2}.$$

If $p < 1/(k+2)\big(k/(3k+2)n - \frac{1}{2}(k+1)/(3k+2)(3k^2+8k+1)\big)$, then the graph $G' = G(p', r', t'; k)$ with parameters $p' = p+k+1$, $r' = r - \frac{1}{2}(k+1)$, $t' = t - \frac{1}{2}(k+3)$ has more edges:

$$|E(G')| = |E(G)| + \frac{1}{4}(k+2)^2(3k+2)\Big[-p + \frac{1}{k+2}\Big(\frac{k}{3k+2}n - \frac{k+1}{2(3k+2)}(3k^2+8k+1)\Big)\Big].$$

If $p > 1/(k+2)\bigl(k/(3k+2)n + \frac{1}{2}(k+1)/(3k+2)(3k^2+8k+9)\bigr)$, then for parameters $p' = p - (k+1)$, $r' = r + \frac{1}{2}(k+1)$, $t' = t + \frac{1}{2}(k+3)$ the graph $G' = G(p', r', t'; k)$ has more edges:

$$|E(G')| = |E(G)| + \frac{1}{4}(k+2)^2(3k+2)\Bigl[p - \frac{1}{k+2}\Bigl(\frac{k}{3k+2}n + \frac{k+1}{2(3k+2)}(3k^2+8k+9)\Bigr)\Bigr].$$

Since by the assumption of the present lemma the number of vertices is large enough, in all the cases the graph G' exists. □

The next theorem gives, for any fixed n and k, the best choice for parameters p, r, t that maximizes the number of edges in $G(p, r, t; k)$.

Theorem 10. *Let k, n be positive integers such that $n \geqslant (k+2)(3k+2)$. The graph $G(p, r, t; k)$ achieves the maximum number of edges for given fixed n and k, when $|r - t| \leqslant 1$ and p is an integer from the interval*

$$I = \Bigl\langle \frac{1}{k+2}\Bigl(\frac{k}{3k+2}n - \frac{k+1}{2(3k+2)}(3k^2+8k+c)\Bigr), \frac{1}{k+2}\Bigl(\frac{k}{3k+2}n + \frac{k+1}{2(3k+2)}(3k^2+8k+8+c)\Bigr)\Bigr\rangle$$

such that $k+1$ divides $n - p$, and $c = 0$ if k is even, $c = -1$ if k is odd and $(n - p(k+2))/(k+1)$ is odd, and $c = 1$ if k is odd and $(n - p(k+2))/(k+1)$ is even.

Proof. Recall that $n = p(k+2) + (r+t)(k+1)$, which implies that $k+1$ divides $n - p$. From Lemma 8 and Lemma 9 it follows that if $G(p, r, t; k)$ has the maximum number of edges for fixed n, k; then $|r - t| \leqslant 1$ and p is an integer from the interval I. Now we prove that if there is more than one parameter p such that $k+1$ divides $n-p$ and $p \in I$, then the number of edges of $G(p, r, t; k)$ for any such choice of p is the same. We observe that it suffices to consider only two different values of p satisfying the conditions, since $|I| \leqslant k+2$ and $k+1$ must divide $n - p$. Therefore, let p and p' be two successive integers of I such that $k+1$ divides $n - p$ and $p' = p + k + 1$. Let r, t and r', t' be integers such that $n = p(k+2) + (r+t)(k+1)$, $|r - t| \leqslant 1$ and $n = p'(k+2) + (r'+t')(k+1)$, $|r' - t'| \leqslant 1$. We show that $G = G(p, r, t; k)$ and $G' = G(p', r', t'; k)$ have the same number of edges. By a calculation similar to that in the proof of Lemma 9, we have

$$|E(G')| = |E(G)| + \frac{1}{4}(k+2)^2(3k+2)\Bigl[-p + \frac{1}{k+2}\Bigl(\frac{k}{3k+2}n - \frac{k+1}{2(3k+2)}(3k^2+8k+c)\Bigr)\Bigr]$$

and $c = 0$ if k is even, $c = -1$ if k is odd and $(n - p(k+2))/(k+1)$ is odd, and $c = 1$ if k is odd and $(n - p(k+2))/(k+1)$ is even. Since $p \in I$, we have $|E(G')| \leqslant |E(G)|$. On the other hand,

$$\begin{aligned}|E(G)| &= |E(G')| \\ &\quad + \frac{1}{4}(k+2)^2(3k+2)\Big[p' - \frac{1}{k+2}\Big(\frac{k}{3k+2}n + \frac{k+1}{2(3k+2)}(3k^2+8k+8+c)\Big)\Big]\end{aligned}$$

and $c = 0$ if k is even, $c = -1$ if k is odd and $(n - p'(k+2))/(k+1)$ is odd, and $c = 1$ if k is odd and $(n - p'(k+2))/(k+1)$ is even. Since $p' \in I$, we have $|E(G')| \geqslant |E(G)|$. Thus $|E(G')| = |E(G)|$. □

Using Lemma 8 and Theorem 10 we can calculate the maximum number of edges in the graph $G(p, r, t; k)$. Thus we obtain the following result.

Theorem 11. *Let $k \geqslant 1$ and $n \geqslant (k+2)(3k+2)$ and let G be a locally k-tree graph of order n with the maximum number of edges. Then*

$$|E(G)| \geqslant \frac{k+1}{2(3k+2)}\,n^2 + \frac{3k(k+1)}{2(3k+2)}\,n + c(k)$$

for a constant $c = c(k)$.

Every locally tree graph is locally acyclic. Erdős and Simonovits [6] showed that the maximum-size locally acyclic graphs are precisely the nearly-balanced complete bipartite graphs (or Mantel-Turán's graphs) with maximum matching being added to one partite side chosen so that the matching is as large as possible. Hence, if n is the order, the size of the graph is $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil + \lfloor (n+1)/4 \rfloor \leqslant \frac{1}{4}n^2 + \frac{1}{4}n$. Therefore Lemma 9 implies the following

Corollary 5. *Let $n \geqslant 15$ and let G be a locally tree graph of order n with maximum size. Then*

$$\frac{1}{5}n^2 + \frac{3}{5}n + c \leqslant |E(G)| < \frac{1}{4}n^2 + \frac{1}{4}n,$$

where c is a constant.

Remark 3. For small n (i.e., $k+1 \leqslant n < (k+2)(3k+2)$) we can obtain a maximal locally k-tree graph of order n with large number of edges using Lemma 3 and the $(k+1)$-join substitution applied to the Mantel-Turán graph.

6. Concluding remarks

In Section 2 the minimum-size locally k-tree graphs of order n for $k \geqslant 0$ have been characterized. In Section 4 the construction which gives a lower bound for the maximum size of locally k-tree graphs of order n for large n and for $k \geqslant 1$ has been described. The problem of finding the maximum size of locally k-tree graphs of order n for $k = 0$ is solved by Mantel' Theorem [14] of 1906 on the largest size of triangle-free graphs. For $k \geqslant 1$ the problem is still open.

Problem 1. What is the maximum size of locally k-tree graphs of order n for $k \geqslant 1$?

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