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INTERPOLATION OF BOUNDED SEQUENCES

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Abstract. This paper deals with an interpolation problem in the open unit disc $\mathbb{D}$ of the complex plane. We characterize the sequences in a Stolz angle of $\mathbb{D}$, verifying that the bounded sequences are interpolated on them by a certain class of not bounded holomorphic functions on $\mathbb{D}$, but very close to the bounded ones. We prove that these interpolating sequences are also uniformly separated, as in the case of the interpolation by bounded holomorphic functions.

Keywords: interpolating sequence, Carleson's theorem, uniformly separated, Blaschke product, Lipschitz class

MSC 2010: 30D50, 30E05

1. Introduction

Let l^∞ be the space of all bounded sequences of complex numbers. We write (z_n) for any sequence in $\mathbb{D}$. As usual, we will put c for all positive constants. $\mathrm{Lip}_{(z_n)}$ denotes the space of all complex functions Φ on (z_n) such that

$$|\Phi(z_k)-\Phi(z_m)|\leqslant c|z_k-z_m|,\quad \forall k,m\in\mathbb{N}.$$

H^∞ is the space of all bounded holomorphic functions on $\mathbb{D}$. We denote by A the disc algebra and by Lip the Lipschitz class on $\mathbb{D}$, that is,

$$\mathrm{Lip}=\{f\in A\colon\ |f(z)-f(w)|\leqslant c|z-w|,\ \forall z,w\in\mathbb{D}\}.$$

Recall that $f\in\mathrm{Lip}$ is equivalent to $f'\in H^\infty$.

Let $\psi(z,w)=|z-w|/|1-\overline{z}w|$ be the pseudo-hyperbolic distance between $z,w\in\mathbb{D}$. The Blaschke product in $\mathbb{D}$ with zeros at (z_n) is the function in H^∞ defined by

$$B(z)=\prod_{n\in\mathbb{N}}\frac{|z_n|}{z_n}\frac{z-z_n}{1-\overline{z}_nz}.$$

We denote by B_m and $B_{m,k}$ the Blaschke products in $\mathbb{D}$ with zeros at $(z_n) \setminus \{z_m\}$ and $(z_n) \setminus \{z_m, z_k\}$, respectively. The Stolz angle with vertex at the point $\eta \in \partial\mathbb{D}$ and aperture $\beta > 0$ is the set

$$S_\beta(\eta) = \{z \in \mathbb{D}\colon\ |z - \eta| < (1 + \beta)(1 - |z|)\}.$$

Finally, (z_n) is called uniformly separated if

$$|B_m(z_m)| \geqslant c, \quad \forall m \in \mathbb{N}.$$

The well known Carleson's theorem ([2]) asserts for a sequence (z_n) that given any $(w_n) \in l^\infty$, there exists $f \in H^\infty$ such that $f(z_n) = w_n$, $\forall n \in \mathbb{N}$, if and only if (z_n) is uniformly separated. If $h \in H^\infty$ and (z_n) is the zero set of the function h, then $h = Bg$, where B is the Blaschke product in $\mathbb{D}$ with zeros at (z_n) and g is a function in H^∞ not vanishing on (z_n). Now, if we differentiate B and 'integrate' g to 'compensate', rather, take $f \in \mathrm{Lip}$, then it is possible to pose a new interpolation problem for a bounded sequence, that consists of finding an interpolating function in the form $B'f$, instead of a function in H^∞. More precisely,

Definition 1. (z_n) is called a balanced interpolating sequence, if given any $(w_n) \in l^\infty$, there exists $f \in \mathrm{Lip}$ such that $(B'f)(z_n) = w_n$, $\forall n \in \mathbb{N}$.

Our result is the following one:

Theorem 1. (z_n) *in a Stolz angle is a balanced interpolating sequence if and only if it is uniformly separated.*

In view of this theorem, the uniformly separated sequences continue being the interpolating sequences for l^∞, though we consider a weaker space of interpolating functions.

2. Proof of the theorem

Proof. Let (z_n) be a balanced interpolating sequence in a Stolz angle $S_\beta(\eta)$. For $m \in \mathbb{N}$ fixed, let (w_n) be defined by: $w_m = 1$; $w_k = 0$, if $k \neq m$. As $(w_n) \in l^\infty$, we take $f \in \mathrm{Lip}$ such that $(B'f)(z_n) = w_n$, $\forall n \in \mathbb{N}$.

For a given $f \in \mathrm{Lip}$ vanishing on $(z_n) \setminus \{z_m\}$, it is proved in [4] that

$$|f(z)| \leqslant c|z - z_k||B_{m,k}(z)|, \quad \forall k \in \mathbb{N},\ k \neq m.$$

Writing this inequality for $z = z_m$ and taking z_k as the point of (z_n) nearest to z_m in the Euclidean distance, say $z_{m'}$, we have

$$\frac{1}{|B'(z_m)|} \leqslant c|z_m - z_{m'}|\,|B_{m,m'}(z_m)|, \quad \forall m \in \mathbb{N}. \tag{1}$$

It is straightforward to obtain

$$|B'(z_m)| = \frac{|B_m(z_m)|}{1-|z_m|^2}, \quad \forall m \in \mathbb{N}. \tag{2}$$

Using (2), the expression (1) becomes

$$\frac{|B_m(z_m)|}{1-|z_m|^2}|z_m - z_{m'}|\,|B_{m,m'}(z_m)| \geqslant c, \quad \forall m \in \mathbb{N}.$$

Since $1-|z_m|^2 > 1-|z_m| > c|z_m - \eta|$ and $|B_{m,m'}(z_m)| < 1$, then we have

$$|B_m(z_m)| \geqslant c\,\frac{|z_m - \eta|}{|z_m - z_{m'}|}, \quad \forall m \in \mathbb{N}.$$

From this inequality, it follows immediately that $|B_m(z_m)| \geqslant c$, $\forall m \in \mathbb{N}$, that is, (z_n) is uniformly separated.

Now, let (z_n) be uniformly separated in a Stolz angle and $(w_n) \in l^\infty$. We take $h \in H^\infty$ such that $h(z_n) = w_n$, $\forall n \in \mathbb{N}$ and see that $h/B' \in \mathrm{Lip}_{(z_n)}$.

We will use that any $g \in H^\infty$ satisfies the following inequalities (see [1] for (5)):

$$|g(z) - g(w)| \leqslant c\psi(z,w), \quad \forall z,w \in \mathbb{D}. \tag{3}$$

$$|g'(z)| \leqslant \frac{c}{1-|z|^2}, \quad \forall z \in \mathbb{D}. \tag{4}$$

$$|g'(z)(1-|z|^2) - g'(w)(1-|w|^2)| \leqslant c\psi(z,w), \quad \forall z,w \in \mathbb{D}. \tag{5}$$

For $z_k, z_m \in (z_n)$, the triangle inequality gives

$$\Big|\frac{h(z_k)}{B'(z_k)} - \frac{h(z_m)}{B'(z_m)}\Big| \leqslant \frac{|h(z_k) - h(z_m)|}{|B'(z_m)|} + \frac{|h(z_k)|\,|B'(z_m) - B'(z_k)|}{|B'(z_k)|\,|B'(z_m)|}. \tag{6}$$

Taking into account (2), for a uniformly separated sequence it holds

$$|B'(z_i)| \geqslant \frac{c}{1-|z_i|^2}, \quad \forall i \in \mathbb{N},$$

and then, the sum in (6) is bounded by

$$c|h(z_k) - h(z_m)|(1-|z_m|^2) + c|h(z_k)||B'(z_m) - B'(z_k)|(1-|z_k|^2)(1-|z_m|^2). \tag{7}$$

Using (3), the first summand in (7) is bounded by

$$c\psi(z_k, z_m)(1-|z_m|^2) \leqslant c\psi(z_k, z_m)(1-|z_m|) \leqslant c|z_k - z_m|.$$

By the triangle inequality, the second summand in (7) is bounded by

$$c\{|B'(z_k)|(1-|z_k|^2)|(1-|z_k|^2)-(1-|z_m|^2)| \\ + |B'(z_m)(1-|z_m|^2) - B'(z_k)(1-|z_k|^2)|(1-|z_k|^2)\},$$

and using (4) and (5), by

$$c\left\{\big|\,|z_m|-|z_k|\,\big|(|z_m|+|z_k|) + \psi(z_k, z_m)(1-|z_k|^2)\right\} \leqslant c|z_k - z_m|.$$

Thus $h/B' \in \mathrm{Lip}_{(z_n)}$. For a sequence (z_n) in a Stolz angle, it is proved in [3] that given any $\Phi \in \mathrm{Lip}_{(z_n)}$, there exists $f \in \mathrm{Lip}$ such that $f(z_n) = \Phi(z_n)$, $\forall n \in \mathbb{N}$, if and only if (z_n) is the union of two uniformly separated sequences. Hence, there exists $f \in \mathrm{Lip}$, such that $f(z_n) = h(z_n)/B'(z_n)$, $\forall n \in \mathbb{N}$, that is, $(fB')(z_n) = w_n$, $\forall n \in \mathbb{N}$, and consequently, (z_n) is a balanced interpolating sequence. $\square$

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