Robert Rałowski; Szymon Żeberski
On nonmeasurable images

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 2, 423–434

Czechoslovak Mathematical Journal, 60 (135) (2010), 423–434

ON NONMEASURABLE IMAGES

Robert Rałowski, Szymon Żeberski, Wrocław

(Received November 4, 2008)

Abstract. Let $(X, \mathbb{I})$ be a Polish ideal space and let T be any set. We show that under some conditions on a relation $R \subseteq T^2 \times X$ it is possible to find a set $A \subseteq T$ such that $R(A^2)$ is completely $\mathbb{I}$-nonmeasurable, i.e, it is $\mathbb{I}$-nonmeasurable in every positive Borel set. We also obtain such a set $A \subseteq T$ simultaneously for continuum many relations $(R_\alpha)_{\alpha<2^\omega}$. Our results generalize those from the papers of K. Ciesielski, H. Fejzić, C. Freiling and M. Kysiak.

Keywords: nonmeasurable set, Bernstein set, Polish ideal space

MSC 2010: 03E35, 03E75, 28A99

1. Motivations

Let us start with an old result obtained by Sierpiński in [7]. He showed that there exist two subsets of reals A, B of Lebesgue measure zero such that their algebraic sum, i.e. $A + B = \{a + b\colon\ a \in A,\ b \in B\}$ is nonmeasurable. Similar fact is true in Baire category case.

Sierpiński's result was generalized to other σ-algebras and σ-ideals of subsets of $\mathbb{R}$. Kharazishvili in [4] proved that if the ideal $\mathbb{I}$ is not closed under algebraic sums and $\mathbb{A}$ is a σ-algebra such that the quotient Boolean algebra $\mathbb{A}/\mathbb{I}$ satisfies c.c.c. then there exist sets $A, B \in \mathbb{I}$ such that $A + B \notin \mathbb{I}$. A similar result was obtained by Cichoń and Jasiński in [1]. They proved that if $\mathbb{I}$ is a σ-ideal with coanalytic base then there exist $A, B \in \mathbb{I}$ such that $A + B$ is not $\mathbb{I}$-measurable i.e. does not belong to the σ-field generated by Borel sets and the ideal $\mathbb{I}$.

Ciesielski, Fejzic and Freiling proved in [3] a stronger version of Sierpiński's theorem. Namely, they showed that if $C \subseteq \mathbb{R}$ has the property that $C + C$ has positive outer measure then there exists $A \subseteq C$ such that $A + A$ is nonmeasurable. Kysiak in [5] generalized this result showing that if $\mathbb{I}$ is a σ-ideal of subsets of $\mathbb{R}$ and $\mathbb{A}$ is a σ-algebra of subsets of $\mathbb{R}$ such that $\mathbb{I} \subseteq \mathbb{A}$ and each set from $\mathbb{A} \setminus \mathbb{I}$ contains a perfect subset then for every $C \in \mathbb{I}$ such that $C + C \notin \mathbb{I}$ we can find $A \subseteq C$ such that $A + A \notin \mathbb{A}$.

In this paper we obtain generalizations of above results in few meanings. We replace addition by any binary relation satisfying some conditions and we replace $\mathbb{I}$-nonmeasurability by complete $\mathbb{I}$-nonmeasurability.

2. Definitions and notations

We use standard set-theoretical notation. If X is a set then we denote the family of all subsets of X by $\mathscr{P}(X)$. If κ is a cardinal number then $[X]^\kappa$ denotes the family of all subsets of X of size κ. Similarly, $[X]^{\leqslant\kappa}$ denotes the family of all subsets of X of size less or equal to κ.

If the space X is fixed and $A \subseteq X$ then by A^c we denote the complement of A, i.e. $A^c = X \setminus A$.

We say that a family $\mathbb{I} \subseteq \mathscr{P}(X)$ is an ideal if it is closed under finite unions and taking subsets. If additionally, $\mathbb{I}$ is closed under countable unions then we say that $\mathbb{I}$ is σ-ideal.

We say that X is a Polish space if X is a metric space with complete metric and a countable dense subset. By Borel(X) we denote the smallest σ-field generated by open subsets of X.

Definition (Polish ideal space)**.** We say that $(X, \mathbb{I})$ is a *Polish ideal space* iff X is an uncountable Polish space and $\mathbb{I} \subseteq \mathscr{P}(X)$ is a σ-ideal with a Borel base containing all singletons. Let us recall that the ideal $\mathbb{I}$ has Borel base iff

$$(\forall I \in \mathbb{I})(\exists B \in \mathbb{I})(B \supseteq I \land B \in \mathrm{Borel}(X)).$$

Definition 2.1 (completely $\mathbb{I}$-nonmeasurable set)**.** Let $(X, \mathbb{I})$ be a Polish ideal space. Let $A \subseteq X$. We say that A is *completely $\mathbb{I}$-nonmeasurable* whenever

$$(\forall B \in \mathscr{B}_+(\mathbb{I}))(A \cap B \neq \emptyset \land A^c \cap B \neq \emptyset),$$

where $\mathscr{B}_+(\mathbb{I}) = \mathrm{Borel}(X) \setminus \mathbb{I}$.

Let us recall that if $(X, \mathbb{I})$ is a Polish ideal space then $A \subseteq X$ is $\mathbb{I}$-nonmeasurable iff A does not belong to the σ-algebra generated by Borel sets and the ideal $\mathbb{I}$.

Completely $\mathbb{I}$-nonmeasurable sets have been investigated in particular in [2], [6], [8].

If $(X, \mathbb{I})$ is a Polish ideal space where $\mathbb{I} = [X]^{\leqslant\omega}$ then a set A is completely $\mathbb{I}$-nonmeasurable iff A is a Bernstein set and in the case when $\mathbb{I}$ is an ideal of Lebesgue null sets, a set A is completely $\mathbb{I}$-nonmeasurable iff A is saturated with respect to the Lebesgue measure.

Definition 2.3. Let $(X, \mathbb{I})$ be a Polish ideal space. We say that $\mathbb{I}$ has the *hole property* if

$$(\forall Z \subseteq X)(\exists B \in \mathrm{Borel}(X))(B \supseteq Z \wedge$$
$$(\forall B' \in \mathrm{Borel}(X))(B' \supseteq Z \to B \setminus B' \in \mathbb{I})).$$

In this case we define (see [6], [8])

$$[Z]_{\mathbb{I}} = \text{ minimal (mod } \mathbb{I}\text{) Borel set containing } Z,$$
$$]Z[_{\mathbb{I}} = \text{ maximal (mod } \mathbb{I}\text{) Borel set contained in } Z.$$

Let us remark that an ideals $\mathbb{I}$ has the hole property if the Boolean algebra $\mathrm{Borel}(X)/\mathbb{I}$ is complete. In particular, all σ-ideals which satisfy c.c.c. have also the hole property. So, the classical ideals of null sets $\mathbb{L}$ and meager sets $\mathbb{K}$ have the hole property.

Let us notice that if $\mathbb{I}$ has the hole property then for every $\mathbb{I}$-nonmeasurable set A, there exists a Borel $\mathbb{I}$-positive set $B \in \mathrm{Borel}(X) \setminus \mathbb{I}$ such that A is completely $\mathbb{I}$-nonmeasurable in the space $X \cap B$. (It is enough to put $B = [A]_{\mathbb{I}} \setminus]A[_{\mathbb{I}}$.)

Now let A, B be any sets and let $R \subseteq A \times B$ be a relation. Then for $X \subseteq A$ and $Y \subseteq B$ we define

$$R(X) = \{y \in B \colon\ \exists x \in X\ (x, y) \in R\},$$
$$R^{-1}(Y) = \{x \in A \colon\ \exists y \in Y\ (x, y) \in R\}.$$

If $x \in A$ and $y \in B$ then

$$R(x) = R(\{x\}) \text{ and } R^{-1}(y) = R^{-1}(\{y\}).$$

3. Results

In [3] authors have shown that for any subset $C \subseteq \mathbb{R}$ there exists a set $A \subseteq C$ such that $A+A$ is a Bernstein set in $R = \{x \in C+C \colon\ |\{(a, b) \in C\times C \colon\ x = a+b\}| = 2^\omega\}$.

The next theorem generalizes the above result in few meanings. We deal with any relation instead of the function $+$, we obtain the result for any ideal $\mathbb{I}$ having the hole property and we do not require that the image of the relation is measurable.

Theorem 3.1. *Let T be any set, λ be a cardinal number such that $\lambda < 2^\omega$ or $\lambda = 2^\omega$ for regular 2^ω. Let $(X, \mathbb{I})$ be a Polish ideal space and assume $\mathbb{I}$ has the hole property. Let $R \subseteq T^2 \times X$ be a binary relation satisfying the following conditions:*

(1) $[R(T^2)]_{\mathbb{I}} = X$,

(2) $|R^{-1}(x)| = 2^\omega$ *for $\mathbb{I}$-almost all $x \in X$,*

(3) $|R \cap S| < \lambda$ *for every* S *of the form* $\{a\} \times T \times \{x\}$ *or* $T \times \{a\} \times \{x\}$, *where* $x \in X$ *and* $a \in T$,
(4) *for every different* $x, y \in X$, $|R_{\square}^{-1}(x) \cap R_{\square}^{-1}(y)| < \lambda$, *where* $R_{\square}^{-1}(x) = \{(a,b), (b,a), (a,a), (b,b) \colon (a,b) \in R^{-1}(x)\}$.
(5) $|R((a,b))| < \lambda$ *for every* $a, b \in T$.

Then there exists a set $A \subseteq T$ *such that* $R(A^2)$ *is completely* $\mathbb{I}$*-nonmeasurable in* X.

P r o o f. Let us consider two families of positive (modulo $\mathbb{I}$) Borel sets, namely

$$\mathscr{B}_0 = \{B \in \mathscr{B}_+(\mathbb{I}) \colon B \subseteq]R(T^2)[_{\mathbb{I}}\},$$
$$\mathscr{B}_1 = \{B \in \mathscr{B}_+(\mathbb{I}) \colon B \subseteq X \setminus]R(T^2)[_{\mathbb{I}}\}.$$

Let us fix an enumerations

$$\mathscr{B}_0 = \{B_\alpha^0 \colon \alpha < \lambda_0\}, \quad \mathscr{B}_1 = \{B_\alpha^1 \colon \alpha < \lambda_1\}.$$

Since there are continuum many Borel sets, without loss of generality, we can assume that $\lambda_0 = \lambda_1 = 2^\omega$.

We will construct a sequence

$$\big\langle (A_\xi, D_\xi) \in \mathscr{P}(T) \times \mathscr{P}(]R(T^2)[_{\mathbb{I}}) \colon \xi < 2^\omega \big\rangle$$

possessing the following properties:

1: $(\forall \xi < 2^\omega)(\forall i \in \{0,1\})(R(A_\xi \times A_\xi) \cap B_\xi^i \neq \emptyset)$,
2: $(\forall \xi < 2^\omega)(D_\xi \cap B_\xi^0 \neq \emptyset)$,
3: $(\forall \xi < 2^\omega)(|A_\xi| < \max\{|\xi|^+, \lambda\} \wedge |D_\xi| < \max\{|\xi|^+, \lambda\})$,
4: $(\forall \xi < \eta < 2^\omega)(A_\xi \subseteq A_\eta \wedge D_\xi \subseteq D_\eta)$,
5: $(\forall \eta < 2^\omega)\Big(\bigcup_{\xi<\eta} R(A_\xi \times A_\xi) \cap \bigcup_{\xi<\eta} D_\xi = \emptyset\Big)$.

Suppose we are at an α-th step of the construction. It means we have a sequence $((A_\xi, D_\xi) \colon \xi < \alpha)$. We will construct A_α and D_α.

Put $A^\alpha = \bigcup_{\xi<\alpha} A_\xi$ and $D^\alpha = \bigcup_{\xi<\alpha} D_\xi$. Since $|D^\alpha| < 2^\omega$, there exists $c^0 \in B_\alpha^0 \setminus D^\alpha$. Fix $c^1 \in B_\alpha^1 \cap R(T^2)$. Fix $d \in D^\alpha$. Then $|\{(a,b) \in R^{-1}\{c^0, c^1\} \colon \{(a,a), (a,b), (b,a), (b,b)\} \cap R^{-1}\{d\} \neq \emptyset\}| < \lambda$. Thus, we have that the set

$$W = \{(a,b) \in R^{-1}\{c^0, c^1\} \colon \{(a,a), (a,b), (b,a), (b,b)\} \cap R^{-1}D^\alpha \neq \emptyset\}$$

has size less than $\max\{|\alpha|^+, \lambda\}$.

Now, let

$$E_1 = \{v \in T \colon \exists u \in A^\alpha \ (u,v) \cap R^{-1}D^\alpha \neq \emptyset\},$$
$$E_2 = \{v \in T \colon \exists u \in A^\alpha \ (v,u) \cap R^{-1}D^\alpha \neq \emptyset\}.$$

Let $E = E_1 \cup E_2$. Then by (3) we have $|E| < \max\{|\alpha|^+, \lambda\}$, thus we have $|R^{-1}D^\alpha \cap (E \times T \cup T \times E)| < \max\{|\alpha|^+, \lambda\}$.

Now let us pick some elements $a \in A^\alpha$ and $d \in D^\alpha$. Then by (3) we have

$$|\{b \in T\colon\ \{(a,b),(b,a)\} \cap R^{-1}\{d\} \neq \emptyset\}| < \lambda.$$

Then we have

$$|R^{-1}D^\alpha \cap (A^\alpha \times T \cup T \times A^\alpha)| < \max\{|\alpha|^+, \lambda\}.$$

But

$$|R^{-1}\{c^0\} \cap (A^\alpha \times T \cup T \times A^\alpha)| < \max\{|\alpha|^+, \lambda\},$$
$$|R^{-1}\{c^1\} \cap (A^\alpha \times T \cup T \times A^\alpha)| < \max\{|\alpha|^+, \lambda\},$$

so we can choose elements

$$(a^0, b^0) \in R^{-1}\{c^0\} \setminus ((E \times T \cup T \times E) \cup W \cup \Delta) \neq \emptyset,$$
$$(a^1, b^1) \in R^{-1}\{c^1\} \setminus ((E \times T \cup T \times E) \cup W \cup \Delta) \neq \emptyset,$$

where $\Delta = \{(u,u)\colon\ u \in T\}$.

Now take $A_\alpha = A^\alpha \cup \{a^0, b^0, a^1, b^1\}$. Since $|A_\alpha| < 2^\omega$ we have $|R(A_\alpha \times A_\alpha)| < 2^\omega$. By (5) we can find $d \in B_\alpha \setminus R(A_\alpha^2)$. Put $D_\alpha = D^\alpha \cup \{d\}$.

Our construction is completed.

Put $A = \bigcup_{\alpha<2^\omega} A_\alpha$. The set $R(A^2)$ has a nonempty intersection with every positive Borel set from X. What is more, $R(A^2) \cap D = \emptyset$, where $D = \bigcup_{\alpha<2^\omega} D_\alpha$. This shows that the complement of the set $R(A^2)$ has a nonempty intersection with every positive Borel set included in $]R(T^2)[_{\mathbb{I}}$. This finishes the proof that $R(A^2)$ is completely $\mathbb{I}$-nonmeasurable. $\square$

Theorem 3.1 requires dealing with ideal $\mathbb{I}$ which have the hole property. One can ask if a similar result is true in general. Naturally, we have to change the assumption (1). One way of doing this is to assume
$(1')$ $X \setminus R(T^2) \in \mathbb{I}$.
If additionally, we replace relation by function we get the following generalization of a result from [3].

Corollary 3.1. *Let T be any set, X an uncountable Polish space and λ be a cardinal number such that $\lambda < 2^\omega$ or $\lambda = 2^\omega$ for regular 2^ω. Let $f\colon T \times T \to X$ be a function satisfying the following conditions:*

1. $f(T \times T) = X$,
2. $|f^{-1}(x)| = 2^\omega$ *for all but countably many* $x \in X$,
3. $|f^{-1}(x) \cap S| < \lambda$ *for every* $x \in X$ *and* S *of the form* $\{a\} \times T$, $T \times \{a\}$, *where* $a \in T$,
4. *for every* $x \neq y \in X$, $|f_\square^{-1}(x) \cap f_\square^{-1}(y)| < \lambda$, *where* $f_\square^{-1}(x) = \{(b, a), (a, a), (a, b), (b, b)\colon (a, b) \in f^{-1}(x)\}$.

Then there exists $A \subseteq T$ such that $f(A \times A)$ is a Bernstein set in X.

Corollary 3.2. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a symmetric C^1 function such that $f_x \neq 0$ almost everywhere. (f_x denotes the derivative of f with respect to the first coordinate.) If $f(\mathbb{R}^2) = \mathbb{R}$ then there exists a subset $A \subseteq \mathbb{R}$ such that $f(A^2)$ is a Bernstein set in $\mathbb{R}$.*

By considerations analogous to those in Theorem 3.1 we get

Theorem 3.2. *Let λ be a cardinal number such that $\lambda < 2^\omega$ or $\lambda = 2^\omega$ for regular 2^ω. Let T_1, T_2 be any sets, $(X, \mathbb{I})$ any Polish ideal space and suppose that $\mathbb{I}$ has the hole property. Let $R \subseteq (T_1 \times T_2) \times X$ be a binary relation satisfying the following conditions:*

1. $[R(T_1 \times T_2)]_{\mathbb{I}} = X$,
2. $|R^{-1}(x)| = 2^\omega$ *for* $\mathbb{I}$*-almost all* $x \in X$,
3. $|R^{-1}(x) \cap S| < \lambda$ *for* S *of the form* $\{a\} \times T_2$, $T_1 \times \{b\}$, *where* $a \in T_1$, $b \in T_2$,
4. $|R(a, b)| < \lambda$ *for every* $a \in T_1$, $b \in T_2$.

Then there exist $A \subseteq T_1$ and $B \subseteq T_2$ such that $R(A \times B)$ is completely $\mathbb{I}$-nonmeasurable in the space X.

Now, we will deal with continuum many relations simultaneously.

Theorem 3.3. *Let λ be a cardinal number such that $\lambda < 2^\omega$ or $\lambda = 2^\omega$ for regular 2^ω. Let T be any set, $(X, \mathbb{I})$ a Polish ideal space and let $(R_\alpha)_{\alpha < 2^\omega} \in (\mathscr{P}(T^2 \times X))^{2^\omega}$ be a sequence of binary relations satisfying the following conditions for every α, β :*

1. $X \setminus R_\alpha(T^2) \in \mathbb{I}$,
2. $|R_\alpha^{-1}(x)| = 2^\omega$ *for* $\mathbb{I}$*-almost all* $x \in X$,
3. $|R_\alpha \cap S| < \lambda$ *for every* S *of the form* $\{a\} \times T \times \{x\}$, $T \times \{a\} \times \{x\}$, *where* $a \in T$, $x \in X$.
4. *for every* $x \neq y \in X$, $|R_{\alpha\square}^{-1}(x) \cap R_{\beta\square}^{-1}(y)| < \lambda$, *where* $R_\square^{-1}(x) = \{(a, b), (b, a), (a, a), (b, b)\colon (a, b) \in R^{-1}(x)\}$.
5. $|R_\alpha(a, b)| < \lambda$ *for every* $a, b \in T$.

Then there exists $A \subseteq T$ such that for every $\alpha < 2^\omega$ the set $R_\alpha(A^2)$ is completely $\mathbb{I}$-nonmeasurable in X.

P r o o f. Let us enumerate all $\mathbb{I}$-positive Borel sets in such a way that every $\mathbb{I}$-positive Borel set appears continuum many times:

$$\mathscr{B}_+(\mathbb{I}) = \{B_\alpha\colon\ \alpha < 2^\omega\}.$$

We will construct a sequence

$$\langle (A_\xi, D_\xi) \in \mathscr{P}(T) \times \mathscr{P}(X)\colon\ \xi < 2^\omega \rangle$$

having the following properties:

1: $(\forall \xi < 2^\omega)(\forall \zeta \leqslant \xi)(R_\zeta(A_\xi \times A_\xi) \cap B_\xi \neq \emptyset)$,

2: $(\forall \xi < 2^\omega)(D_\xi \cap B_\xi \neq \emptyset)$,

3: $(\forall \xi < 2^\omega)(|A_\xi| < \max\{|\xi|^+, \lambda\} \wedge |D_\xi| < \max\{|\xi|^+, \lambda\})$,

4: $(\forall \xi < \eta < 2^\omega)(A_\xi \subseteq A_\eta \wedge D_\xi \subseteq D_\eta)$,

5: $(\forall \eta < 2^\omega)(\forall \zeta < \eta)\Big(\Big(\bigcup\limits_{\xi<\eta} R_\zeta(A_\xi \times A_\xi) \setminus R_\zeta(A_\zeta \times A_\zeta)\Big) \cap \bigcup\limits_{\xi<\eta} D_\xi = \emptyset\Big)$.

Suppose we are at an α-th step of the construction. This means we have already constructed a sequence $\langle (A_\xi, D_\xi)\colon\ \xi < \alpha\rangle$. We will construct A_α and D_α.

Put $A^\alpha = \bigcup\limits_{\xi<\alpha} A_\xi$ and $D^\alpha = \bigcup\limits_{\xi<\alpha} D_\xi$. Fix $\xi \leqslant \alpha$. Since $|D^\alpha| < 2^\omega$, there exists $c^\xi \in (B_\alpha \setminus D^\alpha) \cap R_\xi(T^2)$. Fix $d \in D^\alpha$. Then for every $\eta \leqslant \alpha$

$$|\{(a,b) \in R_\xi^{-1}\{c^\xi\}\colon\ \{(a,a),(a,b),(b,a),(b,b)\} \cap R_\eta^{-1}\{d\} \neq \emptyset\}| < \lambda.$$

Thus, the set

$$W = \{(a,b) \in R_\xi^{-1}\{c^\xi\}\colon\ \Big\{(a,a),(a,b),(b,a),(b,b)\} \cap \bigcup_{\eta\leqslant\alpha} R_\eta^{-1} D^\alpha \neq \emptyset\Big\}$$

has size less than $\max\{|\alpha|^+, \lambda\}$.

Now, let

$$E_1 = \Big\{v \in T\colon\ \exists u \in A^\alpha\ (u,v) \cap \bigcup_{\zeta\leqslant\alpha} R_\zeta^{-1} D^\alpha \neq \emptyset\Big\},$$

$$E_2 = \Big\{v \in T\colon\ \exists u \in A^\alpha\ (v,u) \cap \bigcup_{\zeta\leqslant\alpha} R_\zeta^{-1} D^\alpha \neq \emptyset\Big\}.$$

Let $E = E_1 \cup E_2$. Then by (3) we have $|E| < \max\{|\alpha|^+, \lambda\}$ and thus $|R_\xi^{-1} D^\alpha \cap (E \times T \cup T \times E)| < \max\{|\alpha|^+, \lambda\}$.

Now let us pick some elements $a \in A^\alpha$ and $d \in D^\alpha$. Then by (3) we have

$$|\{b \in T\colon \{(a,b),(b,a)\} \cap R_\xi^{-1}\{d\} \neq \emptyset\}| < \lambda.$$

Then we have

$$|R^{-1}D^\alpha \cap (A^\alpha \times T \cup T \times A^\alpha)| < \max\{|\alpha|^+, \lambda\}.$$

But

$$|R_\xi^{-1}\{c^\xi\} \cap (A^\alpha \times T \cup T \times A^\alpha)| < \max\{|\alpha|^+, \lambda\},$$

so we can choose an element $(a^\xi, b^\xi) \in R_\xi^{-1}\{c^\xi\} \setminus ((E \times T \cup T \times E) \cup W \cup \Delta) \neq \emptyset$, where $\Delta = \{(u,u)\colon\ u \in T\}$. We do the same for all $\xi \leqslant \alpha$.

Now take $A_\alpha = A^\alpha \cup \bigcup_{\xi \leqslant \alpha} \{a^\xi, b^\xi\}$. Since $|A_\alpha| < 2^\omega$, we have $\left|\bigcup_{\xi \leqslant \alpha} R_\xi(A_\alpha \times A_\alpha)\right| < 2^\omega$. We can find $d \in B_\alpha \setminus \bigcup_{\xi \leqslant \alpha} R_\xi(A_\alpha^2)$. Put $D_\alpha = D^\alpha \cup \{d\}$.

Our construction is completed.

Put $A = \bigcup_{\alpha < 2^\omega} A_\alpha$. For every $\xi < 2^\omega$ the set $R_\xi(A^2)$ has a nonempty intersection with every positive Borel set from X. What is more, $R_\xi(A^2) \cap D \subseteq R_\xi(A_\xi^2)$, where $D = \bigcup_{\alpha < 2^\omega} D_\alpha$. Recall that $R_\xi(A_\xi^2)$ has size smaller than continuum. So the complement of the set $R_\xi(A^2)$ has nonempty intersection with every positive Borel set. This completes the proof that $R_\xi(A^2)$ is completely $\mathbb{I}$-nonmeasurable. $\square$

In Theorem 3.3 and Theorem 3.1 we have used assumption (4) which looks technically. The other disadvantage of condition (4) is that it's hard to find natural applications different from symmetric relations. The next theorem has more readable assumptions which provide us a little wider class of applications. The pay of simplicity is that in applications we will omit the ideal of countable sets which gives a Bernstein set. Instead of it we deal with ideals $\mathbb{L}$, $\mathbb{K}$ and get completely $\mathbb{L}, \mathbb{K}$-nonmeasurable sets.

Theorem 3.4. *Let T be any set, $(X, \mathbb{I})$ a Polish ideal space and let λ be a cardinal number such that $\lambda < 2^\omega$ or $\lambda = 2^\omega$ for regular 2^ω. Assume that $(R_\alpha)_{\alpha < 2^\omega} \in (\mathscr{P}(T^2 \times X))^{2^\omega}$ is a sequence of binary relations satisfying the following conditions for every α :*

(1) *$|R_\alpha^{-1}(x)| = 2^\omega$ for $\mathbb{I}$-almost all $x \in X$,*
(2) *$|R_\alpha \cap S| < \lambda$ for every S of the form $\Delta, \{a\} \times T \times \{x\}$, $T \times \{a\} \times \{x\}$ where $a \in T$, $x \in X$.*
(3) *for every $\mathbb{I}$-positive Borel set $B \subseteq X$, $|R_\alpha^{-1}(B) \cap \{a\} \times T| = 2^\omega$ for some $a \in T$,*
(4) *$|R_\alpha(a,b)| < \lambda$ for every $a, b \in T$.*

Then there exists $A \subseteq T$ such that for every $\alpha < 2^\omega$ the set $R_\alpha(A^2)$ is completely $\mathbb{I}$-nonmeasurable in X.

P r o o f. We proceed as in the proof of Theorem 3.3. We will construct a transfinite sequence

$$\langle (A_\xi, D_\xi) \in \mathscr{P}(T) \times \mathscr{P}(X)\colon\ \xi < 2^\omega \rangle$$

satisfying conditions 1 :–5 : (from Theorem 3.3).

Suppose we are at an α-th step of construction and we have constructed a sequence $\langle (A_\xi, D_\xi)\colon\ \xi < \alpha) \rangle$. As before put $A^\alpha = \bigcup\limits_{\xi<\alpha} A_\xi$ and $D^\alpha = \bigcup\limits_{\xi<\alpha} D_\xi$. Using assumptions (2) and (3) there exists $a^\xi \in T$ such that

$$(R_\xi^{-1}(B_\alpha) \cap \{a^\xi\} \times T) \setminus \Big(\Delta \cup \bigcup_{\eta<\alpha} R_{\eta\square}^{-1}(D^\alpha)\Big) \neq \emptyset$$

and choose any b^ξ such that (a^ξ, b^ξ) is in this set. We can do this with every $\xi < \alpha$. Now repeating arguments from the proof of Theorem 3.3 put $A_\alpha = A^\alpha \cup \bigcup\limits_{\xi\leqslant\alpha} \{a^\xi, b^\xi\}$. By (4) we can find $d_\alpha \in B_\alpha \setminus \bigcup\limits_{\xi\leqslant\alpha} R_\xi(A_\alpha^2)$ and then we put $D_\alpha = D^\alpha \cup \{d_\alpha\}$. It completes our construction. Rest of the proof is the same as in Theorem 3.3. $\square$

Before application let us formulate two technical claims.

Claim 3.1. Assume that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^1, onto. Assume that $B \subseteq \mathbb{R}$ is $\mathbb{L}$-positive Borel set. Then $f^{-1}(B)$ is $\mathbb{L}$-positive Borel set on the plane.

Claim 3.1 is a consequence of a well known fact that the image of null set under a Lipschitz function is null.

Claim 3.2. Assume that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is C^1, onto and its partial derivatives do not vanish almost everywhere. Assume that $B \subseteq \mathbb{R}$ is $\mathbb{K}$-positive Borel set. Then $f^{-1}(B)$ is $\mathbb{K}$-positive Borel set on the plane.

We leave a proof of Claim 3.2 to the reader.

Corollary 3.3. *There exists a set $A \subseteq \mathbb{R}$ such that $f(A \times A)$ is completely $\mathbb{L}$-nonmeasurable for every $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is C^1, onto.*

P r o o f. There are continuum many functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ which are C^1. Assumptions $(1), (2), (4)$ of Theorem 3.4 are fulfilled. To see that condition (3) of Theorem 3.4 is true it is enough to use Claim 3.1 and Fubini theorem. $\square$

Corollary 3.4. *There exists a set $A \subseteq \mathbb{R}$ such that $f(A \times A)$ is completely $\mathbb{K}$-nonmeasurable for every $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is C^1, onto and its partial derivatives do not vanish almost everywhere.*

The proof is similar to the proof of Corollary 3.3. We have to use Claim 3.2 instead of Claim 3.1 and Kuratowski-Ulam theorem instead of Fubini theorem.

In [3] authors have shown that there a set C such that $C + C$ is an interval and for any $A \subseteq C$ the set $A + A$ is not a Bernstein set. However, we can obtain the following result.

Theorem 3.5. *Let T_1, T_2 be any sets and $(X, \mathbb{I})$ be a Polish ideal space and let $f\colon T_1 \times T_2 \to X$ be any function satisfying the conditions*
(1) $f(T_1 \times T_2) = X$,
(2) $|f^{-1}(x)| \leqslant \omega$ *for $\mathbb{I}$-almost all $x \in X$,*
(3) *for every $\mathbb{I}$-positive Borel set $B \subseteq X$ we can find continuum many $a \in T_1$ such that $\{a\} \times T_2 \cap f^{-1}(B)$ has size continuum.*
Then there exist $A \subseteq T_1$ and $B \subseteq T_2$ such that $f(A \times B)$ is completely $\mathbb{I}$-nonmeasurable in X. Moreover, if $T_1 = T_2$, then there exists $A \subseteq T_1$ such that $f(A \times A)$ is completely $\mathbb{I}$-nonmeasurable.

Proof. Fix an enumeration $\mathscr{B}_+(\mathbb{I}) = \{I_\xi\colon \xi < 2^\omega\}$ of all positive Borel subsets of X. We will construct a transfinite sequence:

$$\langle (a_\xi, b_\xi, c_\xi, d_\xi) \in T_1 \times T_2 \times I_\xi \times I_\xi\colon \xi < 2^\omega \rangle$$

such that
(1) $(\forall \xi < 2^\omega)(f(a_\xi, b_\xi) = c_\xi)$,
(2) $(\forall \xi < 2^\omega) f(A_\xi \times B_\xi) \cap D_\xi = \emptyset$,
where $A_\xi = \{a_\zeta\colon \zeta < \xi\}$, $B_\xi = \{b_\zeta\colon \zeta < \xi\}$, $D_\xi = \{d_\zeta\colon \zeta < \xi\}$.

Assume we are at α-th step of the construction. We have

$$A_\alpha = \{a_\zeta\colon \zeta < \alpha\},\ B_\alpha = \{b_\zeta\colon \zeta < \alpha\},\ D_\alpha = \{d_\zeta\colon \zeta < \alpha\}.$$

Let

$$U_\alpha = \{a \in T_1\colon (\exists b \in T_2)(f(a, b) \in D_\alpha)\}$$

and

$$V_\alpha = \{b \in T_2\colon (\exists a \in T_1)(f(a, b) \in D_\alpha)\}.$$

Then $|U_\alpha| \leqslant |\alpha| < 2^\omega$ and $|V_\alpha| \leqslant |\alpha| < 2^\omega$. Since $|I_\alpha| = 2^\omega$ and for every $x \in X$ we have $|f^{-1}(\{x\})| \leqslant \omega$, $I_\alpha \setminus f(U_\alpha \times V_\alpha) \neq \emptyset$. So, by assumption (3) there exists

$(a, b) \in T_1 \times T_2$ such that $f(a, b) \in I_\alpha \setminus D_\alpha$. Moreover, there exists $d_\alpha \in I_\alpha \setminus (f(X_\alpha \cup \{a\}, Y_\alpha \cup \{b\})$. This completes α-step of the construction.

Now, put $A = \{a_\zeta \colon \zeta < 2^\omega\}$, $B = \{b_\zeta \colon \zeta < 2^\omega\}$, $C = \{c_\zeta \colon \zeta < 2^\omega\}$, $D = \{d_\zeta \colon \zeta < 2^\omega\}$. By the construction, the sets C and D intersect all $\mathbb{I}$-positive Borel sets. Moreover, $C \subseteq f(A \times B)$ and $D \cap f(A \times B) = \emptyset$. This shows that $f(A \times B)$ is completely $\mathbb{I}$-nonmeasurable.

To prove the second assertion let us assume $T_1 = T_2 = T$. We build a transfinite sequence

$$\langle (a_\xi, b_\xi, d_\xi) \in T \times T \times I_\xi \colon \xi < 2^\omega \rangle$$

satisfying

(1) $(\forall \xi < 2^\omega)(f(a_\xi, b_\xi) \in I_\xi$,

(2) $(\forall \xi < 2^\omega) f(A_\xi \times A_\xi) \cap D_\xi = \emptyset$,

where $A_\xi = \bigcup\limits_{\zeta < \xi} \{a_\zeta, b_\zeta\}$ and $D_\xi = \{d_\zeta \colon \zeta < \xi\}$.

Assume we are at an α-step of the construction. We have

$$A_\alpha = \bigcup_{\zeta < \alpha} \{a_\zeta, b_\zeta\}, \quad D_\alpha = \{d_\zeta \colon \zeta < \alpha\}.$$

Let

$$V_\alpha = \{a \in T \colon (\exists b \in T)(f(a, b) \in D_\alpha \vee f(b, a) \in D_\alpha)\}.$$

By the second assumption, we have $|V| < 2^\omega$. By the third assumption, $|\{a \in T \colon |\{b \in T\ f(a, b) \in I_\alpha\}| = 2^\omega\}| = 2^\omega$ and thus $\{a \in T \colon |\{b \in T\ f(a, b) \in I_\alpha\} \setminus V_\alpha \neq \emptyset$. So, there exist $a, b \in T \setminus V_\alpha$ such that $f(a, b) \in I_\alpha$ and for every $z \in T$ we have $f(a, z), f(b, z), f(z, a), f(z, b) \notin D_\alpha$. Now let $a_\alpha = a$, $b_\alpha = b$. Let d_α be any element of the set $I_\alpha \setminus f(A_{\alpha+1} \times A_{\alpha+1}) \neq \emptyset$. This completes the α-step of the construction. Now, put $A = \bigcup\limits_{\alpha < 2^\omega} A_\alpha$. It is clear that $f(A \times A)$ is completely $\mathbb{I}$-nonmeasurable. $\square$

Immediately we get a corollary.

Corollary 3.5. *Let $(X, \mathbb{I}_X)$, $(Y, \mathbb{I}_Y)$ and $(Z, \mathbb{I}_Z)$ be any Polish ideal spaces. Let us assume that $f \colon X \times Y \to Z$ is a function having the properties*

(1) *$f(X \times Y) = Z$,*

(2) *$|f^{-1}(z)| \leqslant \omega$ for $\mathbb{I}_Z$-almost all $z \in Z$,*

(3) *for every $\mathbb{I}_Z$-positive Borel set $B \subseteq Z$, $f^{-1}(B)$ has positive inner measure with respect to the family* $\mathrm{Borel}(X \times Y) \setminus (\mathbb{I}_X \times \mathbb{I}_Y)$.

Then there exist $A \subseteq X$ and $B \subseteq Y$ such that $f(A \times B)$ is completely $\mathbb{I}_Z$-nonmeasurable.

P r o o f. By Mycielski theorem for every set $D \subseteq X \times Y$ of positive measure (with respect to the family $\mathrm{Borel}(X \times Y) \setminus (\mathbb{I}_X \times \mathbb{I}_Y)$) there exist perfect sets $P \subseteq X$,

$Q \subseteq Y$ such that $P \times Q \subseteq D$. Now, we can apply Theorem 3.5 and get the desired conclusion. □

Acknowledgement. Authors would like to thank the referee for vital improvement of the presentation of the paper.

References

[1] *J. Cichoń and A. Jasiński*: A note on algebraic sums of sets of reals. Real Anal. Exchange *28* (2003), 493–497.

[2] *J. Cichoń, M. Morayne, R. Rałowski, Cz. Ryll-Nardzewski and Sz. Żeberski*: On nonmeasurable unions. Topology and its Applications *154* (2007), 884–893.

[3] *K. Ciesielski, H. Fejzić and C. Freiling*: Measure zero sets with non-measurable sum. Real Anal. Exchange *27* (2001/02), 783–793.

[4] *A. Kharazishvili*: Some remarks on additive properties of invariant σ-ideals on the real line. Real Anal. Exchange *21* (1995/96), 715–724.

[5] *M. Kysiak*: Nonmeasurable algebraic sums of sets of reals. Colloquium Mathematicum *102* (2005), 113–122.

[6] *R. Rałowski and Sz. Żeberski*: Complete nonmeasurability in regular families. Houston Journal in Mathematics *34* (2008), 773–780.

[7] *W. Sierpiński*: Sur la question de la measurabilite de la base de M. Hamel. Fundamenta Mathematicae *1* (1920), 105–111.

[8] *Sz. Żeberski*: On completely nonmeasurable unions. Mathematical Logic Quarterly *53* (2007), 38–42.

Authors' addresses: Robert Rałowski, Szymon Żeberski, Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland, e-mail: `Robert.Ralowski@pwr.wroc.pl`, `Szymon.Zeberski@pwr.wroc.pl`.